Mixed-Integer Nonlinear Optimisation: Bounds Tightening

Benoit Chachuat
b.chachuat@imperial.ac.uk

Ruth Misener
r.misener@imperial.ac.uk

Computational Optimisation Group
Centre for Process Systems Engineering
Department of Computing

Imperial College London

computational optimisation group

Centre for Process Systems Engineering

2017/05/16

Bounds Tightening

1. Feasibility-Based Bounds Tightening

2. Optimality-Based Bounds Tightening

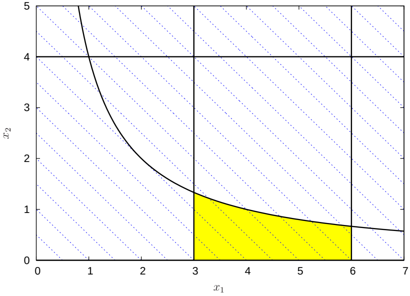
Feasibility-Based Bounds Tightening (FBBT)

Constraint Propagation

Idea. Exploit information from the general constraints $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ in order to reduce the variable range $[\mathbf{x}^{\mathrm{L}}, \mathbf{x}^{\mathrm{U}}]$

Motivating Example. For the following (nonconvex) NLP problem:

1. Reduce the range of variables x_2 using constraint propagation

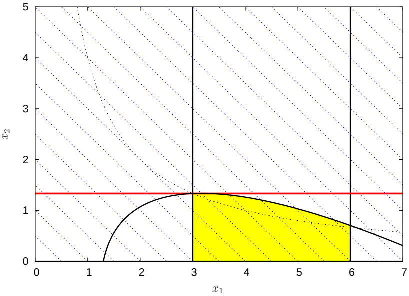


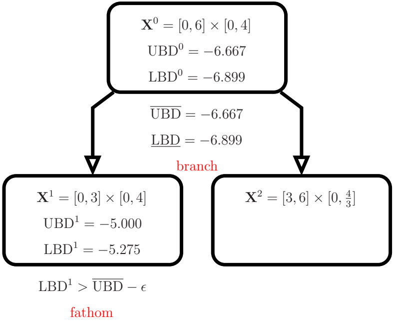
$$\min_{x_1, x_2} \; -x_1 - x_2$$
$$\text{s.t.} \quad x_1 x_2 \leq 4$$
$$(x_1, x_2) \in [3, 6] \times [0, 4]$$

Feasibility-Based Bounds Tightening (FBBT) [cont'd]

2. Construct and solve the corresponding convex NLP relaxation

$$\min_{x_1, x_2} \; -x_1 - x_2$$
$$\text{s.t.} \quad (x_1 + x_2)^2 - 9x_1 - \tfrac{4}{3}x_2 \leq -10$$
$$(x_1, x_2) \in [3, 6] \times [0, \tfrac{4}{3}]$$





➥ Constraint propagation helps B&B to terminate faster!

Feasibility-Based Bounds Tightening (FBBT)

Constraint Propagation

Idea. Exploit information from the general constraints $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ in order to reduce the variable range $[\mathbf{x}^{\mathrm{L}}, \mathbf{x}^{\mathrm{U}}]$

$$
\begin{aligned}
\min_{\mathbf{x}}\ & f(\mathbf{x}) \\
\text{s.t. } & \mathbf{g}(\mathbf{x}) \leq \mathbf{0} \\
& \mathbf{x}^{\mathrm{L}} \leq \mathbf{x} \leq \mathbf{x}^{\mathrm{U}}
\end{aligned}
\quad \xrightarrow{\text{decomp.}} \quad
\begin{aligned}
\min_{\mathbf{v}}\ & v_{\mathrm{obj}} \\
\text{s.t. } & \mathbf{A}\,\mathbf{v} = \mathbf{b} \\
& v_k = v_i v_j, \quad \forall (i,j,k) \in \mathcal{B} \\
& v_k = \frac{v_i}{v_j}, \quad \forall (i,j,k) \in \mathcal{F} \\
& v_k = \varphi(v_i), \quad \forall (i,k) \in \mathcal{U} \\
& \mathbf{v}^{\mathrm{L}} \leq \mathbf{v} \leq \mathbf{v}^{\mathrm{U}}
\end{aligned}
$$

- Refine bounds for variables participating in linear constraints and bilinear/fractional/univariate terms
- Apply interval analysis

Feasibility-Based Bounds Tightening (FBBT) [cont'd]

