Mixed-Integer Nonlinear Optimisation: Introduction & Applications

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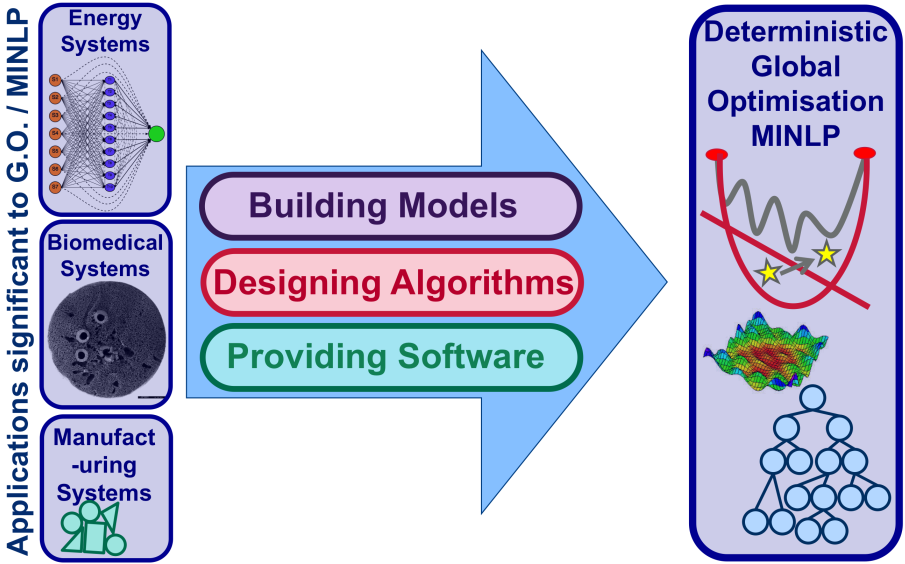
Creating Effective Solution Strategies



1. Mixed-Integer Nonlinear Optimisation
   - Toy Example
   - Mathematical Definition
   - Industrial Interest & Relevance

2. MINLP Applications
   - Impact of missing the global solution
   - Pooling Problem: Intermediate Blending
   - Concave Cost: Economies of Scale
   - Heat Recovery Networks: Nonlinear Nature of Heat Exchange
   - Other Examples

3. Challenges

Toy Example: Mixed-Integer Nonlinear Optimisation

$$\max_{x_1,\,x_2} \quad x_1 + x_2$$

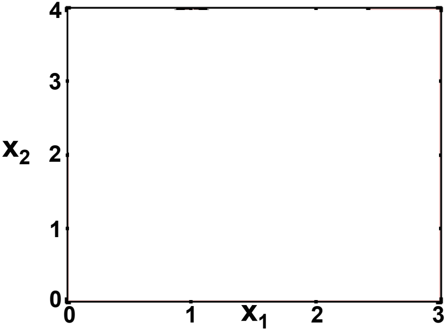
$$8 \cdot x_1^3 - 2 \cdot x_1^4 - 8 \cdot x_1^2 + x_2 \leq 2$$

$$32 \cdot x_1^3 - 4 \cdot x_1^4 - 88 \cdot x_1^2 + 96 \cdot x_1 + x_2 \leq 36$$

$$x_1 \in [0, 3]$$

$$x_2 \in \{0,\ 1,\ 2,\ 3,\ 4\}$$

Rigorously guarantee best answer



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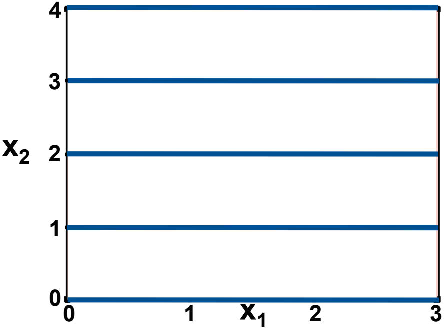

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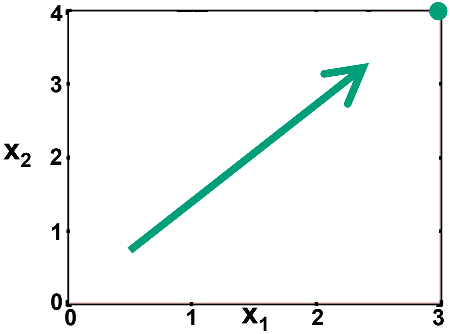

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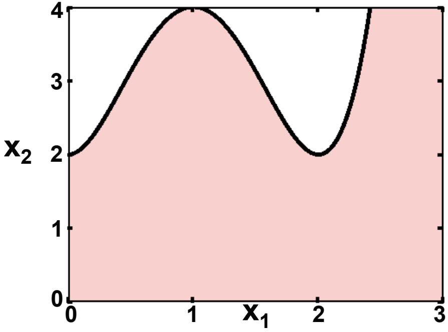

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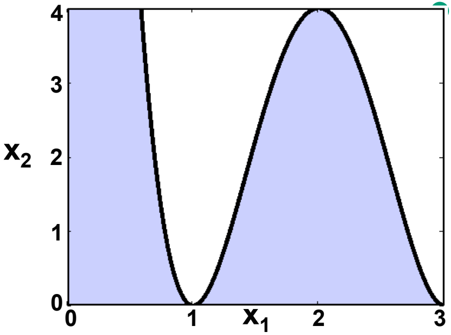

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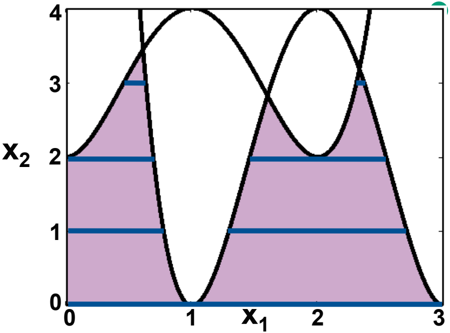

