Maximum Cut Algorithms

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Selected Topics on Combinatorial Optimization

Vienna Graduate School on Computational Optimization

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Outline

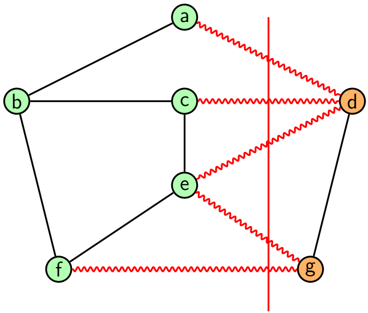
Literature

- Dahn, C., N. M. Kriege, and P. Mutzel: A fixed-parameter algorithm for the max-cut problem on embedded 1-planar graphs, IWOCA 2018, LNCS 10979, Springer, 2018, 141–152
- F. Hadlock: Finding a maximum cut of a planar graph in polynomial time, SIAM Journal on Computing, 4(3), 1975, 221–225
- F. Liers and G. Pardella: Computational Optimization and Applications 51:(1), 2012, 323–344
- M. X. Goemans and D. P. Williamson: Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, Journal of the ACM (42) (6), 1995, 1115–1145

Definition (Cut in a Graph)

Let $G = (V, E, c)$ be an undirected graph with edge weights $c(e)$ for all edges $e \in E$.

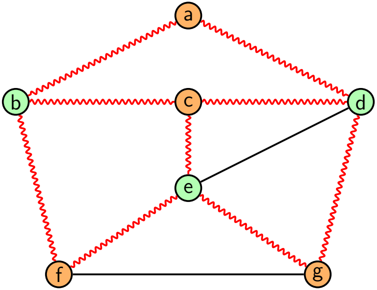
- $S \subseteq V$ defines a cut $F \subseteq E$ in $G = (V, E, c)$ with $F = (S \times \overline{S}) \cap E$
- value of F: *sum over all edge weights* in F



value: 5

Definition (Max-Cut Problem)

Given: An undirected, weighted graph $G = (V, E, c)$.
Find: A cut $F \subseteq E$ in G with maximum value.



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Complexity of the Max-Cut Problem

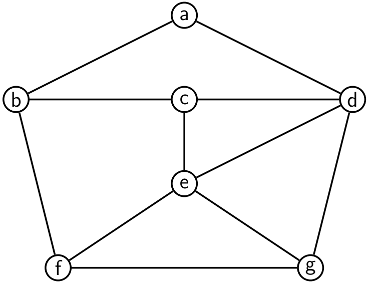
- undirected graphs: NP-hard
- planar graphs: polynomial time solvable [Hadlock 1875]
- embedded k-almost planar graphs: polynomial time solvable [Dahn, Kriege, Mutzel 2018]
- embedded 1-planar graphs: Fixed-Parameter-Tractable [Dahn, Kriege, Mutzel 2018]

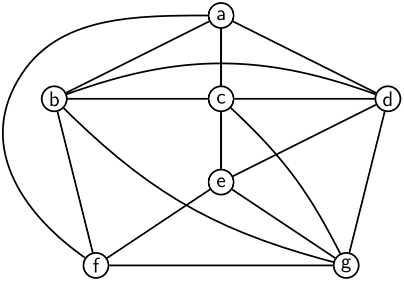
Definition (planar graph)

- *no edges crossing*

Definition (1-planar graph)

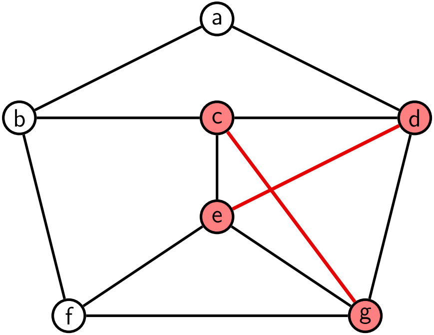
- *at most 1 crossing per edge*





Definition (k-almost planar graph)

- 1-planar
- *with at most k crossings*



a 1-almost planar graph

note: $k \in \mathcal{O}(n)$

Approximation Results

Approximation Results

- Max-Cut has a 0.87856-Approximation (Goemans and Williamson 1994 and Mahajan and Ramesh 1995)
- Max-Cut has no 0.94-Approximation under the assumption that $\mathrm{P} \neq \mathrm{NP}$ (Håstad 1996)

For the unweighted case (all edge weights 1) there exist simple $\frac{1}{2}$-approximation algorithms:

Simple $\frac{1}{2}$-Approximation Algorithms for Unweighted Case

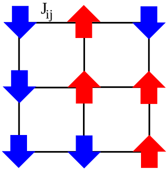
- simple randomized algorithm with expected approximation factor $\frac{1}{2}$
- simple local search algorithm with approximation factor $\frac{1}{2}$

