Piecewise Parametric Structure in the Pooling Problem – from Sparse Strongly-Polynomial Solutions to NP-Hardness

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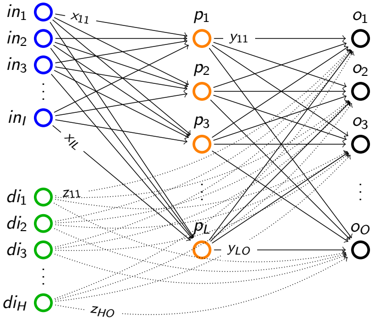
Globally Optimizing Pooling Problems



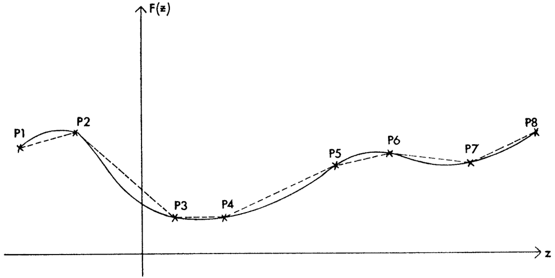
Previous work

- **Explore P / NP boundary** Alfaki and Haugland [2013] • Haugland [2016] • Haugland and Hendrix [2016] • Boland et al. [2016]
- **MIP Approximation** Dey and Gupte [2015] • Gupte et al. [2016]
- **Cutting Planes** D'Ambrosio et al. [2011] • Gupte et al. [2016]
- **Finding Patterns** Ceccon et al. [2016]

This work: Parameterize with respect to pool concentration

- Develop **strongly** polynomial algorithms for several pooling subproblems;
- Use **patterns of dominating topologies** to find active network structure and motivate problem sparsity.

How to solve problems multiplying quality times flowrate?



Haverly [1978]

Solve pooling problem via sequential linear programming, i.e., approximate nonconvex bilinear terms as linear.

Beale et al. [1965]

- **In Iron-Making** Track manganese concentration between two blast furnaces and a sinter plant;
- **To Solve** Reformulate as a separable program and apply a piecewise linear approximation;
- **Key Assumption** Only a few variables are active at once and the associated functions can be approximated linearly.

Standard Pooling Network p-Formulation

Objective $\max_{x_{il}, y_{lj}, z_{ij}, p_{lk}} \sum_{(l,j)\in T_Y} d_j \cdot y_{lj} + \sum_{(i,j)\in T_Z} d_j \cdot z_{ij} - \sum_{(i,l)\in T_X} \gamma_i \cdot x_{il} - \sum_{(i,j)\in T_Z} \gamma_i \cdot z_{ij}$

Feed Avail $\left[A_i^L \leq \sum_{l:(i,l)\in T_X} x_{il} + \sum_{j:(i,j)\in T_Z} z_{ij} \leq A_i^U \quad \forall\, i \right.$

Pool Capacity $\left[S_l^L \leq \sum_{i:(i,l)\in T_X} x_{il} \leq S_l^U \quad \forall\, l \right.$

Product Demand $\left[D_j^L \leq \sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \leq D_j^U \quad \forall\, j \right.$

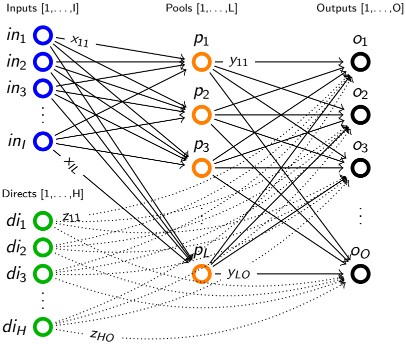
Material Balance $\left[\sum_{i:(i,l)\in T_X} x_{il} - \sum_{j:(l,j)\in T_Y} y_{lj} = 0 \quad \forall\, l \right.$

Quality Balance $\left[\sum_{i:(i,l)\in T_X} C_{ik} x_{il} = p_{lk} \sum_{j:(l,j)\in T_Y} y_{lj} \quad \forall\, l, k \right.$

Product Quality

$$\left[\begin{array}{l} \displaystyle\sum_{l:(l,j)\in T_Y} p_{lk} y_{lj} \\ \displaystyle + \sum_{i:(i,j)\in T_Z} C_{ik} z_{ij} \end{array} \right. \begin{cases} \geq P_{jk}^L \left(\displaystyle\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \right) \\ \leq P_{jk}^U \left(\displaystyle\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \right) \end{cases} \quad \forall\, j, k$$

Bounds $[x_{il},\ y_{lj},\ z_{ij} \geq 0 \ \forall\ i,\ l,\ j$



Standard Pooling Network Restricted p-Formulation

Objective $\max_{x_{il}, y_{lj}, z_{ij}, p_l} \sum_{(l,j)\in T_Y} d_j \cdot y_{lj} + \sum_{(i,j)\in T_Z} d_j \cdot z_{ij} - \sum_{(i,l)\in T_X} \gamma_i \cdot x_{il} - \sum_{(i,j)\in T_Z} \gamma_i \cdot z_{ij}$

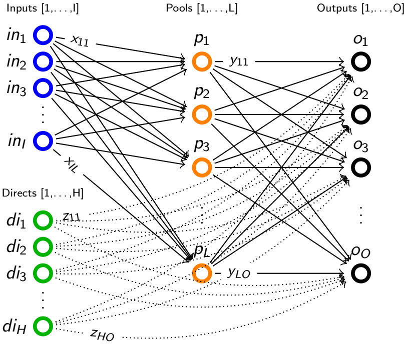
Product Demand $\left[\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \leq D_j^U \quad \forall\, j \right.$

Material Balance $\left[\sum_{i:(i,l)\in T_X} x_{il} - \sum_{j:(l,j)\in T_Y} y_{lj} = 0 \quad \forall\, l \right.$

Quality Balance $\left[\sum_{i:(i,l)\in T_X} C_i x_{il} = p_l \sum_{j:(l,j)\in T_Y} y_{lj} \quad \forall\, l \right.$