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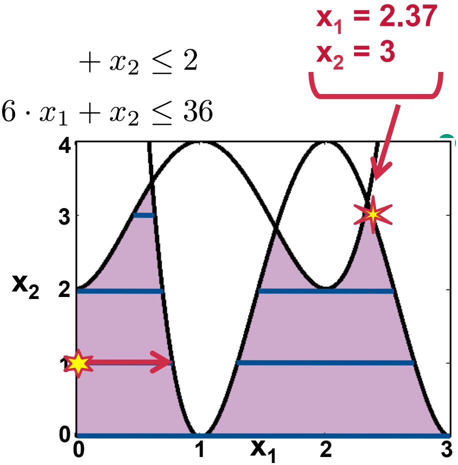

Mixed-Integer Nonlinear Optimisation (MINLP) [1/2]

$$
\begin{aligned}
\min_{\mathbf{x}} \quad & f_0(\mathbf{x}) \\
\text{s.t.} \quad & b_i^{\mathrm{LO}} \le f_i(\mathbf{x}) \le b_i^{\mathrm{UP}} \quad \forall\, i \in \mathcal{M} := \{1, \ldots, M\} \\
& x_j^{\mathrm{LO}} \le \quad x_j \le x_j^{\mathrm{UP}} \quad \forall\, j \in \mathcal{N} := \{1, \ldots, N\} \\
& x_j \in \mathbb{Z} \qquad\qquad\qquad \forall\, j \in \mathcal{I} \subseteq \mathcal{N}
\end{aligned}
\tag{MINLP}
$$

- **Sets** $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{I}$ represent constraints, variables, and discrete variables, respectively;
- **Functions** $f_i : \mathbb{R}^N \mapsto \mathbb{R} \;\forall\, i \in \{0, \ldots, M\}$;
- **Parameters**
  Constraint bounds $b_i^{\mathrm{LO}} \in \mathbb{R} \cup \{-\infty\}$, $b_i^{\mathrm{UP}} \in \mathbb{R} \cup \{+\infty\}$;
  Variable bounds $x_j^{\mathrm{LO}} \in \mathbb{R} \cup \{-\infty\}$, $x_j^{\mathrm{UP}} \in \mathbb{R} \cup \{+\infty\}$.

Mixed-Integer Nonlinear Optimisation (MINLP) [2/2]

We assume that we can infer finite bounds on the variables $\mathbf{x}$ and that the image of f_i is finite on $\mathbf{x}$. Typical expressions for $f_0(\mathbf{x})$ and $f_i(\mathbf{x})$ are:

$$
\begin{aligned}
f_i(\mathbf{x}) = c_i + a_i^T \, \mathbf{x} + \mathbf{x}^T \; Q_i \; \mathbf{x} \\
+ \sum_{s \in \mathcal{S}} c_{s,\,i} \cdot \prod_{j \in \mathcal{N}} x_j^{p_{s,\,i,\,j}} + \sum_{j \in \mathcal{N}} c_{e,\,i,\,j} \; e^{x_j} + \sum_{j \in \mathcal{N}} c_{\ell,\,i,j} \; \log x_j
\end{aligned}
$$

where the powers $p_{s,\,i,\,j}$ and coefficients c_i, a_i, Q_i, $c_{s,\,i}$, $c_{e,\,i,\,j}$, $c_{\ell,\,i,\,j}$ are constant reals; $s \in \mathcal{S}$ indexes the signomial terms.

Industrial Interest & Relevance?

$$\begin{aligned} \min\ & c^{\mathrm{T}}\, \mathbf{x} + \\ \text{s.t. }\ & A_i\, \mathbf{x} + \qquad\qquad = b_i \quad i = 1, \ldots, I \\ & \text{Some of } x_j \text{ integer} \end{aligned}$$

- Widely used in industry MIP

Industrial Interest & Relevance?

$$\min c^{\mathsf{T}} \mathbf{x} + \mathbf{x}^{\mathsf{T}} Q_0 \mathbf{x}$$

$$\text{s.t. } A_i \mathbf{x} + \qquad = b_i \quad i = 1, \ldots, I$$

$$\text{Some of } x_j \text{ integer}$$

- Widely used in industry — MIP
- Commercial [IBM CPLEX, 2014] — MIQP
  - Passed academic codes from Imperial & CMU in 2015 [ISMP]

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- Strong commercial goal — MIQCQP

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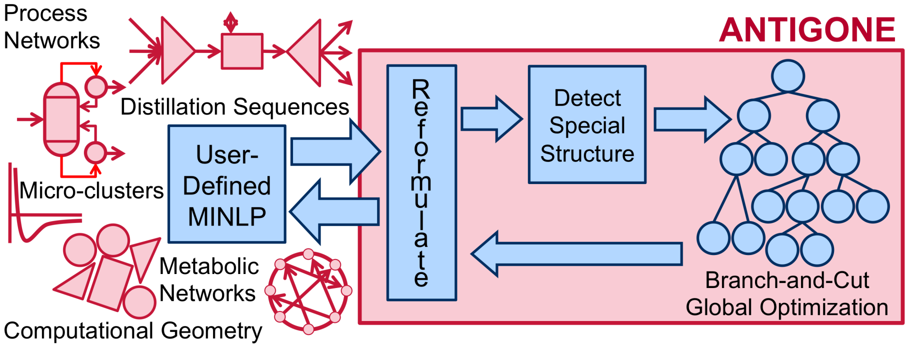
- Widely used in industry — MIP
- Commercial [IBM CPLEX, 2014] — MIQP
  - Passed academic codes from Imperial & CMU in 2015 [ISMP]
- Strong commercial goal — MIQCQP
- Wait-&-See — MINLP

Globally Optimizing MINLP: Computational System



ANTIGONE

Algorithms for coNTinuous / Integer Global Optimization of Nonlinear Equations [Misener, Floudas, GAMS GmbH]

Misener & Floudas, ANTIGONE, *J Global Optimization*; 2014

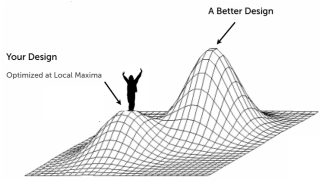
Finding the global solution: Does it matter?