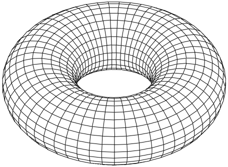
Motivation

Applications

- Quadratic 0/1-Optimization $\min\{x^T Qx + q^T x \mid x \in \{0,1\}^n\}$ for a matrix Q and a vector q
- Via minimization for designing electronic circuits
- Graph partitioning helps for Divide-and-Conquer approaches for big data into 2 or k parts of similar size, e.g. distributed computing or if the graph is too large to be handeled
- Break minimization with the home/away allocation for sport leagues
- Energy minimization for Ising Spin Glasses in Physics
- Adiabatic Quantum Computing

Ising Spin Glasses

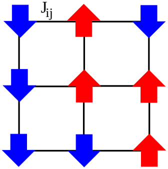


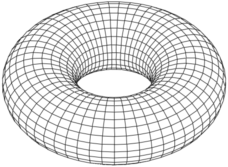


Ising Spin Glasses

- Ising Spin Glasses are certain alloys, e.g., AuFe or $Rb_2Cu_{1-x}Co_xF_4$.
- We can think of a pure material polluted by some iron atoms.
- Slowly reduce the temperature of the alloy to absolute zero.
- We are interested in the magnetic moments of the spins (iron atoms).
- This state of minimal energy is called the ground state.

Ising Spin Model Problem





Definition (Ising Spin Model Problem (ISM))

- **Given:** $G = (V, E)$ with weights h_i $(i \in V)$ and J_{ij} $((i,j) \in E)$
- **Find:** Values for the spin variables $s_i \in \{-1, +1\}$, $i \in V$, so that

$$E(s) = \sum_{i \in V} h_i s_i + \sum_{(i,j) \in E} J_{ij} s_i s_j$$

is minimized. $\leftarrow$ ground state

(ISM) equivalent to QUBO, MAX W2SAT, MAX CUT

Transformation to Max-Cut

We identify a spin configuration with a partition $V = V^+ \cup V^-$ ($V^+ \cap V^- = \emptyset$), so that $V^+ = \{i \in V \mid s_i = +1\}$ and $V^- = \{i \in V \mid s_i = -1\}$.

$$
\begin{aligned}
H(S) &= \sum_{\{i,j\} \in E} J_{ij} s_i s_j \\
&= \sum_{\{i,j\} \in E,\ i,j \in V^+} J_{ij} + \sum_{\{i,j\} \in E,\ i,j \in V^-} J_{ij} - \sum_{\{i,j\} \in E,\ i \in V^+, j \in V^-} J_{ij}
\end{aligned}
$$

and thus

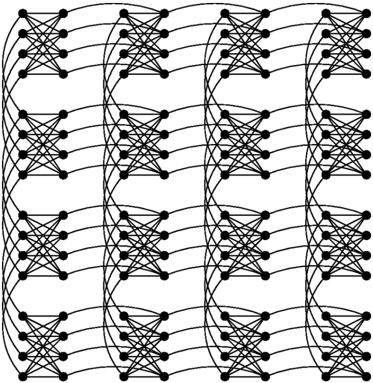
$$
H(S) - \sum_{\{i,j\} \in E} J_{ij} = -2 \sum_{\{i,j\} \in E,\ i \in V^+,\ j \in V^-} J_{ij}.
$$

Setting $c_{ij} = -J_{ij}$ for all adjacent pairs of spins leads to Max-Cut.

Adiabatic Quantum Computing

Structure of Adiabatic Quantum Computer by D-Wave:

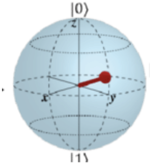
4x4 Chimera grid

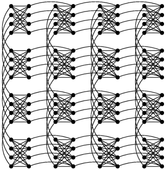


D-Wave 2017: 2000Q with 2048 Qubits (16x16 Chimera grid)

mit Jnger (Köln), Reinelt (Heidelberg), Rinaldi (Rom); Stollenwerk, Lobe (DLR), Kaibel (Magdeburg); McGeoch (D-Wave)

Adiabatic Quantum Computing





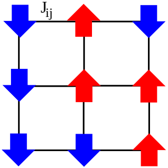
Idee: Adiabatic Quantum Computing

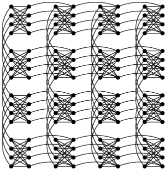
- construct system (A) with (unknown) ground state, which corresponds to the solution of my problem
- construct system (B) whose ground state can be prepared and measured experimentally
- transfer system (B) slowly to system (A) and measure the ground state ← Adiabatic Theorem