Product Quality
$$\left[\begin{array}{l} \displaystyle\sum_{l:(l,j)\in T_Y} p_l y_{lj} \\ \displaystyle + \sum_{i:(i,j)\in T_Z} C_i z_{ij} \end{array} \right. \begin{cases} \geq P_j^L \left(\displaystyle\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \right) \\ \leq P_j^U \left(\displaystyle\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \right) \end{cases} \quad \forall\, j$$

Bounds $\left[x_{il},\ y_{lj},\ z_{ij} \geq 0 \ \forall\ i,\ l,\ j \right.$



Assumptions

- No bounds on feed availability or pool capacity. No lower bounds and finite upper bounds on product demand. Only 1 quality k.
- The problem is feasible and profitable ($f^* > 0$)
- **Known:** NP-hard [Haugland, 2016]

Restricted p-Formulation for One Pool, One Output

Objective $\max_{\{x_i\},y,\{z_i\},p} f = d \cdot y + \sum_{i:(i,1)\in T_Z} d \cdot z_i - \sum_{i:(i,1)\in T_X} \gamma_i \cdot x_i - \sum_{i:(i,1)\in T_Z} \gamma_i \cdot z_i$

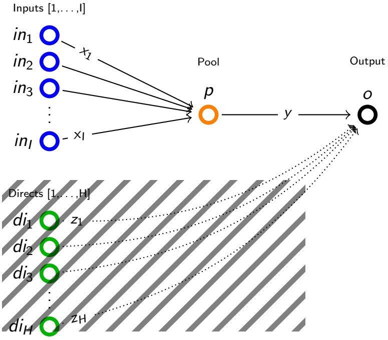
Product Demand $\left[y + \sum_{i:(i,1)\in T_Z} z_i \leq D^U \right.$

Material Balance $\left[\sum_{i:(i,1)\in T_X} x_i - y = 0 \right.$

Quality Balance $\left[\sum_{i:(i,1)\in T_X} C_i x_i = p \cdot y \right.$

Product Quality $\left[p \cdot y + \sum_{i:(i,1)\in T_Z} C_i \cdot z_i \right. \begin{cases} \geq \mathbf{P^L}\left(\mathbf{y} + \sum_{\mathbf{i:(i,1)\in T_Z}} \mathbf{z_i}\right) \\ \leq P^U\left(y + \sum_{i:(i,1)\in T_Z} z_i\right) \end{cases}$

Bounds $[x_i,\ y,\ z_i \geq 0\ \forall\ i$



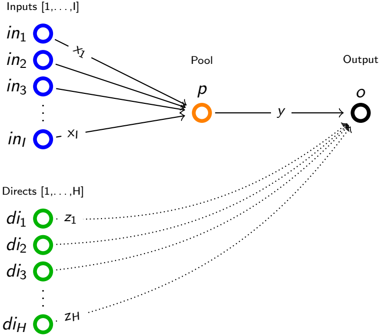
Known result

Can solve with a polynomial number of LPs of polynomial size [Haugland and Hendrix, 2016, Boland et al., 2016]

Restricted p-Formulation for One Pool, One Output

$$\max_{\{x_i\},\{z_i\}} f = d \cdot D^U - \sum_{i \in T_X} \gamma_i \cdot x_i - \sum_{i \in T_Z} \gamma_i \cdot z_i$$

$$\begin{cases} \sum_{i \in T_X} x_i + \sum_{i \in T_Z} z_i = D^U \\ P^L \le \left(\sum_{i \in T_X} C_i x_i + \sum_{i \in T_Z} C_i z_i \right) \Big/ \left(\sum_{i \in T_X} x_i + \sum_{i \in T_Z} z_i \right) \le P^U \\ p = \sum_{i \in T_X} C_i x_i \Big/ \sum_{i \in T_X} x_i \\ 0 \le x_i, \ \forall\ i \in T_X \qquad 0 \le z_i, \ \forall\ i \in T_Z \end{cases}$$



Solve by treating in turn:

- No directs (only inputs and quality constraints dropped)
- No inputs (only directs and quality constraints kept)
- Inputs + directs (quality constraints present)

Definition - Active set

Active set

Set A of nodes from the feed layer is:

- An **input active set** if $A \subseteq T_X,\ x_i > 0\ \forall i \in A,\ x_i = 0\ \forall i \in T_X \setminus A,\ z_i = 0\ \forall i \in T_Z.$
- A **direct active set** if $A \subseteq T_Z,\ z_i > 0\ \forall i \in A,\ z_i = 0\ \forall i \in T_Z \setminus A,\ x_i = 0\ \forall i \in T_X.$
- A **mixed active set** if $A \subseteq T_X \cup T_Z,\ x_i > 0\ \forall i \in A \cap T_X \neq \emptyset,\ z_i > 0\ \forall i \in A \cap T_Z \neq \emptyset, z_i = 0\ \forall i \in T_Z \setminus A,\ x_i = 0\ \forall i \in T_X \setminus A.$

For an active set A in One Pool-One Output, the objective function f,

$$f = \begin{cases} f_A(p) = dD - \sum_{i \in A \cap T_X} \gamma_i x_i - \sum_{i \in A \cap T_Z} \gamma_i z_i, & A \text{ an input/mixed active set,} \\ f_A = dD - \sum_{i \in A} \gamma_i z_i, & A \text{ a direct active set,} \end{cases}$$

Let $h = dD - f$ denote the cost function associated with objective function f.

Definition - Feasibility

Feasibility with respect to product quality constraints

A One Pool-One Output active set is **feasible** if the product quality bounds $[P^L, P^U]$ are met, i.e. the second constraint holds. An infeasible active set is not a valid One Pool-One Output solution and is therefore strictly dominated by any feasible active set.

Definition - Dominance

Dominance and breakpoints between active sets

Let A_1, A_2 be feasible input/mixed active sets. Let $f_A^*(p)$ be the optimal solution to One Pool-One Output and $h_A^*(p)$ its corresponding optimal cost, assuming active set A and fixed p.