GO provides certainty (within the limits of the model)

Safety & Risk	Finding the worst-case solution
Thermodynamics	Finding the correct physical solution
Modelling	Finding the parameters that give the best interpretation of the physics

Why settle for 2^{nd} (or 3^{rd}, or 4^{th}) best?

GO can provide enhanced performance (technical or economic)



Applications of Global Optimisation

In engineering applications, accurate mathematical modelling may require **discrete decisions** and **nonlinear relationships**

- Engineering design & manufacturing
  - Product & process design
  - Production planning/scheduling/logistics
- Computational chemistry
  - Chemical & phase equilibria
  - Molecular design
- Biochemistry & biochemical sciences:
  - Molecular (protein) structure prediction
  - Diagnosis, e.g., cancer

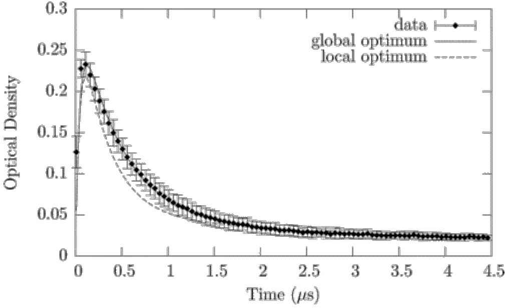


Figure 1: Singer et al., 2006

Ex 1: Missing the Global Solution [1/2] From C Adjiman

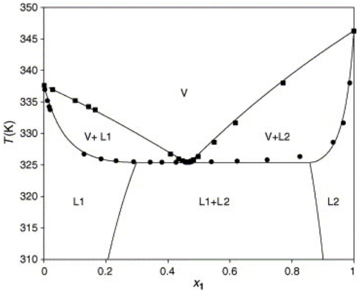
- Consider the mixture of 2,3-dimethyl-2-butene (1) and methanol (2);
- At atmospheric pressure, the mixture exhibits a homogeneous azeotrope (Uusi-Kyyny et al., 2004);
- Uusi-Kyyny et al. (2004) used parameter estimation to develop a SRK/NRTL model for this mixture:

minimise	**Deviations between model and experiments**
s.t.	**Phase equilibrium at each experimental condition, e.g. temperature, pressure, total composition**

- This parameter estimation problem is very difficult!
  - Calculating phase equilibrium requires globally minimising the Gibbs free energy. The problem has a highly nonconvex set of constraints.
- A convenient approach:
  - Write phase equilibrium based on local optimality (equality of chemical potentials or fugacities);
  - Even this case, this is still a nonconvex problem

Ex 1: Missing the Global Solution [2/2] From C Adjiman

- Local optimisation predicts a heterogeneous azeotrope: The model is wrong (Xu et al., 2005), even on a qualitative level.



- Potential for entirely wrong design and wasted effort.
- How can we do better?
  - Incorporate a final phase stability check after the parameter estimation by solving a global optimisation problem. (What could go wrong?)
  - Formulate the parameter estimation problem so that such an answer is not feasible (Bollas et al., 2009).

Ex 2: Missing the Global Solution [1/2] From C Adjiman

Kinetic mechanism of reaction of cyclohexanedienyl radical with O_2 in cyclohexane?

$$
\begin{array}{ll}
(CH_3)_3CO + 1{,}4\text{-}C_6H_8 & \rightarrow c\text{-}C_6H_7 + (CH_3)_3COH \\
c\text{-}C_6H_7 + O_2 & \leftrightarrow p\text{-}C_6H_7OO \\
c\text{-}C_6H_7 + O_2 & \leftrightarrow o\text{-}C_6H_7OO \\
o\text{-}C_6H_7OO & \rightarrow C_6H_6 + HO_2 \\
2\ c\text{-}C_6H_7 & \rightarrow \text{products}
\end{array}
$$

Taylor et al. (2004) report extensive, high-quality, time-dependent data.

- This is a challenging optimisation problem!

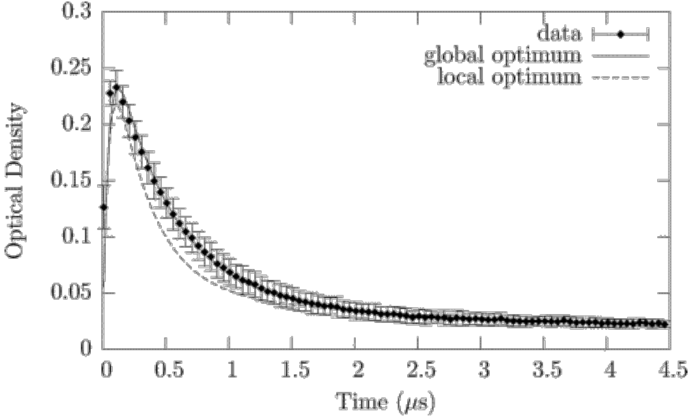
  Minimise **Deviations between model and experiments**

  s.t. **Dynamic model of the experiments**

  - The constraints are given by differential equations;
  - The rate expressions are highly nonlinear.
- A convenient approach: Solve the problem with a local solver

Ex 2: Missing the Global Solution [2/2] From C Adjiman

- Results of a local and global optimisation (Singer et al., 2006)



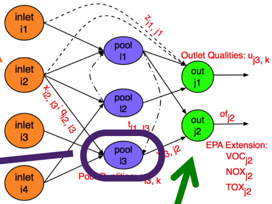
- Initial analysis of the local optimum led to the erroneous dismissal of the proposed mechanism.
  - Wasted time & effort developing an alternative mechanism!