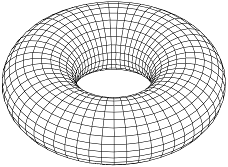
D-Wave: 2017: 2000Q with 2048 Qubits (16x16 Chimera grid)

Source:
https://www.researchgate.net/publication321133310_Molecular_Spin_Qudits_for_Quantum_Algorithms/figures?lo=1

Quantum Annealing







Definition (Ising Spin Model Problem (ISM))

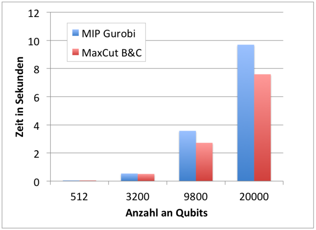
- **Given:** $G = (V, E)$ with weights h_i ($i \in V$) and J_{ij} ($(i, j) \in E$)
- **Find:** Values for the spin variables $s_i \in \{-1, +1\}$, $i \in V$, so that

$$E(s) = \sum_{i \in V} h_i s_i + \sum_{(i,j) \in E} J_{ij} s_i s_j$$

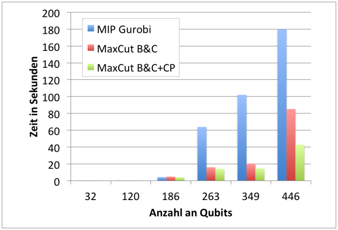
is minimized. ← ground state

(ISM) equivalent to QUBO, MAX W2SAT, MAX CUT

Experimental Evaluation (Exact Solution)



random QUBO instances



random ISM instances

- D-Wave Quantum Annealer (C_8 with 500 Qubits) vs. QP CPLEX [McGeoch,Wang 2013]
- Experiments with MIP-solver CPLEX [Dash/Puget 2015]
- Our MaxCut Branch-and-Cut vs. strengthened MaxCut B&C by cutting planes ← provable exact solutions

Simple Randomized Max-Cut Algorithm

Algorithm Randomized Max-Cut

1. $V_1 := V_2 := \emptyset$
2. **for** all $v \in V$
3. with probability $\frac{1}{2}$ set $V_1 = V_1 \cup \{v\}$
   otherwise set $V_2 = V_2 \cup \{v\}$
4. **Return** V_1, V_2

Theorem

Algorithm Randomized Max-Cut computes a cut in graph $G = (V, E)$ containing an expected number of $|E|/2$ edges. Hence, for an unweighted graph it provides an expected approximation factor of $\frac{1}{2}$. The algorithm runs in linear time $O|V|$.

Proof

We show $\mathrm{E}\,(\#\text{cut edges}) = \mathrm{E}\,(w(V_1, V_2)) = |E|/2$

Define indicator variables X_e for $e \in E$

$$\text{with } X_e := \begin{cases} 1 & \text{if } |e \cap V_1| = |e \cap V_2| = 1 \\ 0 & \text{else} \end{cases}$$

Observation $w(V_1, V_2) = \sum_{e \in E} X_e$

$$\begin{aligned} \mathrm{E}\,(w(V_1, V_2)) &= \mathrm{E}\left(\sum_{e \in E} X_e\right) \\ &= \sum_{e \in E} \mathrm{E}\,(X_e) \\ &= \sum_{e \in E} \mathrm{Prob}\,(X_e = 1) \\ &= |E| \cdot \mathrm{Prob}\,(X_e = 1)\,, \text{ for an edge } e = (u, v) \end{aligned}$$

Expected Number of Edges in the Cut

We have

$$
\begin{aligned}
& \mathrm{E}\left(w(V_1, V_2)\right) \\
= \;& |E| \cdot \mathrm{Prob}\left(X_e = 1\right), \text{ for an edge } e = (u, v) \\
= \;& |E| \cdot \mathrm{Prob}\left(((u \in V_1) \wedge (v \in V_2)) \vee ((u \in V_2) \wedge (v \in V_1))\right) \\
= \;& |E| \cdot \left(\mathrm{Prob}\left((u \in V_1) \wedge (v \in V_2)\right) + \mathrm{Prob}\left((u \in V_2) \wedge (v \in V_1)\right)\right) \\
= \;& |E| \cdot \left(\mathrm{Prob}\left(u \in V_1\right) \cdot \mathrm{Prob}\left(v \in V_2\right) + \mathrm{Prob}\left(u \in V_2\right) \cdot \mathrm{Prob}\left(v \in V_1\right)\right) \\
= \;& |E| \cdot \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right) = |E| \cdot \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{|E|}{2}
\end{aligned}
$$

Since $\mathrm{OPT} \leq |E|$, we get an expected approximation factor of $\frac{1}{2}$. $\square$

Local Search for Unweighted Max-Cut

General idea of local search

1. Find a feasible initial solution.
2. Repeat until convergence: Find a solution in the neighbourhood with better objective value.