- Set A_1 **dominates** A_2 **at** p (in the sense of maximized objective function profitability) when,

$$\boldsymbol{A}_1 \succeq_p \boldsymbol{A}_2 \quad \Leftrightarrow \quad f_{A_1}^*(p) \geq f_{A_2}^*(p) \quad \Leftrightarrow \quad h_{A_1}^*(p) \leq h_{A_2}^*(p).$$

- Pool concentration p is a **breakpoint between** A_1 **and** A_2 if:

$$\boldsymbol{A}_1 \asymp_p \boldsymbol{A}_2 \quad \Leftrightarrow \quad f_{A_1}^*(p) = f_{A_2}^*(p) \quad \Leftrightarrow \quad h_{A_1}^*(p) = h_{A_2}^*(p).$$

- The dominance relation also extends to direct active sets, but in this case f is not parametric on p. Consequently, when comparing two direct active sets, dominance is established similarly but independent of p, and as such no breakpoints exist. Thus, for fixed p, a total order can be established over the set of all possible active sets.

Definition - Dominant active sets

Dominant active sets and dominance breakpoints

Let $\mathcal{A}^*(p)$ be the **dominant active set** (overall) at p if

$$\mathcal{A}^*(p) = \underset{\{\mathcal{A}_I, \mathcal{A}_D, \mathcal{A}_M\}}{\arg\max} (f^*_{\mathcal{A}_I}(p), f^*_{\mathcal{A}_D}, f^*_{\mathcal{A}_M}(p)) = \underset{\{\mathcal{A}_I, \mathcal{A}_D, \mathcal{A}_M\}}{\arg\min} (h^*_{\mathcal{A}_I}(p), h^*_{\mathcal{A}_D}, h^*_{\mathcal{A}_M}(p))$$

and the optimal objective solution is $f^* = \max_p f^*_{\mathcal{A}^*(p)}(p)$, where:

- $\mathcal{A}_I$ is the **dominant input active set at** p if $\mathcal{A}_I(p) = \underset{A \subseteq T_X}{\arg\max} f^*_A(p)$.
- $\mathcal{A}_M$ is the **dominant mixed active set at** p if $\mathcal{A}_M(p) = \underset{A \subseteq T_X \cup T_Z,\ A \cap T_X \neq \emptyset,\ A \cap T_Z \neq \emptyset}{\arg\max} f^*_A(p)$.
- $\mathcal{A}_D$ is the **dominant direct active set** if $\mathcal{A}_D = \underset{A \subseteq T_Z}{\arg\max} f^*_A$.

A **dominance breakpoint** is the p value where the dominant active set changes:

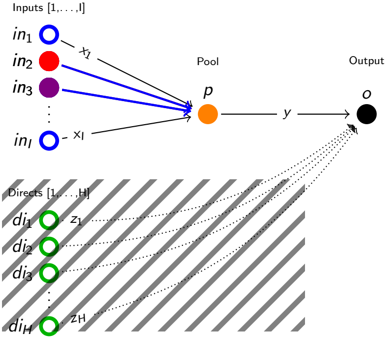
$$\forall \epsilon > 0 \ \mathcal{A}^*(p - \epsilon) = A_1 \neq A_2 = \mathcal{A}^*(p + \epsilon) \text{ and } A_1 \asymp_p A_2.$$

Input & mixed dominance breakpoints are similarly defined but $\mathcal{A}^*$ replaced by $\mathcal{A}_I$ & $\mathcal{A}_M$. Denote the input & mixed dominance breakpoints sets $\mathcal{B}_I$ & $\mathcal{B}_M$.

Only Input $\rightarrow$ Pool Flows (No Directs)

$$\max_{\{x_i\}} f = d \cdot D^U - \sum_{i \in T_X} \gamma_i \cdot x_i$$

$$\begin{cases} \sum_{i \in T_X} x_i = D^U \\ p = \sum_{i \in T_X} C_i x_i \Big/ \sum_{i \in T_X} x_i \\ 0 \leq x_i, \ \forall\, i \in T_X \end{cases}$$



For fixed p, consider 2 input nodes $\mathbf{in_2}$, $\mathbf{in_3}$ where $\mathbf{C_2} \leq p \leq \mathbf{C_3}$. Then the flows are:

$$\mathbf{x_2} = \frac{\mathbf{D^U(p - C_3)}}{\mathbf{C_2 - C_3}}, \quad \mathbf{x_3} = \frac{\mathbf{D^U(p - C_2)}}{\mathbf{C_3 - C_2}},$$

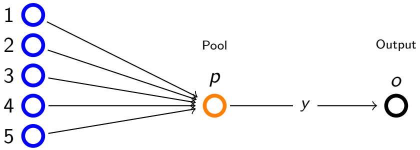
More generally, input pair $\{\mathbf{i}, \mathbf{j}\}$ has the best return on investment for a fixed p when:

$$\{\mathbf{i}, \mathbf{j}\} = \operatorname*{arg\,min}_{\mathbf{x} = \{\mathbf{i},\mathbf{j}\} \subseteq \mathbf{T_X}} \gamma_{\mathbf{x}}(\mathbf{p}) = \frac{\gamma_{\mathbf{i}}(\mathbf{p} - \mathbf{C_j}) + \gamma_{\mathbf{j}}(\mathbf{C_i} - \mathbf{p})}{\mathbf{C_i} - \mathbf{C_j}}$$

and $\{\mathbf{i}, \mathbf{j}\}$ is the **dominant input active set** (over any cardinality). The **breakpoint** in dominance of $\{\mathbf{i}, \mathbf{j}\}$ w.r.t. a given $\{\mathbf{k}, \mathbf{l}\}$ occurs at points

$$p_{\{\mathbf{i},\mathbf{j}\},\{\mathbf{k},\mathbf{l}\}} = \left(\frac{\gamma_{\mathbf{j}} \mathbf{C_i} - \gamma_{\mathbf{i}} \mathbf{C_j}}{\mathbf{C_i} - \mathbf{C_j}} - \frac{\gamma_{\mathbf{l}} \mathbf{C_k} - \gamma_{\mathbf{k}} \mathbf{C_l}}{\mathbf{C_k} - \mathbf{C_l}} \right) \Big/ \left(\frac{\gamma_{\mathbf{i}} - \gamma_{\mathbf{j}}}{\mathbf{C_i} - \mathbf{C_j}} - \frac{\gamma_{\mathbf{k}} - \gamma_{\mathbf{l}}}{\mathbf{C_k} - \mathbf{C_l}} \right) \forall \begin{cases} \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l} \in T_X, \\ \{\mathbf{i}, \mathbf{j}\} \neq \{\mathbf{l}, \mathbf{k}\} \end{cases}$$