Pooling Problem: Intermediate Blending

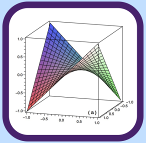
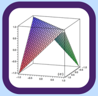
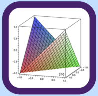
Know input species concentrations (*e.g.*, sulfur)

Variables

$$\sum_{i\in\mathrm{In}} c_{\mathrm{Sulfur},\,i}\cdot f_i = c_{\mathrm{Sulfur,\,Out}}\cdot\sum_{o\in\mathrm{Out}} f_o$$



Have output species requirements (e.g., laws)

Common nonlinear term: x × y

Standard Pooling Network p-Formulation

Objective $\displaystyle\max_{x_{il},y_{lj},z_{ij},p_{lk}} \sum_{(l,j)\in T_Y} d_j \cdot y_{lj} + \sum_{(i,j)\in T_Z} d_j \cdot z_{ij} - \sum_{(i,l)\in T_X} \gamma_i \cdot x_{il} - \sum_{(i,j)\in T_Z} \gamma_i \cdot z_{ij}$

Feed Avail $\left[A_i^L \le \displaystyle\sum_{l:(i,l)\in T_X} x_{il} + \sum_{j:(i,j)\in T_Z} z_{ij} \le A_i^U \quad \forall\ i \right.$

Pool Capacity $\left[S_l^L \le \displaystyle\sum_{i:(i,l)\in T_X} x_{il} \le S_l^U \quad \forall\ l \right.$

Product Demand $\left[D_j^L \le \displaystyle\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \le D_j^U \quad \forall\ j \right.$

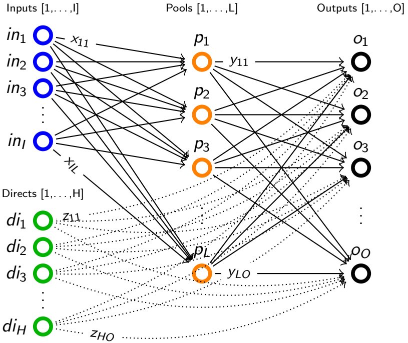
Material Balance $\left[\displaystyle\sum_{i:(i,l)\in T_X} x_{il} - \sum_{j:(l,j)\in T_Y} y_{lj} = 0 \quad \forall\ l \right.$

Quality Balance $\left[\displaystyle\sum_{i:(i,l)\in T_X} C_{ik} x_{il} = p_{lk} \sum_{j:(l,j)\in T_Y} y_{lj} \quad \forall\ l, k \right.$

Product Quality

$$\left[\sum_{l:(l,j)\in T_Y} p_{lk} y_{lj} + \sum_{i:(i,j)\in T_Z} C_{ik} z_{ij} \begin{cases} \ge P_{jk}^L \left(\displaystyle\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \right) \\ \le P_{jk}^U \left(\displaystyle\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \right) \end{cases} \right. \quad \forall\ j, k$$

Bounds $\left[x_{il},\ y_{lj},\ z_{ij} \ge 0\ \forall\ i,\ l,\ j \right.$



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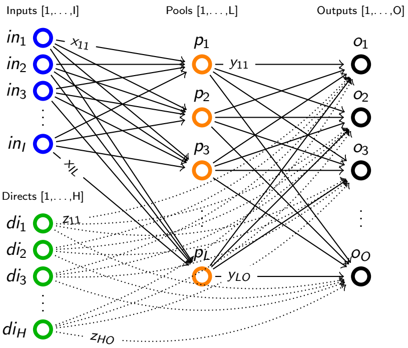

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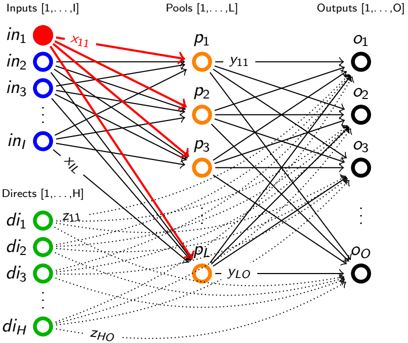

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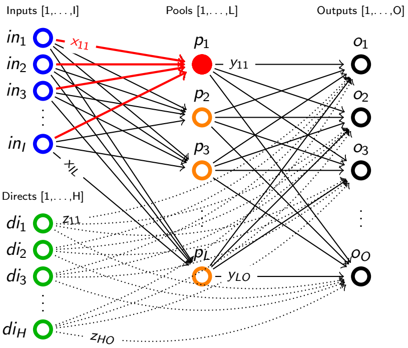
Product Demand $\left[D_j^L \le \sum\limits_{l:(l,j)\in T_Y} y_{lj} + \sum\limits_{i:(i,j)\in T_Z} z_{ij} \le D_j^U \quad \forall\, j \right.$

Material Balance $\left[\sum\limits_{i:(i,l)\in T_X} x_{il} - \sum\limits_{j:(l,j)\in T_Y} y_{lj} = 0 \quad \forall\, l \right.$

Quality Balance $\left[\sum\limits_{i:(i,l)\in T_X} C_{ik} x_{il} = p_{lk} \sum\limits_{j:(l,j)\in T_Y} y_{lj} \quad \forall\, l, k \right.$

