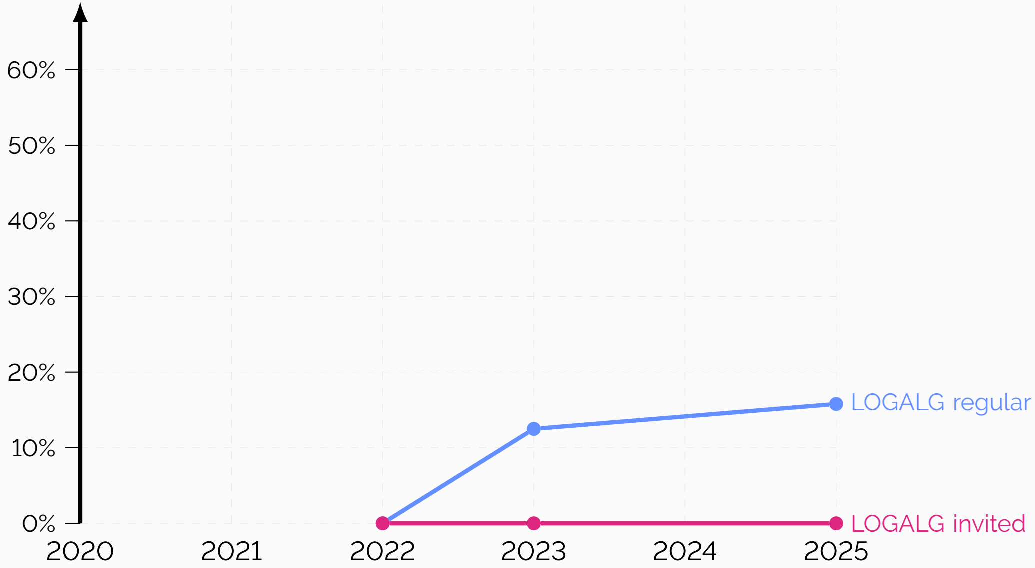
The Parameterized Complexity of Learning Monadic Second-Order Logic

Steffen van Bergerem, Martin Grohe, and Nina Runde

Workshop on Logic, Graphs, and Algorithms 2025

Women in Logic, Graphs, and Algorithms



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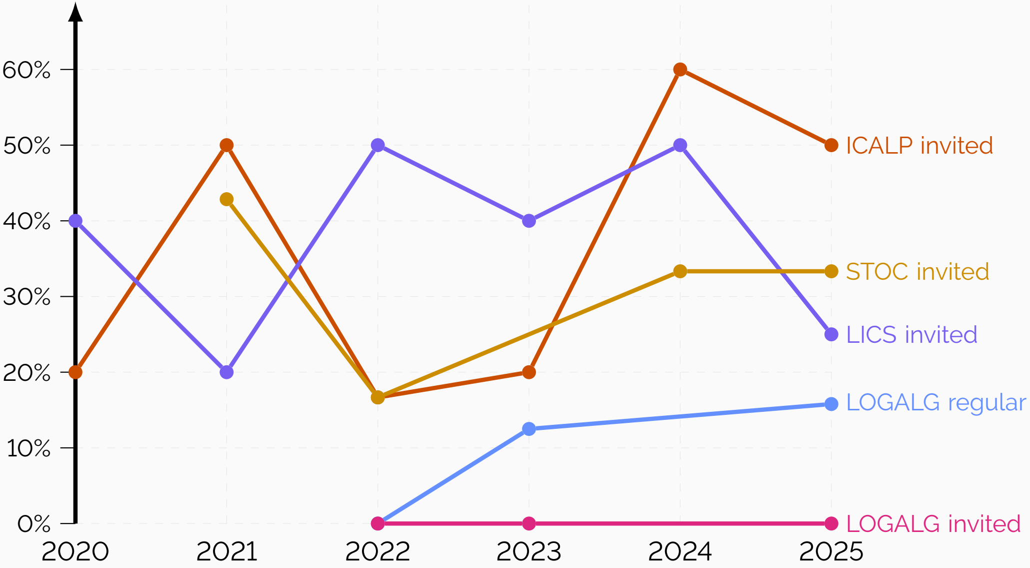

Women in Logic, Graphs, and Algorithms

Sandra Kiefer

descriptive complexity theory

Weisfeiler–Leman algorithm

Nicole Schweikardt

counting logics

FO enumeration for nowhere dense classes

Isolde Adler

nowhere dense = stable = dependent

graph decompositions

Nina Runde

MSO learning

homomorphism reconstructibility

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Model Checking

Given a graph G and a sentence φ

Decide whether $G \models \varphi$

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Enumeration

Given a graph G and a formula $\varphi(\bar{x})$

Enumerate all tuples $\bar{v}$ with $G \models \varphi(\bar{v})$

Learning

Consistent Learning

Given a graph G, sets of vertices $S_+, S_- \subseteq V(G)$

Output a formula $\varphi(x)$
such that $G \models \varphi(v)$ for all $v \in S_+$ and $G \not\models \varphi(v)$ for all $v \in S_-$