Only Input → Pool Flows: Numerical Example



$(C_1 = 1, \gamma_1 = 135)$, $(C_2 = 2, \gamma_2 = 105)$, $(C_3 = 3, \gamma_3 = 90)$, $(C_4 = 4, \gamma_4 = 111)$, $(C_5 = 5, \gamma_5 = 140)$; $D^U = 100$; $d = 150$;

$$\partial \mathbf{f} / \partial \mathbf{p} = -\mathbf{D}^{\mathbf{U}}(\gamma_{\mathbf{i}} - \gamma_{\mathbf{j}})/(\mathbf{C}_{\mathbf{i}} - \mathbf{C}_{\mathbf{j}})$$

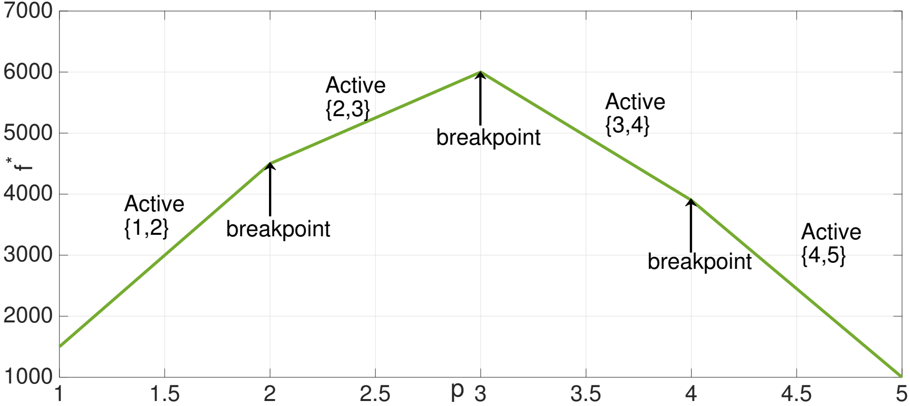
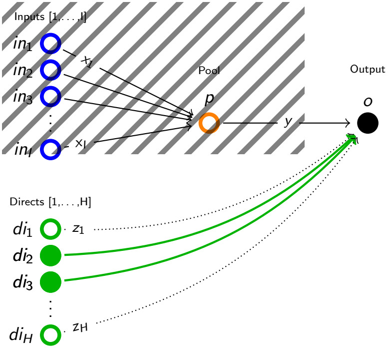


Figure 1: Optimal Objective Function $f^*(p)$ vs. pool concentration p

Only Input → Output Flows (No Pool)

$$\max_{\{z_i\}} f = d \cdot D^U - \sum_{i \in T_Z} \gamma_i \cdot z_i$$

$$\begin{cases} \sum_{i \in T_Z} z_i = D^U \\ P^L \le \sum_{i \in T_Z} C_i z_i \Big/ \sum_{i \in T_Z} z_i \le P^U \\ 0 \le z_i, \ \forall \ i \in T_Z \end{cases}$$



Given θ, the lowest cost direct node, i.e.,

$$\gamma_\theta = \min_{\mathbf{i} \in \mathbf{T_Z}} \gamma_{\mathbf{i}},$$

if $\mathbf{C}_\theta \in [\mathbf{P^L}, \mathbf{P^U}]$ then θ is the dominant direct active set.

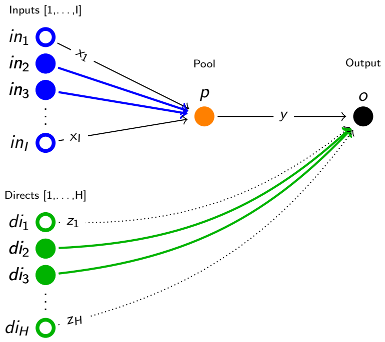
Otherwise a feasible active direct pair $\{\mathbf{i}, \mathbf{j}\}$ dominates any alternative:

$$\{\mathbf{i}, \mathbf{j}\} = \underset{\{\alpha,\beta\} \subseteq \mathbf{T_Z}}{\arg\min} \frac{\gamma_\alpha(\mathbf{P}^\star - \mathbf{C}_\beta) + \gamma_\beta(\mathbf{C}_\alpha - \mathbf{P}^\star)}{\mathbf{C}_\alpha - \mathbf{C}_\beta}, \ \mathbf{P}^\star = \begin{cases} P^L, \text{ if } (C_\alpha - C_\beta)(\gamma_\alpha - \gamma_\beta) > 0 \\ P^U, \text{ if } (C_\alpha - C_\beta)(\gamma_\alpha - \gamma_\beta) < 0 \end{cases}$$

Both Inputs and Directs: Intuition

$$\max_{\{x_i\},\{z_i\}} f = d \cdot D^U - \sum_{i \in T_X} \gamma_i \cdot x_i - \sum_{i \in T_Z} \gamma_i \cdot z_i$$

$$\begin{cases} \sum_{i \in T_X} x_i + \sum_{i \in T_Z} z_i = D^U \\ P^L \leq \left(\sum_{i \in T_X} C_i x_i + \sum_{i \in T_Z} C_i z_i \right) \Big/ \left(\sum_{i \in T_X} x_i + \sum_{i \in T_Z} z_i \right) \leq P^U \\ p = \sum_{i \in T_X} C_i x_i \Big/ \sum_{i \in T_X} x_i \\ 0 \leq x_i, \ \forall\, i \in T_X \qquad 0 \leq z_i, \ \forall\, i \in T_Z \end{cases}$$

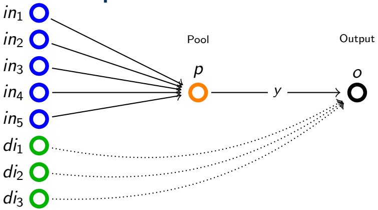


Calculate the breakpoints in strongly polynomial time (details: Optimization Online).