Product Quality
$$\left[\begin{array}{l} \sum\limits_{l:(l,j)\in T_Y} p_{lk} y_{lj} \\ + \sum\limits_{i:(i,j)\in T_Z} C_{ik} z_{ij} \end{array} \begin{cases} \ge P_{jk}^L \left(\sum\limits_{l:(l,j)\in T_Y} y_{lj} + \sum\limits_{i:(i,j)\in T_Z} z_{ij} \right) \\ \le P_{jk}^U \left(\sum\limits_{l:(l,j)\in T_Y} y_{lj} + \sum\limits_{i:(i,j)\in T_Z} z_{ij} \right) \end{cases} \right. \quad \forall\, j, k$$

Bounds $\left[x_{il},\ y_{lj},\ z_{ij} \ge 0 \ \forall\ i,\ l,\ j \right.$



Standard Pooling Network p-Formulation

Objective $\max_{x_{il},y_{lj},z_{ij},p_{lk}} \sum_{(l,j)\in T_Y} d_j \cdot y_{lj} + \sum_{(i,j)\in T_Z} d_j \cdot z_{ij} - \sum_{(i,l)\in T_X} \gamma_i \cdot x_{il} - \sum_{(i,j)\in T_Z} \gamma_i \cdot z_{ij}$

Feed Avail $\left[A_i^L \le \sum_{l:(i,l)\in T_X} x_{il} + \sum_{j:(i,j)\in T_Z} z_{ij} \le A_i^U \quad \forall\ i \right.$

Pool Capacity $\left[S_l^L \le \sum_{i:(i,l)\in T_X} x_{il} \le S_l^U \quad \forall\ l \right.$

Product Demand $\left[\mathbf{D_j^L} \le \sum_{\mathbf{l:(l,j)\in T_Y}} \mathbf{y_{lj}} + \sum_{\mathbf{i:(i,j)\in T_Z}} \mathbf{z_{ij}} \le \mathbf{D_j^U} \quad \forall\ \mathbf{j} \right.$

Material Balance $\left[\sum_{i:(i,l)\in T_X} x_{il} - \sum_{j:(l,j)\in T_Y} y_{lj} = 0 \quad \forall\ l \right.$

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$$\left[\begin{array}{l} \sum_{l:(l,j)\in T_Y} p_{lk} y_{lj} \\ + \sum_{i:(i,j)\in T_Z} C_{ik} z_{ij} \end{array} \right. \left\{ \begin{array}{l} \ge P_{jk}^L \left(\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \right) \\ \le P_{jk}^U \left(\sum_{l:(l,j)\in T_Y} y_{lj} + \sum_{i:(i,j)\in T_Z} z_{ij} \right) \end{array} \right. \quad \forall\ j, k$$

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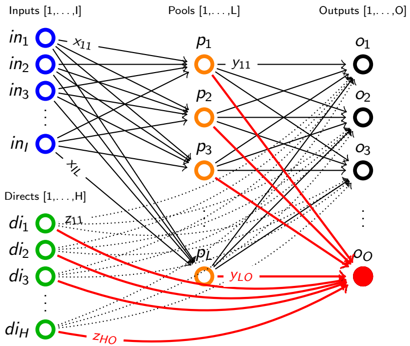

Standard Pooling Network p-Formulation

Objective $\displaystyle \max_{x_{il}, y_{lj}, z_{ij}, p_{lk}} \sum_{(l,j)\in T_Y} d_j \cdot y_{lj} + \sum_{(i,j)\in T_Z} d_j \cdot z_{ij} - \sum_{(i,l)\in T_X} \gamma_i \cdot x_{il} - \sum_{(i,j)\in T_Z} \gamma_i \cdot z_{ij}$

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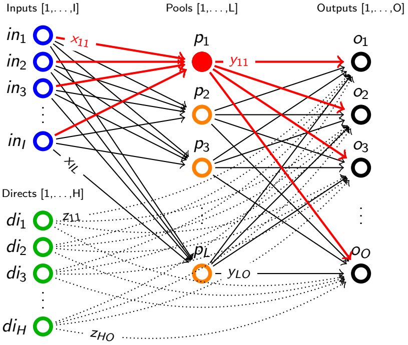
Bounds $[x_{il},\ y_{lj},\ z_{ij} \ge 0\ \forall\ i,\ l,\ j$



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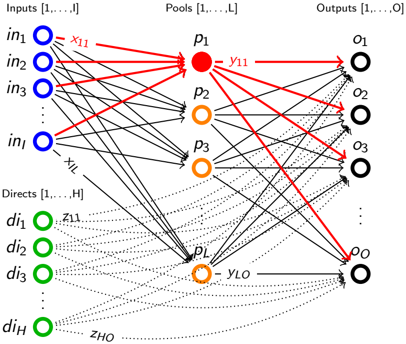

Standard Pooling Network p-Formulation

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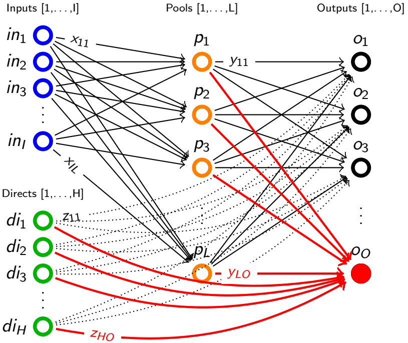

Product Quality $\left[\begin{array}{l} \sum_{l:(l,j)\in T_Y} \mathbf{p_{lk} y_{lj}} \\ + \sum_{i:(i,j)\in T_Z} \mathbf{C_{ik} z_{ij}} \end{array} \right. \left\{ \begin{array}{l} \ge \mathbf{P_{jk}^L} \left(\sum_{l:(l,j)\in T_Y} \mathbf{y_{lj}} + \sum_{i:(i,j)\in T_Z} \mathbf{z_{ij}} \right) \\ \le \mathbf{P_{jk}^U} \left(\sum_{l:(l,j)\in T_Y} \mathbf{y_{lj}} + \sum_{i:(i,j)\in T_Z} \mathbf{z_{ij}} \right) \end{array} \right. \quad \forall\ \mathbf{j}, \mathbf{k}$