Linear Constraint Propagation: $\mathbf{a}^\mathsf{T}\mathbf{v} = \sum_i a_i v_i \leq b$

$$\xrightarrow{\text{reduc.}} \begin{cases} v_i^\mathrm{U} \leftarrow \min\left\{v_i^\mathrm{U}, \frac{1}{a_i}\left(b - \sum_{j\neq i}\min\{a_j v_j^\mathrm{L}, a_j v_j^\mathrm{U}\}\right)\right\}, & \text{if } a_i > 0 \\ v_i^\mathrm{L} \leftarrow \max\left\{v_i^\mathrm{L}, \frac{1}{a_i}\left(b - \sum_{j\neq i}\min\{a_j v_j^\mathrm{L}, a_j v_j^\mathrm{U}\}\right)\right\}, & \text{if } a_i < 0 \end{cases}$$

➥ Similar approach for $=$ and $\geq$ linear constraints

Bilinear Term Propagation: $v_k = v_i v_j$

$$\xrightarrow{\text{reduc.}} \begin{cases} v_i^\mathrm{U} \leftarrow \min\left\{v_i^\mathrm{U}, \max\left\{\dfrac{v_k}{v_j} : v_k \in [v_k^\mathrm{L}, v_k^\mathrm{U}], v_j \in [v_j^\mathrm{L}, v_j^\mathrm{U}]\right\}\right\} \\ v_i^\mathrm{L} \leftarrow \max\left\{v_i^\mathrm{L}, \min\left\{\dfrac{v_k}{v_j} : v_k \in [v_k^\mathrm{L}, v_k^\mathrm{U}], v_j \in [v_j^\mathrm{L}, v_j^\mathrm{U}]\right\}\right\} \end{cases}$$

Univariate Term Propagation: $v_k = \varphi(v_i)$

$$\xrightarrow{\text{reduc.}} \begin{cases} v_i^\mathrm{U} \leftarrow \min\left\{v_i^\mathrm{U}, \max\left\{\varphi(v_k) : v_k \in [v_k^\mathrm{L}, v_k^\mathrm{U}]\right\}\right\} \\ v_i^\mathrm{L} \leftarrow \max\left\{v_i^\mathrm{U}, \min\left\{\varphi(v_k) : v_k \in [v_k^\mathrm{L}, v_k^\mathrm{U}]\right\}\right\} \end{cases}$$

FBBT: Equation Factoring

$$
\left\{
\begin{array}{l}
w_m^{\mathrm{MT,\,LO}} \quad \le \sum_{i=0}^{C} \sum_{j=i}^{C} Q_{m,\,i,\,j} \cdot x_i \cdot x_j + a_m \cdot x \le w_m^{\mathrm{MT,\,UP}} \\
\Downarrow \text{ Factor } x_{i'} \text{ out of the multivariable term} \\
w_m^{\mathrm{MT,\,LO}} \quad \le x_{i'} \cdot \left(\sum_{i=0}^{C} Q_{m,\,i,\,i'} \cdot x_i + a_{i'} \right) + \sum_{\substack{i=0 \\ i \ne i'}}^{C} \sum_{\substack{j=i \\ i \ne i'}}^{C} Q_{m,\,i,\,j} \cdot x_i \cdot x_j \\
\qquad\qquad + a_m \cdot x \le w_m^{\mathrm{MT,\,UP}} \\
\Downarrow \text{ Replace variables with intervals} \\
w_m^{\mathrm{MT,\,LO}} \quad \le \left[\underline{x_{i'}},\, \overline{x_{i'}} \right] \cdot \left(\sum_{i=0}^{C} Q_{m,\,i,\,i'} \cdot \left[\underline{x_i},\, \overline{x_i} \right] + a_{i'} \right) \\
\qquad\qquad + \sum_{\substack{i=0 \\ i \ne i'}}^{C} \sum_{\substack{j=i \\ i \ne i'}}^{C} Q_{m,\,i,\,j} \cdot \left[\underline{w_{i,\,j}^{xx}},\, \overline{w_{i,\,j}^{xx}} \right] + a_m \cdot [\underline{x},\, \overline{x}] \le w_m^{\mathrm{MT,\,UP}} \\
\Downarrow \\
\text{Infer bounds on } x_{i'}
\end{array}
\right.
$$

FBBT: Other expressions

Infer additional FBBT strategies on RLT, McCormick Envelope, and Edge-Concave Facets

Bounds Tightening

1. Feasibility-Based Bounds Tightening

2. Optimality-Based Bounds Tightening

Refresher: KKT Multipliers – Marginal Prices

Relaxed Convex Problem:

$$\text{LBD} = \min_{\mathbf{v}} \ f^{\text{cv}}(\mathbf{v}) \quad \text{s.t. } \mathbf{g}^{\text{cv}}(\mathbf{v}) \le \mathbf{0}$$