For a fixed p, the dominant active set is one of the following:

1. $\mathbf{x} = \{\mathbf{i}, \mathbf{j}\}$ delivering at output concentration $\mathbf{p}$ (equiv. to input $\rightarrow$ pool only).
2. $\mathbf{z} = \{\mathbf{k}, \mathbf{l}\}$ delivering at output concentration $\mathbf{P}^\star$ (equiv. to input $\rightarrow$ output only).
3. $\{\mathbf{x}, \mathbf{q}\} = \{\mathbf{i}, \mathbf{j}, \mathbf{q}\}$ delivering at output concentration $P(\mathbf{x}, \mathbf{q})$, where $\mathbf{x}$ is the blend of nodes $\mathbf{i}, \mathbf{j}$ with concentration $\mathbf{p}$.

Both Inputs and Directs: Example & Algorithm



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$(C_{in_3} = 3, \gamma_{in_3} = 90)$, $(C_{in_4} = 4, \gamma_{in_4} = 111)$,
$(C_{in_5} = 5, \gamma_{in_5} = 140)$;
$(C_{di_1} = 1.5, \gamma_{di_1} = 150)$, $(C_{di_2} = 8, \gamma_{di_2} = 115)$,
$(C_{di_3} = 4.5, \gamma_{di_3} = 120)$,
$D^U = 100$; $d = 150$; $P^L = 1$; $P^U = 3$;

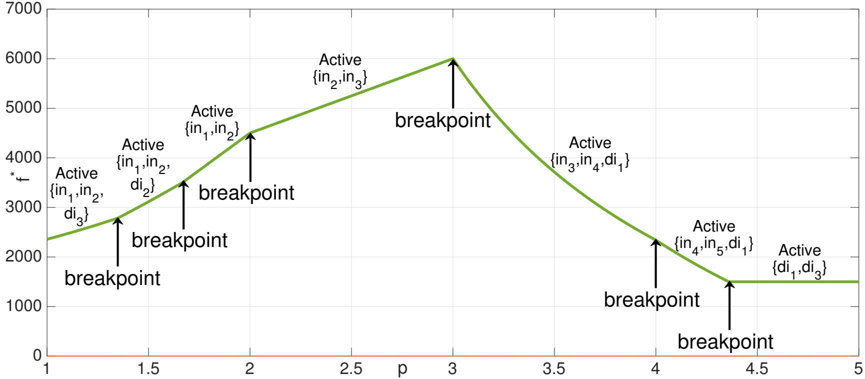


Figure 2: Optimal Objective Function $f^*(p)$ vs. pool concentration p

Both Inputs and Directs: Example & Algorithm

Over a Breakpoint (p) Interval I:

- If a direct-only active set z dominates over I, $f^*(p)$ is **constant** w.r.t. p over I.
- If an input-only active set x dominates over I, $f^*(p)$ is **linear** w.r.t. p over I.
- If a mixed (input and direct) active set m dominates over I, $f^*(p)$ is **monotone and either convex or concave** w.r.t. p over I.

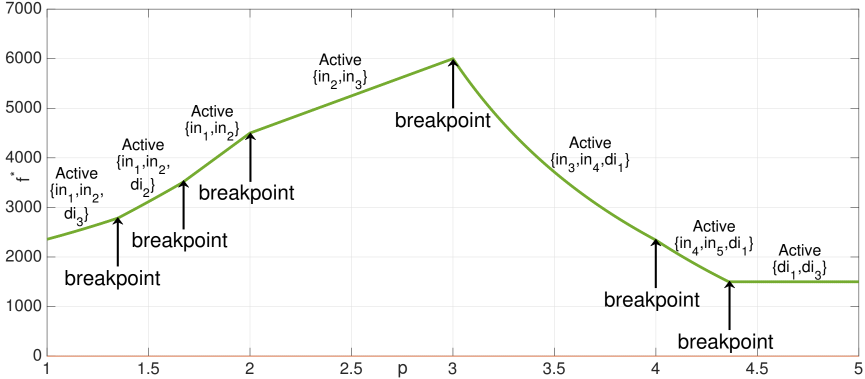


Figure 2: Optimal Objective Function $f^*(p)$ vs. pool concentration p

Both Inputs and Directs: Example & Algorithm

Finding the Global Solution

In strongly polynomial-time, find optimal objective $f^* = \max\left(\max_{p \in B} f^*(p), f_z^*\right)$, where B is the set of all breakpoints between dominant input-only and mixed active sets, and f_z^* is the optimal objective for the dominant direct-only active set.

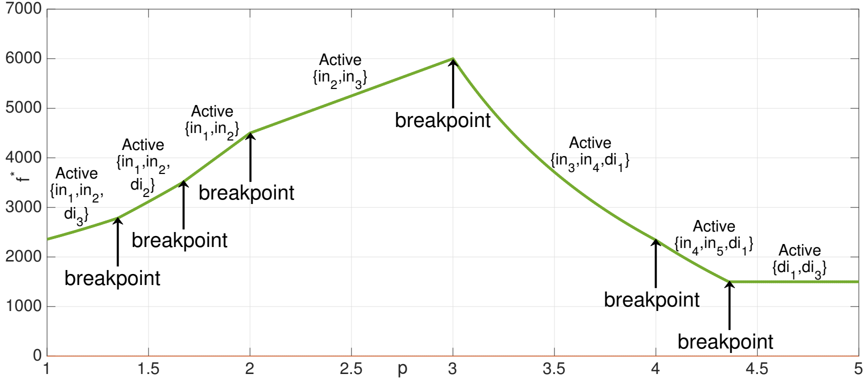


Figure 2: Optimal Objective Function $f^*(p)$ vs. pool concentration p

One Pool, Multiple Outputs: Problem

Objective $\max_{\{x_i\},\{y_j\},\{z_{ij}\},p} f = \sum_{j \in T_Y} d_j \cdot y_j + \sum_{(i,j) \in T_Z} d_j \cdot z_{ij} - \sum_{i \in T_X} \gamma_i \cdot x_i - \sum_{(i,j) \in T_Z} \gamma_i \cdot z_{ij}$