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Standard Pooling Network p-Formulation

Objective $\max_{x_{il},y_{lj},z_{ij},p_{lk}} \sum_{(l,j)\in T_Y} d_j \cdot y_{lj} + \sum_{(i,j)\in T_Z} d_j \cdot z_{ij} - \sum_{(i,l)\in T_X} \gamma_i \cdot x_{il} - \sum_{(i,j)\in T_Z} \gamma_i \cdot z_{ij}$

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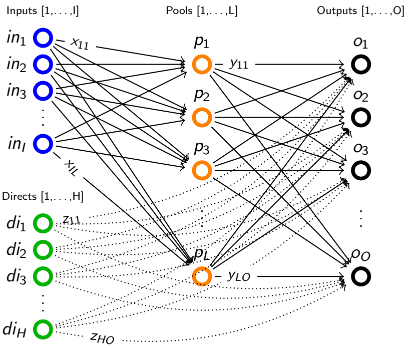
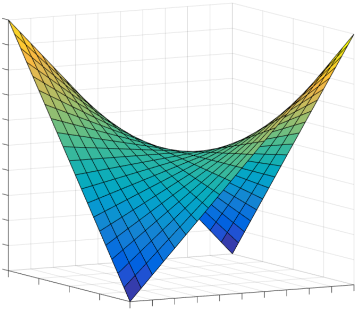
Bounds $\left[x_{il},\ y_{lj},\ z_{ij} \geq 0\ \forall\ i,\ l,\ j \right.$

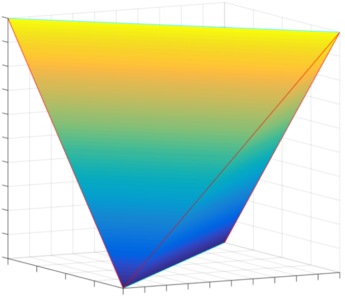


Bilinearities

- Underestimator: McCormick convex hull



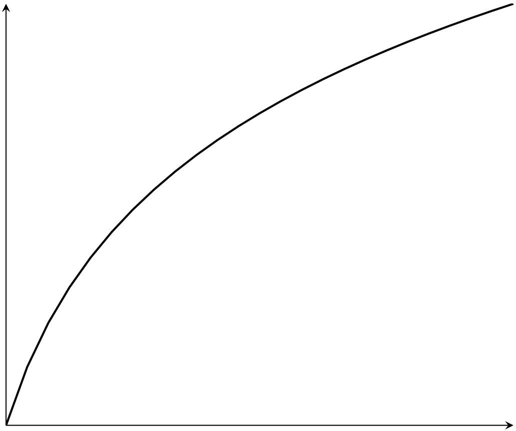
(a) Bilinearity



(b) The McCormick Hull

Concave Cost Functions

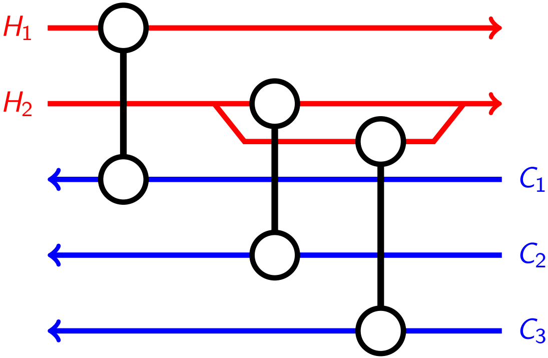
$$\text{Cost} = x^{0.6}$$



Sanity Check. $\Rightarrow$ Why 0.6?

Heat Exchanger Networks

- Heat Exchanger : Device transfering heat between two liquids without mixing
- Heat Exchanger Network: Collection of heat exchangers over a set of hot and cold streams
- Design Problem: Optimise the heat exchanger network superstructure



Calculating the Area of a Heat Exchanger

- The area of a heat exchanger is calculated using

$$A = \frac{q \cdot U}{\Delta\, T_{LMTD}}$$

  where

  - q is the energy transferred
  - U is a constant that depends on the streams
  - $\Delta\, T_{LMTD}$ is the log mean temperature difference, a function that characterises the average temperature difference.

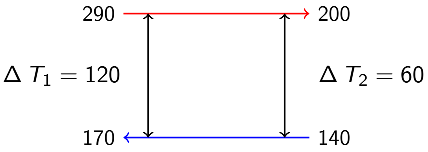
The Log Mean Temperature Difference

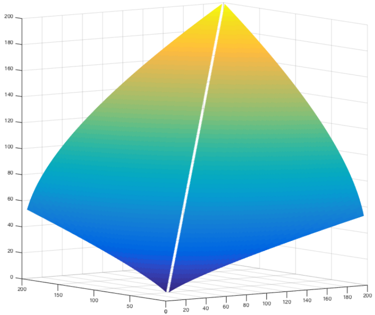
Function Definition

- The log mean temperature difference (LMTD) is defined as

$$\Delta T_{LMTD} = \frac{\Delta T_1 - \Delta T_2}{\ln \left({}^{\Delta T_1}/_{\Delta T_2} \right)}$$

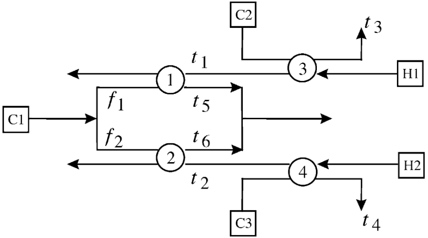
- Has indeterminacies
- Characterises the nonlinear nature of heat exchange