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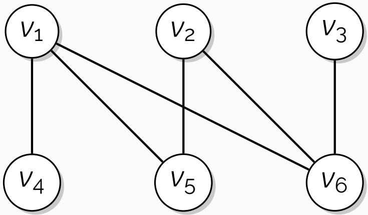

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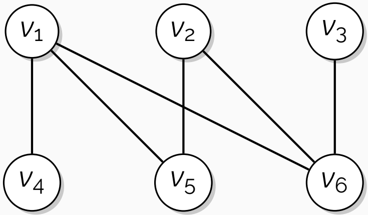

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Output $\varphi(x) = x{=}v_1 \lor x{=}v_3$

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Given a graph G, sets of vertices $S_+, S_- \subseteq V(G)$, and $\ell, q \in \mathbb{N}$

Output an MSO formula $\varphi(x)$ with $qr(\varphi) \leqslant q$ using at most ℓ constants such that $G \models \varphi(v)$ for all $v \in S_+$ and $G \not\models \varphi(v)$ for all $v \in S_-$

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Reject if there is no such formula

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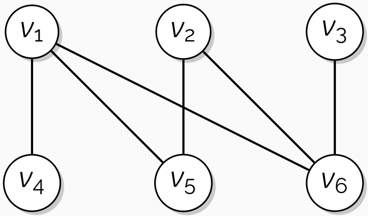

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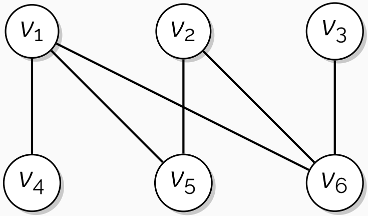

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Output $\varphi(x) = \exists Y\ (\text{bipartite}(Y) \wedge Y(v_1) \wedge Y(x))$

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v. B., Grohe, and Runde, CSL 2025

In general, the MSO consistent-learning problem is **para-NP-hard**.

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The MSO consistent-learning problem is **fixed-parameter linear** on classes of **bounded clique-width**.

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There is an algorithm that solves the MSO consistent-learning problem in time $f(\ell, q, c) \cdot |V(G)|$, where c is the clique-width of G and f is a computable function.

Proof idea

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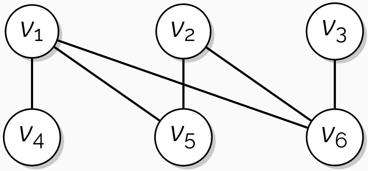

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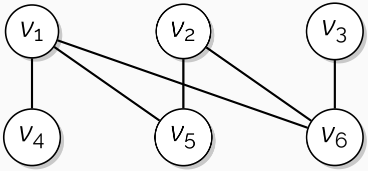

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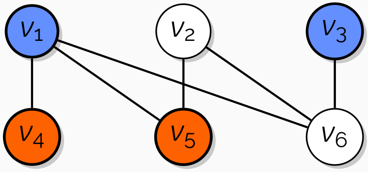

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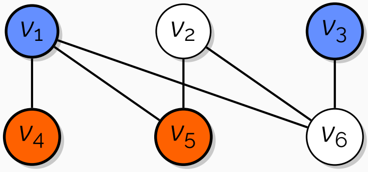

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- with placeholders instead of constants, the number of formulas only depends on $\ell, q \longrightarrow$ try all of them
- encode examples using two colours S_+, S_-
- for formula $\varphi(x, y_1, \ldots, y_\ell)$, check
  $G \models \exists y_1 \ldots \exists y_\ell \forall x \Big((S_+(x) \to \varphi(x, \bar{y})) \land (S_-(x) \to \neg\varphi(x, \bar{y})) \Big)$
- model-checking is fixed-parameter linear for bounded clique-width

Higher-Dimensional Learning

k-Dimensional MSO Consistent Learning

Given a graph G, sets of tuples $S_+, S_- \subseteq (V(G))^k$, and $\ell, q \in \mathbb{N}$

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There is an algorithm that solves the k-dimensional MSO consistent-learning problem in time $(m+1)^{f(k,\ell,q,c)} \cdot |V(G)|^2$, where c is the clique-width of G, $m = |S_+ \cup S_-|$, and f is a computable function.

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This is optimal.

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- Probably Approximately Correct Learning
- assume probability distribution on $(V(G))^k \times \{+,-\}$

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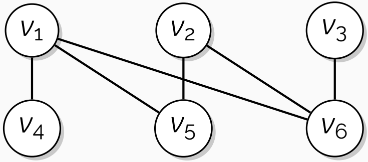
The MSO PAC-learning problem is **fixed-parameter linear** on classes of **bounded clique-width**, even in **higher dimensions** k.

Main Results



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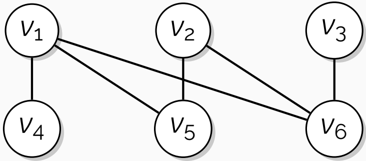

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1. **In general**, the MSO consistent-learning problem is **para-NP-hard**.

On classes of **bounded clique-width**:

2. The MSO **consistent-learning** problem is **fixed-parameter linear**.
3. The k-dim. consistent-learning problem can be solved in time quadratic in the size of the graph, but XP in the number of examples. This is optimal.
4. The MSO **PAC-learning** problem is **fixed-parameter linear**, even in **higher dimensions**.