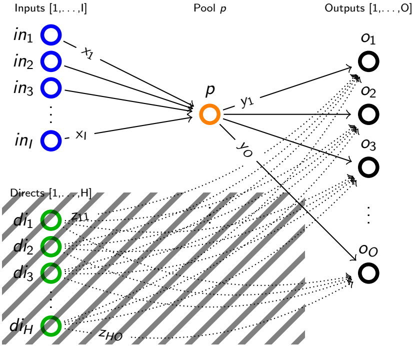
Product Demand $\left[y_j + \sum_{i:(i,j) \in T_Z} z_{ij} \le D_j^U \quad \forall\, j \right.$

Material Balance $\left[\sum_{i \in T_X} x_i - \sum_{j \in T_Y} y_j = 0 \right.$

Quality Balance $\left[\sum_{i \in T_X} C_i x_i = p \sum_{j \in T_Y} y_j \right.$

Product Quality $\left[p \cdot y_j + \sum_{i:(i,j) \in T_Z} C_i \cdot z_{ij} \begin{cases} \ge \mathbf{P_j^L} \left(\mathbf{y_j} + \sum_{\mathbf{i:(i,j) \in T_Z}} \mathbf{z_{ij}} \right) \\ \le P_j^U \left(y_j + \sum_{i:(i,j) \in T_Z} z_{ij} \right) \end{cases} \forall\, j \right.$

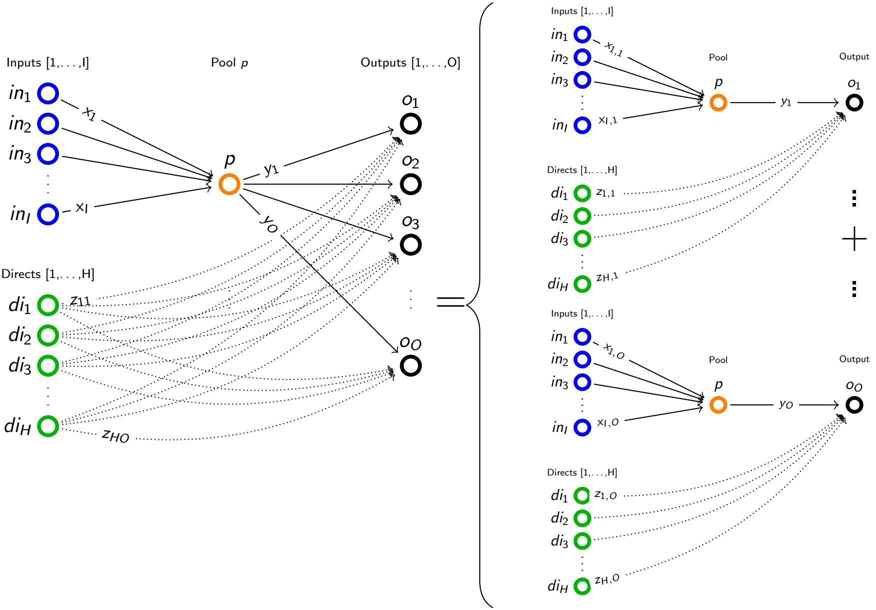
Bounds $\left[x_i,\ y_j,\ z_{ij} \ge 0\ \forall\ i,\ j \right.$



Known result

Can solve with a polynomial number of LPs of polynomial size [Haugland and Hendrix, 2016, Boland et al., 2016]

One Pool, Multiple Outputs: Additivity Intuition



One Pool, Multiple Outputs: Additivity over Outputs

$$\max_{\{x_i\},\{y_j\},\{z_{ij}\},p} f = \sum_{j\in T_Y} d_j\cdot y_j + \sum_{(i,j)\in T_Z} d_j\cdot z_{ij} - \sum_{i\in T_X}\gamma_i\cdot x_i - \sum_{(i,j)\in T_Z}\gamma_i\cdot z_{ij}$$

$$\left[\, y_j + \sum_{i:(i,j)\in T_Z} z_{ij} \le D_j^U \quad \forall\, j \right.$$

$$\left[\, \sum_{i\in T_X} x_i - \sum_{j\in T_Y} y_j = 0 \right.$$

$$\left[\, p\cdot y_j + \sum_{i:(i,j)\in T_Z} C_i\cdot z_{ij} \begin{cases} \ge P_j^L\left(y_j + \sum_{i:(i,j)\in T_Z} z_{ij}\right) \\ \le P_j^U\left(y_j + \sum_{i:(i,j)\in T_Z} z_{ij}\right) \end{cases} \quad \forall\, j \right.$$

$$\left[\, \sum_{i\in T_X} C_i x_i = p\sum_{j\in T_Y} y_j \right.$$

$$\left[\begin{array}{rccl} 0 \le & x_i & & \forall\, i\in T_X \\ 0 \le & y_j & & \forall\, j\in T_Y \\ 0 \le & z_{ij} & & \forall\, (i,j)\in T_Z \\ \min_i C_i \le & p & & \le \max_i C_i \end{array}\right.$$