Design Problem: Heat Exchanger Networks

- **Hot / cold stream pairings minimising the cost?**



Design Problem: Heat Exchanger Networks

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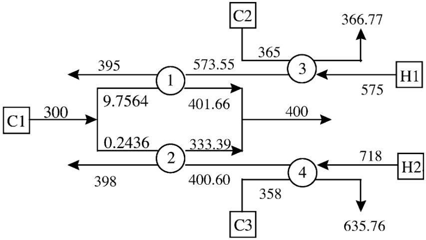

- **Global Optimum:** \$36199
- Local Optima: \$38513, \$39809, \$41836, \$47681

Source: Zamora & Grossmann, *Comput Chem Eng* **22**:1749-1770, 1998

Design Problem: Heat Exchanger Networks

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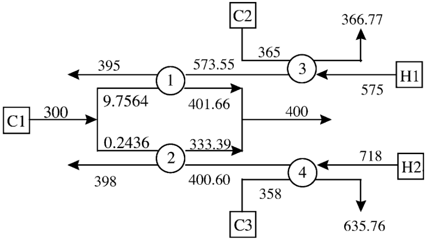

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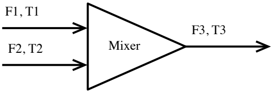
Discrete:	Heat exchanger exists or not?
Nonlinear:	Stream mixing; heat exchanger area calculation; concave cost function; log mean temperature difference

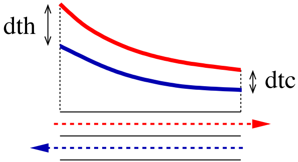
Design Problem: Heat Exchanger Networks

Nonconvex Nonlinearities in HEN Synthesis:

- Cost [power] $\mathbf{A_i}^{\beta}$
- Area [fraction]
  $\mathbf{A_i} = \frac{1}{U} \frac{\mathbf{q}}{\mathbf{\Delta T}_{\mathrm{LMTD}}}$
- Mixing [bilinear]
  $\mathbf{f_1}\ \mathbf{t_1} + \mathbf{f_2}\ \mathbf{t_2} = \mathbf{f_3}\ \mathbf{t_3}$



- $\mathbf{\Delta T}_{\mathrm{LMTD}} = \dfrac{\mathbf{\Delta T_1} - \mathbf{\Delta T_2}}{\log(\mathbf{\Delta T_1}/\mathbf{\Delta T_2})}$
- $\mathbf{\Delta T}_{\mathrm{LMTD}} \approx$
  $\left[\mathbf{\Delta T_1}\ \mathbf{\Delta T_2}\ \frac{\mathbf{\Delta T_1}+\mathbf{\Delta T_2}}{2}\right]^{\frac{1}{3}}$



Discrete:	Heat exchanger exists or not?
Nonlinear:	Stream mixing; heat exchanger area calculation; concave cost function; log mean temperature difference

Design Problem: Heat Exchanger Networks

$$
\begin{aligned}
\min\ & \frac{270}{U_1}\ \frac{q_1}{\Delta T_1} + \frac{720}{U_2}\ \frac{q_2}{\Delta T_2} \\
& + \frac{240}{U_3}\ \frac{q_3}{\Delta T_3} + \frac{900}{U_4}\ \frac{q_4}{\Delta T_4} \\
\text{s.t. } & q_1 = 5.555(t_1 - 395) \\
& q_1 = f_1\ t_5 - 300\ f_1 \\
& q_2 = 3.125(t_2 - 398) \\
& q_2 = f_2\ t_6 - 300\ f_2 \\
& q_3 = 4.545(t_3 - 365) \\
& q_3 = 5.555(575 - t_1) \\
& q_4 = 3.571(t_4 - 358) \\
& q_4 = 3.125(718 - t_2)
\end{aligned}
$$

$$
\begin{aligned}
& q_1 + q_2 = 1000 \\
& q_1 + q_3 = 999.9 \\
& q_2 + q_4 = 1000 \\
& f_1 + f_2 = 10 \\
& \delta t_{1\mathrm{h}} = t_1 - t_5 \\
& \delta t_{1\mathrm{c}} = 95 \\
& \delta t_{2\mathrm{h}} = t_2 - t_6 \\
& \delta t_{2\mathrm{c}} = 98 \\
& \delta t_{3\mathrm{h}} = 575 - t_3 \\
& \delta t_{3\mathrm{c}} = t_1 - 365 \\
& \delta t_{4\mathrm{h}} = 718 - t_4 \\
& \delta t_{4\mathrm{c}} = t_2 - 358
\end{aligned}
$$

$$
\begin{aligned}
& \Delta T_1 = (\delta t_{1\mathrm{h}} - \delta t_{1\mathrm{c}})/(\log\ \delta t_{1\mathrm{h}}/\delta t_{1\mathrm{c}}) \\
& \Delta T_2 = (\delta t_{2\mathrm{h}} - \delta t_{2\mathrm{c}})/(\log\ \delta t_{2\mathrm{h}}/\delta t_{2\mathrm{c}}) \\
& \Delta T_3 = (\delta t_{3\mathrm{h}} - \delta t_{3\mathrm{c}})/(\log\ \delta t_{3\mathrm{h}}/\delta t_{3\mathrm{c}}) \\
& \Delta T_4 = (\delta t_{4\mathrm{h}} - \delta t_{4\mathrm{c}})/(\log\ \delta t_{4\mathrm{h}}/\delta t_{4\mathrm{c}}) \\
& f_1\ t_5 + f_2\ t_6 = 4000 \\
& \delta t_k \geq 5, \quad k = 1\mathrm{h}, 1\mathrm{c}, \ldots, 4\mathrm{h}, 4\mathrm{c} \\
& 0 \leq q_i^{\mathrm{L}} \leq q_i \leq q_i^{\mathrm{U}}, \quad i = 1, \ldots, 4 \\
& 0 \leq t_i^{\mathrm{L}} \leq t_i \leq t_i^{\mathrm{U}}, \quad i = 1, \ldots, 6 \\
& 0 \leq f_i^{\mathrm{L}} \leq f_i \leq f_i^{\mathrm{U}}, \quad i = 1, 2
\end{aligned}
$$

