Coursework
Mixed-Integer Nonlinear Optimisation

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INSTRUCTIONS

1. The assignment is due 26 May.
2. Please email your solutions to `r.misener@imperial.ac.uk` with a copy to `dominique.sundt@univie.ac.at`. Solutions may be completed in LaTeX or hand-written and subsequently scanned. Please note that Imperial automatically deletes messages of size greater than 25Mb.

Question 1 *Convexity of Robust Principal Component Analysis*

The **Introduction & Applications** *slides give the following definition of Robust Principal Component Analysis:*

$$\min_{\mathbf{A},\,\mathbf{E}} \operatorname{rank}(\mathbf{A}) + \lambda \left\|\mathbf{E}\right\|_0$$
$$X = \mathbf{A} + \mathbf{E}$$

where:

Definitions	$\operatorname{rank}(\mathbf{A})$	*Rank of the matrix*
	$\left\|\mathbf{E}\right\|_0 = \#\left\{E_{i,j} \neq 0\right\}$	*# Nonzero elements*

Is this a convex problem or not? Please justify your answer.

Question 2 *Model building for nonlinear optimisation*

Consider the problem of finding a rectangle with minimum area enclosing the set of points $\{\mathbf{p}^{(1)} := (2,3), \mathbf{p}^{(2)} := (4,4), \mathbf{p}^{(3)} := (4,2), \mathbf{p}^{(4)} := (6,2)\}$.

Formulate this optimisation problem as nonlinear program. Is the optimisation problem convex or not?

Hint: See the **Introduction & Applications** *slides for some intuition into the problem.*

Question 3 *Comparing convex underestimators*

Consider the nonlinear function:

$$f(x_1, x_2) = \left(\frac{x_1^2 \cdot x_2 + x_1 \cdot x_2^2}{2} \right)^{1/3}$$

on the domain $x_1, x_2 \in [1, 2]^2$*. Please justify all answers.*

3.1 *What is the true range of this function?*

3.2 *What is the range of this function using natural interval extensions?*

3.3 *Justify an* α *value for a possible* αBB *underestimator.*

3.4 *Set up a set of linear constraints underestimating* $f(x_1, x_2)$ *using factorable programming.*

3.5 *Does* $f(x_1, x_2)$ *have a special structure? What is the tightest relaxation that can be designed?*

Question 4 *Convex envelope of sum / Sum of convex envelopes*

Consider the function:

$$f(x_1, x_2, x_3, x_4) = \alpha_{12} \cdot x_1 \cdot x_2 + \alpha_{13} \cdot x_1 \cdot x_3 + \alpha_{14} \cdot x_1 \cdot x_4$$

where $\alpha_{12}, \alpha_{13}, \alpha_{14} \in \mathbb{R}$ *and* $\mathbf{x} \in [\mathbf{x}^L, \mathbf{x}^U]$ *has finite variable bounds. Could the convex envelope of* $f(x_1, x_2, x_3, x_4)$ *ever strictly dominate the sum of convex envelopes for each individual term? Please justify your answer.*

Question 5 *Mixed-integer formulations for logical constraints*

Express the following constraints using a mixed-integer formulation, e.g. continuous variables, binary variables, and additional parameters.

5.1 *If one constraint evaluates to true, force a second constraint to also evaluate as true:*

$$\mathbf{a_1}^{\mathrm{T}}\mathbf{x} \leq b_1 \implies \mathbf{a_2}^{\mathrm{T}}\mathbf{x} \leq b_2.$$

5.2 *Force a constraint to evaluate as true if and only if a second constraint evaluates as true:*

$$\mathbf{a_1}^{\mathrm{T}}\mathbf{x} \leq b_1 \iff \mathbf{a_2}^{\mathrm{T}}\mathbf{x} \leq b_2.$$

Question 6 *Maximum Underestimation Error*

Consider function $f(x, y) = x \cdot y$ *on a rectangular domain* $[x^L, x^U] \times [y^L, y^U]$*. Assume* $x^L < x^U$ *and* $y^L < y^U$*.*

What is the maximum separation distance of the bilinear term $x \cdot y$ *from its convex envelope* $\max(x^L y + x y^L - x^L y^L, x^U y + x y^U - x^U y^U)$*? Where does this maximum separation distance occur? Please justify your answer.*

Question 7 *Piecewise Linear Underestimation*

Consider the concave function:

$$f(x) = x^{\beta}, x \in [0, x^U],$$

where $x^U > 0$ *and* $0 < \beta < 1$.

7.1 *What is the maximum distance between* $f(x)$ *and its convex hull? Where does this point occur?*

7.2 *Suppose we have evaluated* f *at data points* $\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_N$ *where* $\hat{x}_0 = 0$, $\hat{x}_N = x^U$, *and the remaining points are evenly-spaced so that* $\hat{x}_i - \hat{x}_{i-1} = x^U/N$ *for* $i = 1, \ldots, N$. *Devise a mixed-integer formulation underestimating* f *using the data points* $[(\hat{x}_0, f(\hat{x}_0)), \ldots, (\hat{x}_N, f(\hat{x}_N))]$.

7.3 *Show that the mixed-integer formulation developed in Q7.2 is* sharp, *i.e. its LP relaxation is non-dominated by the convex hull of* f.

7.4 *Suppose now that, instead of evaluating the* $N + 1$ *data points at evenly-spaced intervals, we can evaluate* f *at any* $N + 1$ *data points. How shall we space the points to minimise the maximum underestimation error?*