$$\begin{cases} f(p) = \sum_{j\in T_Y} f_j(p) \\ \Leftrightarrow \max_{\{x_i\},\{y_j\},\{z_{ij}\},p} f = \\ = \max_p \left(\sum_{j\in T_Y} \max_{\{x_{ij}\},\{z_{ij}\}} f_j(p)\right), \\ x_i = \sum_{j\in T_Y} x_{ij} \quad \forall i\in T_X, \\ y_j = \sum_{i\in T_X} x_{ij} \quad \forall j\in T_Y, \\ p = \sum_{i\in T_X} C_i x_i \Big/ \sum_{j\in T_Y} y_j, \end{cases}$$

$$\Longleftrightarrow$$

For j given, $\forall j$:

$$\max_{\{x_{ij}\},\{z_{ij}\}} f_j(p) = d_j\cdot y_j + \sum_{i:(i,j)\in T_Z} d_j\cdot z_{ij} - \sum_{i\in T_X}\gamma_i\cdot x_{ij} - \sum_{i:(i,j)\in T_Z}\gamma_i\cdot z_{ij}$$

$$\left[\, y_j + \sum_{i:(i,j)\in T_Z} z_{ij} \le D_j^U \right.$$

$$\left[\, \sum_{i\in T_X} x_{ij} - y_j = 0 \right.$$

$$\left[\, p\cdot y_j + \sum_{i:(i,j)\in T_Z} C_i\cdot z_{ij} \begin{cases} \ge P_j^L\left(y_j + \sum_{i:(i,j)\in T_Z} z_{ij}\right) \\ \le P_j^U\left(y_j + \sum_{i:(i,j)\in T_Z} z_{ij}\right) \end{cases} \right.$$

$$\left[\, \sum_{i\in T_X} C_i x_{ij} = p\cdot y_j \right.$$

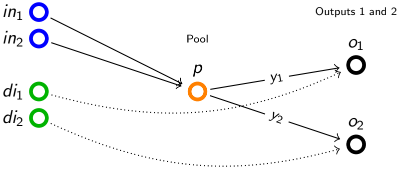
$$\left[\begin{array}{rccl} 0 \le & x_{ij} & & \forall\, i\in T_X \\ 0 \le & y_j & & \\ 0 \le & z_{ij} & & \forall\, i:(i,j)\in T_Z \\ \min_i C_i \le & p & & \le \max_i C_i \end{array}\right.$$

One Pool, Multiple Outputs: Decomposed Problem

$$\max_{\{x_i\},\{y_j\},\{z_{ij}\},p} f = \max_p \left(\sum_{j\in T_Y} \max_{\{x_{ij}\},\{z_{ij}\}} f_j(p) \right),$$

$$\begin{cases} x_i = \sum_{j\in T_Y} x_{ij} \quad \forall i \in T_X, \\ y_j = \sum_{i\in T_X} x_{ij} \quad \forall j \in T_Y, \\ p = \sum_{i\in T_X} C_i x_i \Big/ \sum_{j\in T_Y} y_j, \\ \forall j \in T_Y \quad \begin{cases} \max_{\{x_{ij}\},\{z_{ij}\}} f_j(p) = d_j \cdot D_j^U - \sum_{i\in T_X} \gamma_i \cdot x_{ij} - \sum_{i\in T_Z} \gamma_i \cdot z_{ij}, \\ \begin{cases} \sum_{i\in T_X} x_{ij} + \sum_{i\in T_Z} z_{ij} = D_j^U, \\ P_j^L \le \left(\sum_{i\in T_X} C_i x_{ij} + \sum_{i\in T_Z} C_i z_{ij} \right) \Big/ \left(\sum_{i\in T_X} x_{ij} + \sum_{i\in T_Z} z_{ij} \right) \le P_j^U, \\ p = \sum_{i\in T_X} C_i x_{ij} \Big/ \sum_{i\in T_X} x_{ij}, \\ 0 \le x_{ij},\ \forall\ i \in T_X \qquad 0 \le z_{ij},\ \forall\ i \in T_Z. \end{cases} \end{cases} \end{cases}$$

One Pool, Multiple Outputs: Numerical Example



$(C_{in_1} = 3, \gamma_{in_1} = 6)$, $(C_{in_2} = 1, \gamma_{in_2} = 13)$,
$(C_{di_1} = 1.5, \gamma_{di_1} = 12)$, $(C_{di_2} = 2.5, \gamma_{di_2} = 3)$,
$D_1^U = 200$; $D_2^U = 150$; $d_1 = 15$; $d_2 = 15$;
$P_1^L = 1$; $P_1^U = 2$; $P_2^L = 1$; $P_2^U = 2$;

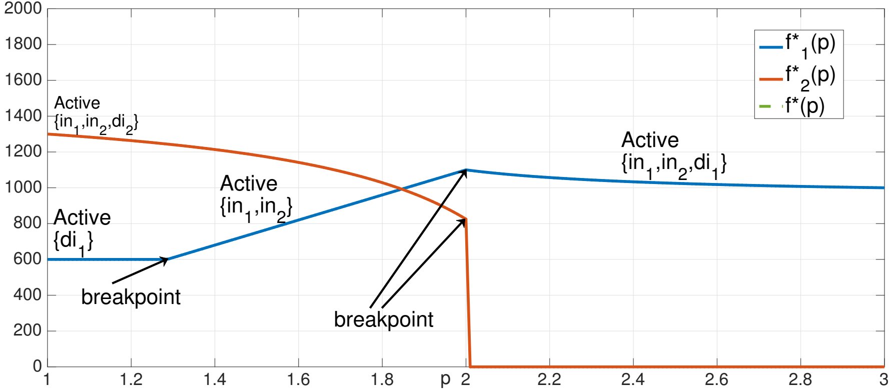
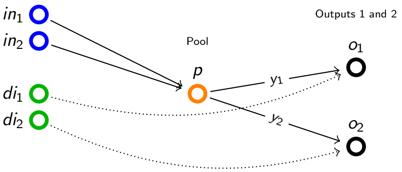