It is necessary to define problem dependent neighbourhoods.

This strategy leads to a local optimum with respect that there is no better solution in the neighbourhood.

Local search for unweighted Max-Cut

Initial solution e.g. $V_1 = \emptyset$, $V_2 = V$

Definition (Neighbourhood $\mathcal{N}$ of a cut)

The 1-Exchange neighbourhood w.r.t. a cut (V_1, V_2) consists of all partitions (V_{1k}, V_{2k}) for $k = 1, 2, \ldots, |V|$ with

1. If $v_k \in V_1$, then $V_{1k} = V_1 \setminus \{v_k\}$ und $V_{2k} = V_2 \cup \{v_k\}$,
2. If $v_k \in V_2$, then $V_{1k} = V_1 \cup \{v_k\}$ und $V_{2k} = V_2 \setminus \{v_k\}$.

Theorem (Local Search for Max-Cut)

For an unweighted Max-Cut instance G with optimum value OPT let $m_{\mathcal{N}}(G)$ the solution value of a local optimum w.r.t $\mathcal{N}$. Then we have

$$m_{\mathcal{N}}(G) \geq \frac{\mathrm{OPT}}{2}.$$

Proof: Analysis of the 1-Exchange for Max-Cut

Let $m := |E|$ and V_1, V_2 the corresponding vertex sets to $m_{\mathcal{N}}$.

Since $\mathrm{OPT} \leq m$ it suffices to show $2m_{\mathcal{N}}(G) \geq m$.

For $W \subseteq V$ let $E(W) = \{(u, v) \in E \mid u, v \in W\}$ and $\delta(W) = \{(u, v) \in E \mid u \in W \text{ and } v \notin W\}$, $m_1 = |E(V_1)|$, and $m_2 = |E(V_2)|$. Then $m = m_1 + m_2 + m_{\mathcal{N}}(G)$.

We can show that $m_1 + m_2 - m_{\mathcal{N}}(G) \leq 0$

Together this leads to

$$\implies \quad m - 2m_{\mathcal{N}}(G) \leq 0$$

$$\implies \quad 2m_{\mathcal{N}}(G) \geq m$$

Proof of $m_1 + m_2 - m_{\mathcal{N}}(G) \leq 0$

(V_1, V_2) is local optimum with value $m_{\mathcal{N}}(G)$ and V_{1k}, V_{2k} let be the neighbourhood partitions. For $v_i \in V$ let

$$\begin{aligned} m_{1i} &= |\{v \mid v \in V_1, (v, v_i) \in E\}|, \\ m_{2i} &= |\{v \mid v \in V_2, (v, v_i) \in E\}|. \end{aligned}$$

$$\Longrightarrow \quad (\forall k) \begin{cases} |\delta(V_{1k})| \leq m_{\mathcal{N}}(G) \\ |\delta(V_{2k})| \leq m_{\mathcal{N}}(G) \end{cases}$$

$$\Longrightarrow \quad \left\{ \begin{array}{l} (\forall v_i \in V_1)\ m_{1i} - m_{2i} \leq 0 \\ (\forall v_j \in V_2)\ m_{2j} - m_{1j} \leq 0 \end{array} \right\} \geq \text{ that many neighbours on the other side}$$

$$\Longrightarrow \quad \begin{cases} \sum_{v_i \in V_1} (m_{1i} - m_{2i}) = 2m_1 - m_{\mathcal{N}}(G) \leq 0 \\ \sum_{v_j \in V_2} (m_{2j} - m_{1j}) = 2m_2 - m_{\mathcal{N}}(G) \leq 0 \end{cases}$$

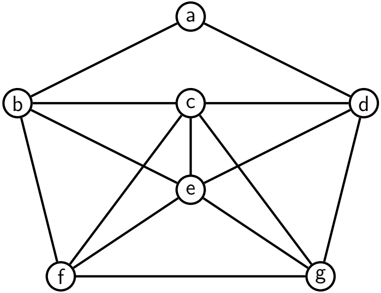
$$\Longrightarrow \quad m_1 + m_2 - m_{\mathcal{N}}(G) \leq 0 \qquad \square$$

FPT-Algorithm for Max-Cut

Input of the Algorithm: Graph $G = (V, E, c)$ with non-negative edge weights and its set of crossing edges in a 1-planar embedding with k crossings (can be extended to arbitrary edge weights)