Discrete: Heat exchanger exists or not?
Nonlinear: Stream mixing; heat exchanger area calculation; concave cost function; log mean temperature difference

MINLP Models in Applications

Nonlinear Aspects

- Pooling
- AC power flow
- Heat exchange
- Concave cost
- Water networks
- Gas networks
- Thermodynamics

Discrete Aspects

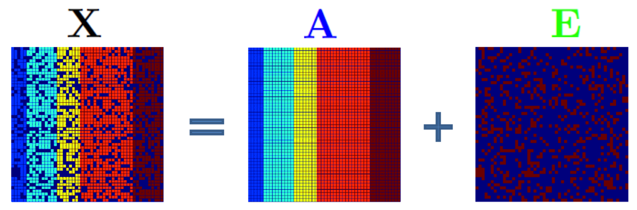
- Piecewise linear approximation
- Logical entities
- Number of units

Next 2 Weeks

Branch & bound for MINLP with convex & nonconvex nonlinearities • Underestimators • Using convexity • Multi-term relaxations • Branching • Bounds tightening • . . .

Example: Robust Principal Component Analysis (PCA)

Challenge Given $X = \mathbf{A} + \mathbf{E}$, recover $\mathbf{A}$ & $\mathbf{E}$



Optimisation problem

What is the lowest rank matrix that agrees with the data up to some sparse error?

$$\min_{\mathbf{A},\,\mathbf{E}} \operatorname{rank}(\mathbf{A}) + \lambda \, \|\mathbf{E}\|_0$$

$$X = \mathbf{A} + \mathbf{E}$$

Example: Convexity of Robust PCA?

Optimisation problem

$$\min_{\mathbf{A},\,\mathbf{E}} \operatorname{rank}(\mathbf{A}) + \lambda \|\mathbf{E}\|_0$$

$$X = \mathbf{A} + \mathbf{E}$$

Definitions	$\operatorname{rank}(\mathbf{A})$	Rank of the matrix
	$\|\mathbf{E}\|_0 = \#\{E_{i,j} \neq 0\}$	# Nonzero elements

Prove that this is a convex program or find a counter example?

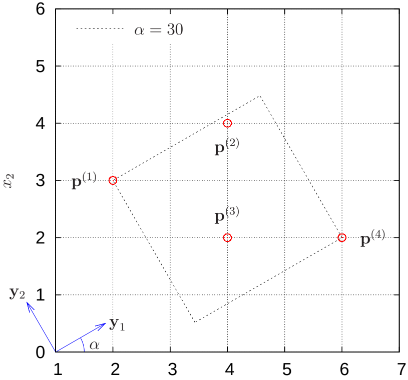
Example: Convex Relaxation of Robust PCA

Sources Candes, Li, Ma, Wright, *J. ACM*, 2011; Sagonas, Panagakis, Zafeiriou, Pantic, Proc. IEEE ICCV. 2015.

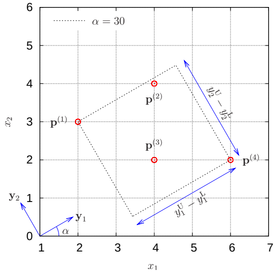
A First Example Problem – Statement

Workshop. Find a rectangle with minimum area enclosing the set of points $\{\mathbf{p}^{(1)} := (2,3), \mathbf{p}^{(2)} := (4,4), \mathbf{p}^{(3)} := (4,2), \mathbf{p}^{(4)} := (6,2)\}$

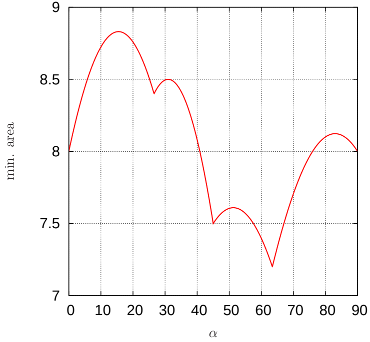
⇒ Formulate this problem as an NLP
Hint: Consider the coordinate system (y_1, y_2) aligned with the axis of the rectangle, as obtained by rotating (x_1, x_2) by the angle α

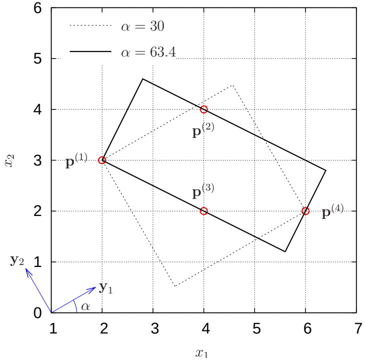


A First Example Problem – NLP Formulation



A First Example Problem – Results





- Multi-Start Gradient Descent: 20,000 Random Initial points

Freq	Area	α	y_1^U	y_1^L	y_2^U	y_2^L
32.6%	8.00	0.0	6.000	2.000	4.000	2.000
30.5%	**7.20**	**63.4**	**5.367**	**3.578**	**-0.447**	**-4.472**
15.2%	8.00	90.0	4.000	2.000	-2.000	-6.000
12.1%	7.50	45.0	5.657	3.536	0.707	-2.828
9.7%	8.40	26.6	6.261	3.130	1.789	-0.894