Figure 3: Optimal Objective Function $f^*(p)$ vs. pool concentration p

One Pool, Multiple Outputs: Numerical Example



$(C_{in_1} = 3, \gamma_{in_1} = 6),\ (C_{in_2} = 1, \gamma_{in_2} = 13),$
$(C_{di_1} = 1.5, \gamma_{di_1} = 12),\ (C_{di_2} = 2.5, \gamma_{di_2} = 3),$
$D_1^U = 200;\ D_2^U = 150;\ d_1 = 15;\ d_2 = 15;$
$P_1^L = 1;\ P_1^U = 2;\ P_2^L = 1;\ P_2^U = 2;$

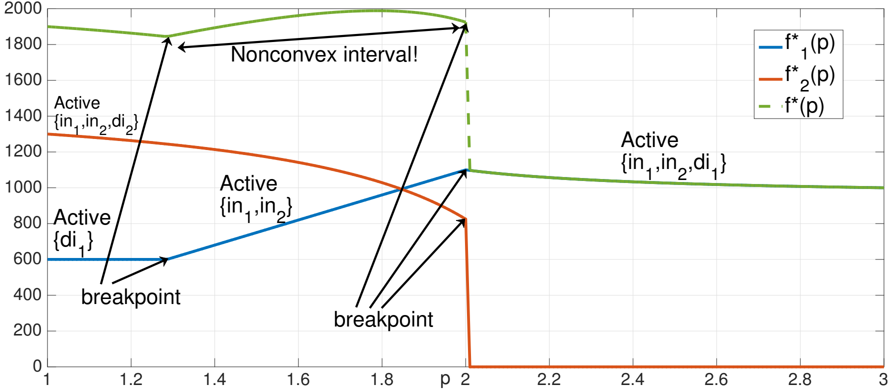


Figure 3: Optimal Objective Function $f^*(p)$ vs. pool concentration p

One Pool, Multiple Outputs: Algorithm

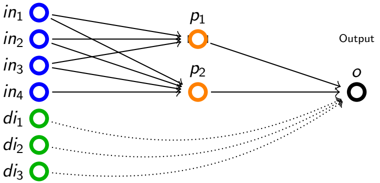
Overcoming Non-Convexity

- $f^*(p)$ can be non-convex, but we have its derivative over a breakpoint interval I (additivity over outputs):

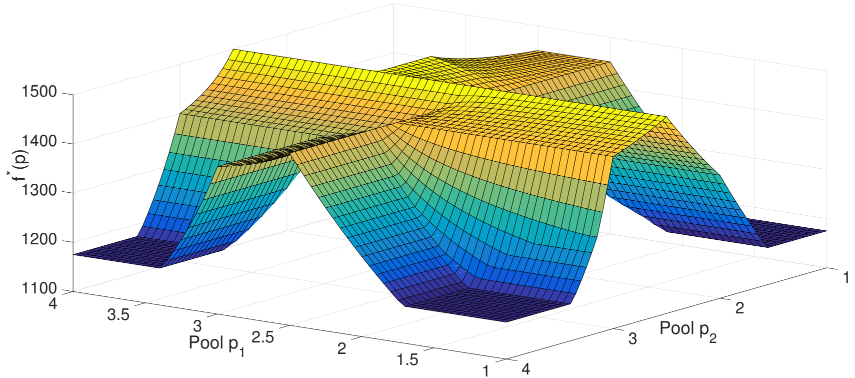
$$\frac{\partial f^*(p)}{\partial p} = \Bigg(\sum_{j \in T_{Y_m}} \frac{D_j^U (C_q^j - P^j(x,q))}{(C_i^j - C_k^j)(p - C_q^j)^2} \cdot \big(C_q^j(\gamma_k^j - \gamma_i^j) + C_j^j(\gamma_i^j - \gamma_q^j) + C_i^j(\gamma_q^j - \gamma_k^j) \big) \Bigg) + C,$$

  where C is a determined constant, j superscript is the output and T_{Y_m} are the outputs where a mixed active set dominates over I.
- $2|T_Y|$-bounded degree polynomial when derivative is $\partial f^*(p)/\partial p = 0$ has rational coefficients and can be solved in polynomial-time by *Lenstra-Lenstra-Lovacz* algorithm.

Two Pools, One Output: Numerical Example



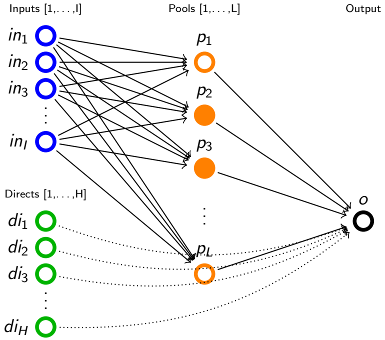
$(C_{in_1} = 3, \gamma_{in_1} = 6)$, $(C_{in_2} = 1, \gamma_{in_2} = 13)$,
$(C_{in_3} = 4, \gamma_{in_3} = 10)$, $(C_{in_4} = 2, \gamma_{in_4} = 9)$;
$(C_{di_1} = 2, \gamma_{di_1} = 10)$, $(C_{di_2} = 6, \gamma_{di_2} = 3)$,
$(C_{di_3} = 4, \gamma_{di_3} = 12)$,
$D^U = 200$; $d = 15$; $P^L = 1$; $P^U = 2.5$;



Multiple Pools: Two Pools, One Output - Why Important?

For a one output problem

- At any given p pool concentrations vector, only a maximum of two pools are active (have incoming and outgoing flow strictly non-zero) at the optimal solution of the problem.
- The L-pool problem considered can be solved by finding the maximum between the optimal objectives of all 2-pool sub-problems contained in a discrete set S, where $|S| = \binom{L}{2}$.



Other Extensions

For a multiple pools / multiple outputs problem

Since every output has a maximum of two active pools feeding into it at optimality, the following cases arise:

- If 2 outputs share 2 active pools: Additivity over outputs as in 1 pool case;
- If 2 outputs share no active pools: Separable problem over outputs;
- P / NP boundary when 2 outputs share only one active pool.

Full pooling problem

- Add back feed availability, pool capacity and lower product demand;
- Relax assumed profitability;
- Detect infeasibility;
- Multiple qualities.

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