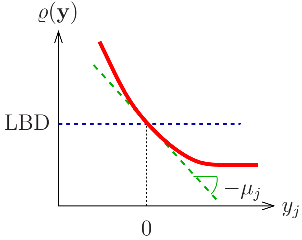
$\xrightarrow{\text{pert.}}$

Perturbation Function:

$$\varrho(\mathbf{y}) = \min_{\mathbf{v}} \ f^{\text{cv}}(\mathbf{v}) \quad \text{s.t. } \mathbf{g}^{\text{cv}}(\mathbf{v}) \le \mathbf{y}$$

Properties:

- ϱ is convex on the set where the perturbed problem has a solution (and provided a C.Q. holds)
- KKT multipliers: $\mu_j = -\left.\dfrac{\partial \varrho}{\partial y_j}\right|_{\mathbf{y}=\mathbf{0}}$
  - μ_j: rate of change in objective solution value caused by a change in constraint j



Special Case: LP

- piecewise affine perturbation function ϱ (no need for C.Q.)
- interpretation of μ_j's as marginal prices

Optimality-based Domain Reduction

Marginal-based Reduction

- Let $\mathbf{v}^k$ be a solution of the relaxed convex problem
- Suppose $g_j^{\mathrm{cv}}(\mathbf{v}^k) = 0$, with $\mu_j^k > 0$

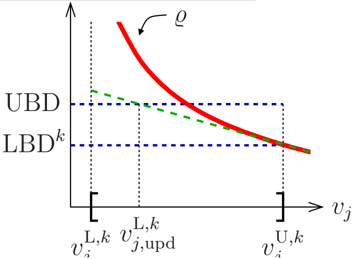
Then, we have: $\boxed{g_j^{\mathrm{cv}}(\mathbf{v}^k) \geq -\dfrac{\mathrm{UBD} - \mathrm{LBD}^k}{\mu_j^k}}$

- Special Case: $g_j^{\mathrm{cv}}(\mathbf{v}) := v_j - v_j^{\mathrm{U},k}$

$$\boxed{v_{j,\mathrm{upd}}^{\mathrm{L},k} \leftarrow \max\left\{v_j^{\mathrm{L},k}, v_j^{\mathrm{U},k} - \frac{\mathrm{UBD}-\mathrm{LBD}^k}{\mu_j^k}\right\}}$$

- Special Case: $g_j^{\mathrm{cv}}(\mathbf{v}) := v_j^{\mathrm{L},k} - v_j$

$$\boxed{v_{j,\mathrm{upd}}^{\mathrm{U},k} \leftarrow \min\left\{v_j^{\mathrm{U},k}, v_j^{\mathrm{L},k} + \frac{\mathrm{UBD}-\mathrm{LBD}^k}{\mu_j^k}\right\}}$$



➥ A variable range may be reduced when it is at its lower/upper bound

➥ Marginal-based reduction can be applied repeatedly — **How?**

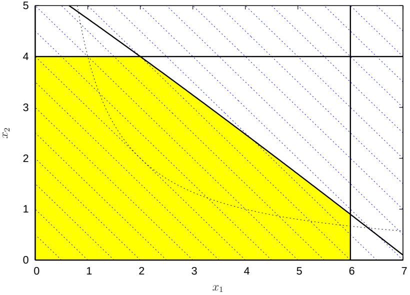
Marginal-based Reduction: Example

Workshop. Apply marginal-based reduction to the following relaxed problem, whose solution point is $x_1^* = 6$, $x_2^* \approx 0.899$, and KKT multipliers for the active constraints $(x_1 + x_2)^2 - 6x_1 - 4x_2 \leq 8$ and $x_2 \leq 6$ are $\mu_1^* \approx 0.102$ and $\mu_2^* \approx 0.204$, respectively. (Consider $\mathrm{UBD} = -6.667$)

$$\min_{x_1, x_2} \; -x_1 - x_2$$

$$\text{s.t.} \quad (x_1 + x_2)^2 - 6x_1 - 4x_2 \leq 8$$

$$(x_1, x_2) \in [0, 6] \times [0, 4]$$



$$\boxed{x_{1,\mathrm{upd}}^{\mathrm{L}} = \quad}$$

Optimality-Based Bounds Tightening (OBBT)

What if none of the variables are at their lower/upper bounds?