Definition (Set of crossing edges)

- embedding of G provides a set of crossing edges X
- crossing $\chi \in X$



$\chi_1 = \{cg, de\}$,
$\chi_2 = \{be, cf\}$,
$X = \{\chi_1, \chi_2\}$

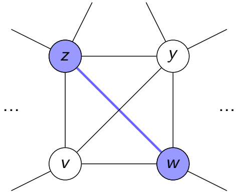
note: $|X| = k$

Max-Cut for 1-planar graphs: Idea

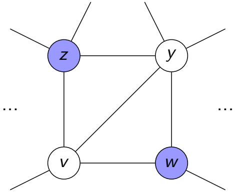
- transform the given 1-planar graph into a planar graph
- use a `MaxCut` algorithm for planar graphs
- transfer the result back onto the original graph

Planarize a crossing

Option 1: Deleting an edge



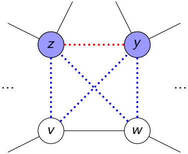
G



$G - wz$

Planarize a crossing

Option 2: Contracting two nodes



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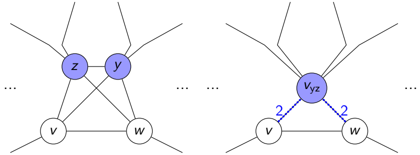


G/yz

Algorithm

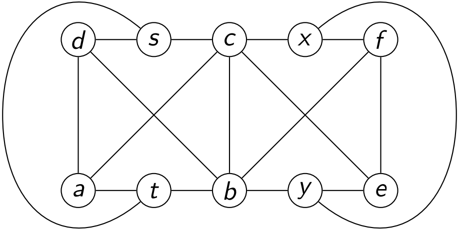
$\mathtt{GewMaxCut}_{\mathtt{k}}(G, \Gamma, X(\Gamma))$

Input: Graph G with $\mathtt{k}$-almost planar drawing Γ, crossing set $X(\Gamma)$ and $\mathtt{k} = |X(\Gamma)|$
Output: Maximal Cut $F \subseteq E_G$

if $\mathtt{k} == 0$ **then**
  $F = \textsc{GewMaxCut}_{\mathtt{pl}}(G, \Gamma)$
else
  choose an element $z = \{e_1 = ac, e_2 = bd\} \in X(\Gamma)$
  $G_1 = G/cd$;
  $G_2 = G/bc$;
  $G_3 = G - e_2$;
  $F_i = \textsc{MaxCut}_{|X_i|}(G_i, \Gamma_i, X_i)$ for $i \in \{1, 2, 3\}$
  $j = \operatorname{argmax}_{1 \leq i \leq 3} c(F_i)$
  $F =$ transfer optimal cut F_j to G
end if
return F

Example: Calculating $\texttt{MaxCut}(H, X)$

Given the 2-almost planar graph H with the set of crossings $X = \{\chi_1, \chi_2\}$:

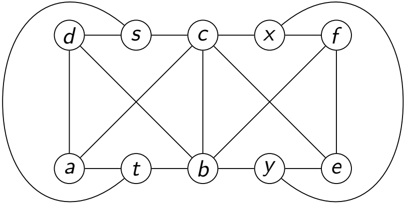


H

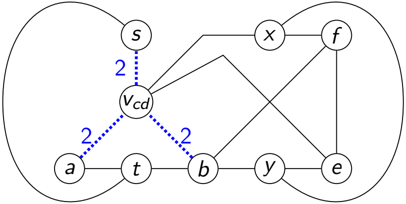
Choose $\chi_1 = \{ac, bd\}$ to be planarized.

Example: Calculating $\texttt{MaxCut}(H, X)$

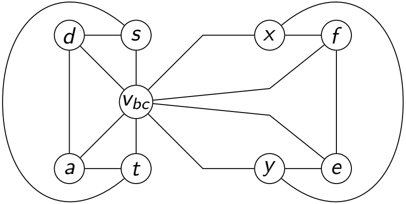
Calculate the three new subgraphs of H w.r.t. χ_1:



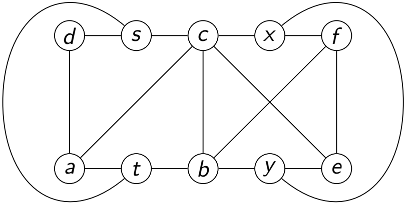
H



$H_1 = H/cd$



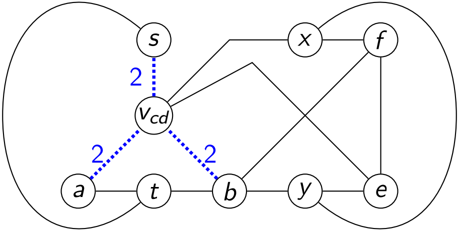
$H_2 = H/bc$



$H_3 = H - bd$

Example: Calculating $\texttt{MaxCut}(H_1, X_1)$

Recursive call for H_1 with $X_1 = \{\chi_2\}$:

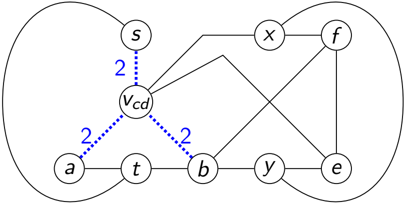


$H_1 = H/cd$

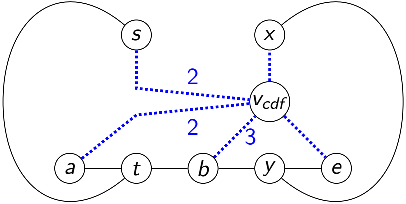
Choose $\chi_2 = \{bf, v_{cd}e\}$ to be planarized

Example: Calculating $\texttt{MaxCut}(H_1, X_1)$

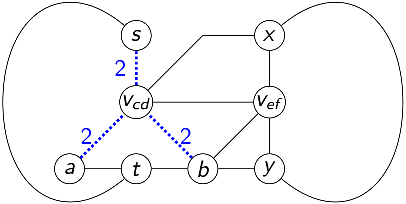
Calculate the three new subgraphs of H_1 w.r.t. χ_2:



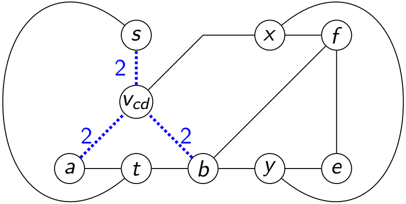
$H_1 = H/cd$



H_1/cf



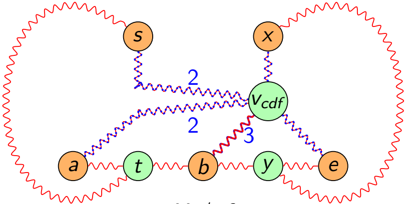
H_1/ef



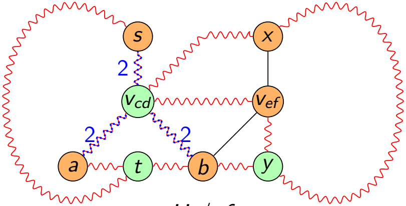
$H_1 - ce$

Example: Calculating $\texttt{MaxCut}(H_1, X_1)$

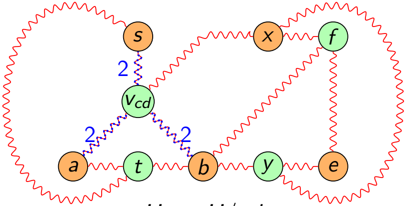
Recursively calculated solutions:



H_1/cf



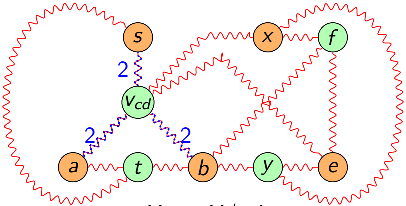
H_1/ef



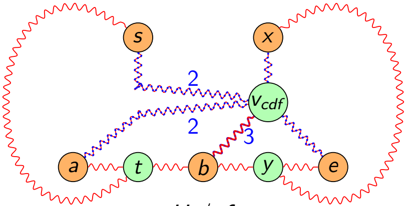
$H_1 = H/cd$

Example: Calculating $\mathtt{MaxCut}(H_1, X_1)$

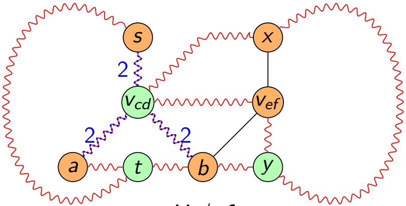
Finding the best solution and transferring it onto H_1:



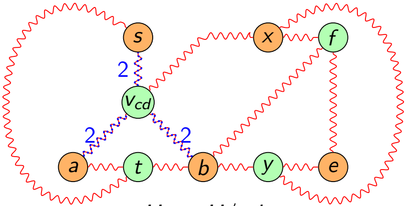
$H_1 = H/cd$



H_1/cf



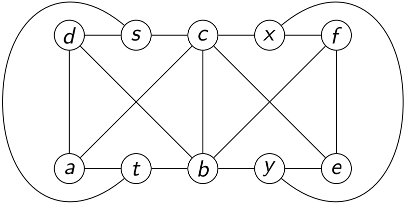
H_1/ef



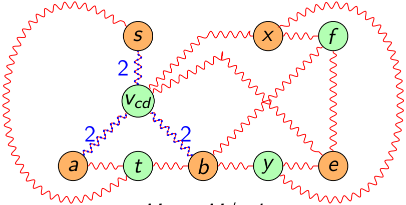
$H_1 = H/cd$

Example: Calculating $\texttt{MaxCut}(H, X)$

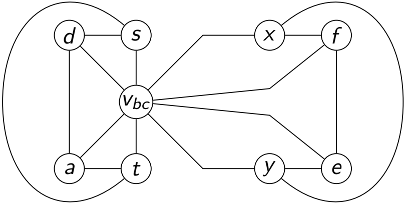
Recursively calculated solution for H_1:



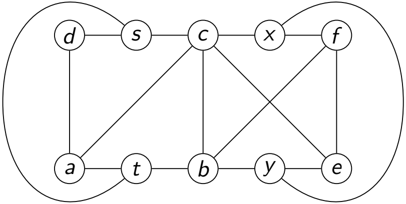
H



$H_1 = H/cd$



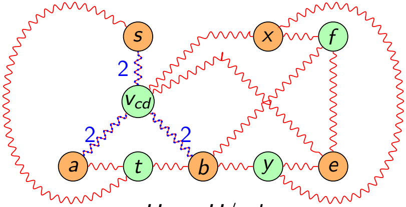
$H_2 = H/bc$



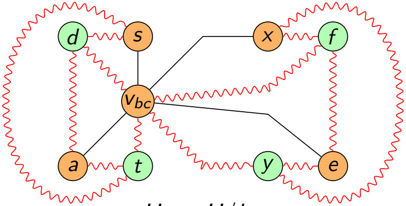
$H_3 = H - bd$

Example: Calculating $\texttt{MaxCut}(H, X)$

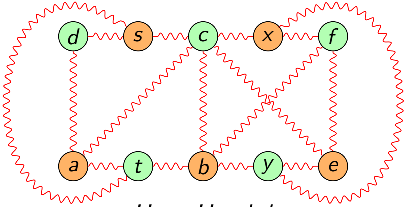
Recursively calculated solutions:



$H_1 = H/cd$



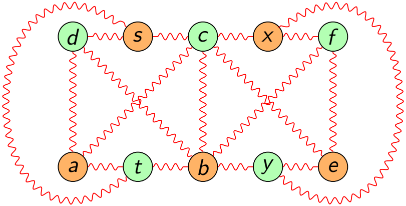
$H_2 = H/bc$



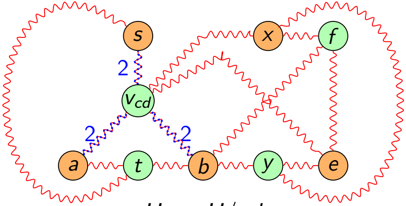
$H_3 = H - bd$

Example: Calculating $\texttt{MaxCut}(H, X)$

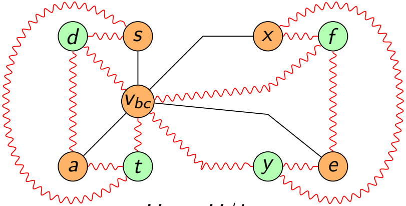
Finding the best solution and transferring it onto H:



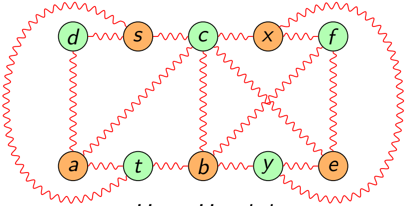
H



$H_1 = H/cd$



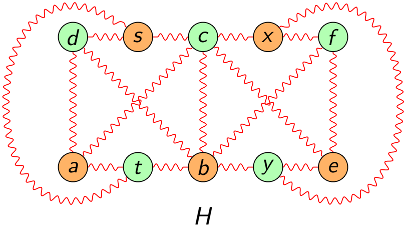
$H_2 = H/bc$



$H_3 = H - bd$

Example: Calculating $\texttt{MaxCut}(H, X)$

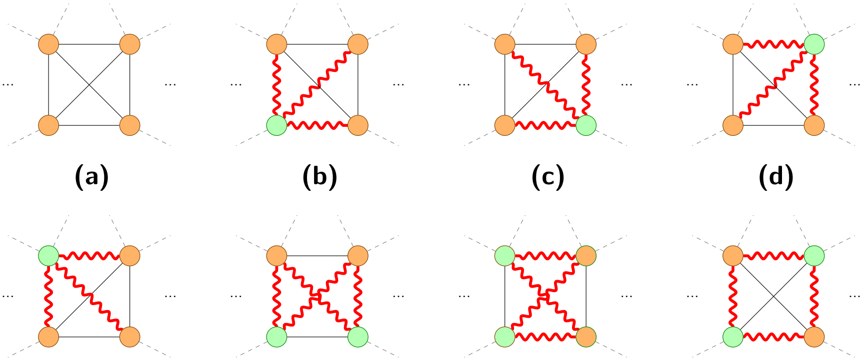
Maximum cut in H:



Value of the maximum cut in H: 17

Partitioning the endpoints of a crossing

8 non-isomorphic partitions of the four endpoints of a crossing:



(a) (b) (c) (d)

(e) (f) (g) (h)

Proof of optimality *(sketch)*

Theorem 1

`MaxCut` computes a maximum cut in a 1-planar graph G given a set of crossing edges X in a 1-planar embedding of G.

Proof by induction over $k = |X|$

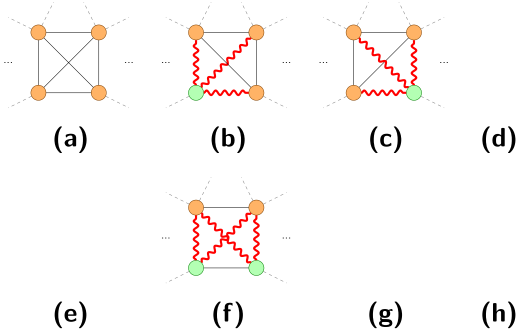
$k = 0$

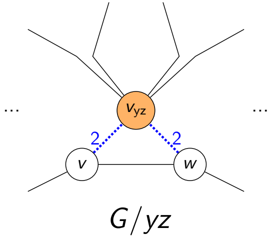
- G is planar

$k > 0$

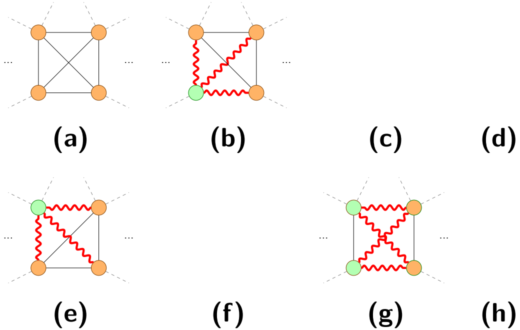
- look at 8 cases (a)-(h)

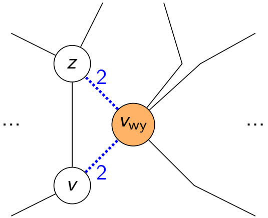
Proof of optimality *(sketch)*





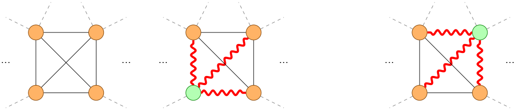
Proof of optimality *(sketch)*



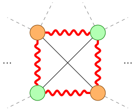


G/wy

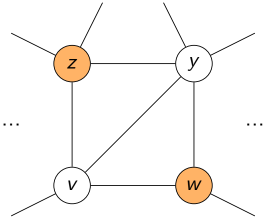
Proof of optimality *(sketch)*



(a) **(b)** **(c)** **(d)**



(e) **(f)** **(g)** **(h)**



$G - wz$

Running time

Theorem

`MaxCut` computes a maximum cut in an embedded 1-planar graph with n nodes and k crossings in time $\mathcal{O}(3^k \cdot n^{3/2} \log n)$.

Sketch of proof:

- no more than 3^k recursive calls
- dominating running time of one call: $\mathcal{O}(n^{3/2} \log n)$
- overall running time: $\mathcal{O}(3^k \cdot n^{3/2} \log n)$

FPT

Theorem

`MaxCut` computes a maximum cut in an embedded 1-planar graph with n nodes and k crossings in time $\mathcal{O}(3^k \cdot n^{3/2} \log n)$.

Corollary

`MaxCut` computes a maximum cut in an embedded k-almost planar graph in polynomial time.

Parameterization

The MAX-CUT problem on embedded 1-planar graphs with k crossings is *fixed-parameter-tractable* with parameter k.

Conclusion and open problems

Conclusion

- MAX-CUT on embedded k-almost planar graphs $\in P$
- MAX-CUT on embedded 1-planar graphs $\in FPT$

Open problems

- Drop the assumption that we are given a 1-planar embedding
- Generalize this approach to k-planar or non restricted graphs

Counter example on a 4-planar graph