- Let UBD be an upper bound on the global solution value
  - e.g., current best optimum (incumbent)

Range Lower Bound:

$$v_{j,\mathrm{upd}}^{\mathrm{L},k} \leftarrow \min_{\mathbf{v}} \; v_j \quad \text{s.t. } \mathbf{g}^{\mathrm{cv}}(\mathbf{v}) \le \mathbf{0}, \quad f^{\mathrm{cv}}(\mathbf{v}) \le \mathrm{UBD}$$

Range Upper Bound:

$$v_{j,\mathrm{upd}}^{\mathrm{U},k} \leftarrow \max_{\mathbf{v}} \; v_j \quad \text{s.t. } \mathbf{g}^{\mathrm{cv}}(\mathbf{v}) \le \mathbf{0}, \quad f^{\mathrm{cv}}(\mathbf{v}) \le \mathrm{UBD}$$

Cons Solution of additional NLP/LP problems

Pros Larger range reductions compared with FBBT

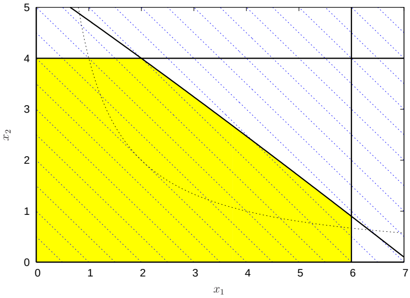
➡ OBBT can be applied repeatedly

➡ Trade-off: computational burden vs. improvement

Optimality-Based Bounds Tightening: Example

Workshop. Apply OBBT to the following relaxed problem, assuming an upper bound of $\mathrm{UBD} = -6.667$ has been obtained.

$$\begin{aligned} \min_{x_1,x_2} \quad & -x_1 - x_2 \\ \text{s.t.} \quad & (x_1+x_2)^2 - 6x_1 - 4x_2 \leq 8 \\ & (x_1,x_2) \in [0,6]\times[0,4] \end{aligned}$$



$$\begin{aligned} \min_{x_1,x_2} \quad & x_1 \\ \text{s.t.} \quad & (x_1+x_2)^2 - 6x_1 - 4x_2 \leq 8 \\ & -x_1 - x_2 \leq -6.667 \\ & (x_1,x_2) \in [0,6]\times[0,4] \end{aligned}$$

$$\boxed{x_{1,\mathrm{upd}}^{\mathrm{L}} = 4.890}$$

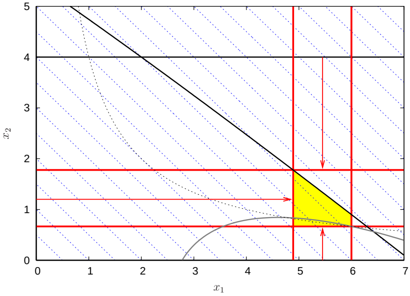
$$\begin{aligned} \max_{x_1,x_2} \quad & x_1 \\ \text{s.t.} \quad & (x_1+x_2)^2 - 6x_1 - 4x_2 \leq 8 \\ & -x_1 - x_2 \leq -6.667 \\ & (x_1,x_2) \in [0,6]\times[0,4] \end{aligned}$$

$$\boxed{x_{1,\mathrm{upd}}^{\mathrm{U}} = 6}$$

Optimality-Based Bounds Tightening: Example

Workshop. Apply OBBT to the following relaxed problem, assuming an upper bound of $\mathrm{UBD} = -6.667$ has been obtained.

$$\min_{x_1,x_2} \; -x_1 - x_2$$
$$\text{s.t.} \quad (x_1+x_2)^2 - 6x_1 - 4x_2 \leq 8$$
$$(x_1,x_2) \in [0,6]\times[0,4]$$



$$\min_{x_1,x_2} \; x_2$$
$$\text{s.t.} \quad (x_1+x_2)^2 - 6x_1 - 4x_2 \leq 8$$
$$-x_1 - x_2 \leq -6.667$$
$$(x_1,x_2) \in [0,6]\times[0,4]$$

$$\boxed{x_{2,\mathrm{upd}}^{\mathrm{L}} = 0.667}$$

$$\max_{x_1,x_2} \; x_2$$
$$\text{s.t.} \quad (x_1+x_2)^2 - 6x_1 - 4x_2 \leq 8$$
$$-x_1 - x_2 \leq -6.667$$
$$(x_1,x_2) \in [0,6]\times[0,4]$$

$$\boxed{x_{2,\mathrm{upd}}^{\mathrm{U}} = 1.777}$$