Dynamic Shortest Path Algorithms

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Outline

Literature

- G. Ramalingam and T. Reps: On the computational complexity of dynamic graph problems. Theoretical Computer Science 158, 1996, 233-277
- C. Demetrescu and G. F. Italiano: Experimental Analysis of Dynamic All Pairs Shortest Path Algorithms, ACM Transactions on Algorithms, vol. 2 (4), 2006, S. 578-601
- C. Demetrescu and G. F. Italiano: A new approach to dynamic all pairs shortest paths. Journal of the ACM 51(6): 968–992 (2004)

There are many recent papers for dynamic algorithms for solving the All-Pairs Shortest Path Problem (social network analysis); they are mainly based on Ramalingam and Reps.

Dynamic Graph Algorithms

Definition (Dynamic Graph Algorithm)

A dynamic algorithm maintains a property P of a weighted graph during dynamic changes of, e.g.,

- insertion of new edges/vertices
- deletion of new edges/vertices
- weight-changes of edges/vertices

Requirement to dynamic algorithms

- fast response of queries concerning property P
- fast update operations, in particularly faster than a static algorithm computing from scratch

Here, we are concerned with edge changes, only.

Motivation

Applications

- Prediction of future states of networks shall be answered quickly
- Centrality indices for social network analysis are based on shortest path
- Social networks change (quickly) over time
- Brain networks change very quickly
- Weights on road networks vary a lot during day time
- Navigation queries shall be answered quickly

Dynamic Complexity Measures

Amortized

Worst Case in the size of the input, amortized over a sequence of operations.

Alternatively

Worst Case in the size of the changes in input and output.

Ramalingam and Reps used the alternative complexity measure.

Dynamic Complexity Measures

Definition

- A vertex v is called modified if the input value has changed or if v was inserted or deleted.
- A vertex v is called affected if v was inserted or it got a new output value.
- Let *CHANGED* be the set of all modified or affected vertices.
- We define $\delta = |CHANGED|$.
- We define $\epsilon = \delta +$number of edges incident to *CHANGED*.

Remarks

- δ is a lower bound for the number of operations needed.
- It is also obvious that all the incident edges to *CHANGED* need to be tested in order to guarantee correctness.

Dynamic Complexity Measures

Definition

- A dynamic algorithm is called full-dynamic if it works for increased edge weights and edge deletions, resp., as well as for edge weight decrease and edge insertions, resp.
- Otherwise it is called incremental (or partially dynamic).

Definition

- A dynamic algorithm is called bounded if the update time for one operation is bounded in ϵ.
- Otherwise it is called unbounded.

Here: a bounded dynamic single-source shortest path algorithm.

Single-Source Shortest Path

Definition (SSSP)

- **Given:** Directed graph $G = (V, E)$ with positive edge weights c and vertex $t \in V$.
- **Find:** Shortest (v, t)-paths in G for all $v \in V$.

Remark: Our definition is different from the standard definition, where a path form s to all other vertices is searched.

Dijkstra's Algorithm

Let M be a very large number and $dist(v)$ the distance matrix.

Q is a priority queue with operations

- InsertQ($v, dist(v)$): inserts v into Q with priority $dist(v)$
- ExtractMinQ(): outputs and then deletes the minimum from Q
- DecreasePrioQ($v, dist(v)$): updates the decreased priority of v in Q to $dist(v)$

Dijkstra's Algorithm

```
1:  function DIJKSTRA
2:      Initialisation: Q = ∅; dist[t] = 0; INSERTQ(t, 0)
3:      for all vertices v ∈ V, v ≠ t do dist(v) = M; INSERTQ(v, M)
4:      while Q ≠ ∅ do
5:          v =EXTRACTMINQ()          // distance of v is minimum
6:          for all edges (u, v) ∈ E  do // edge scanning
7:              if dist(u) > c(u, v) + dist(v) then
8:                  set dist(u) = c(u, v) + dist(v)
9:                  DECREASEPRIOQ(u, dist(u))
10:             end if
11:         end for
12:     end while
13:     return dist(v) for all v ∈ V
14: end function
```

Analysis of Dijkstra's Algorithm

Let $n = |V|$ and $m = |E|$.

The running time of Dijkstra is $T(n, m) =$
$n \cdot T(\textsc{InsertQ}) + n \cdot T(\textsc{ExtractMinQ}() + m \cdot T(\textsc{DecreasePrioQ}())$

Binary Heap for Q: $T(n, m) = O((n + m) \log n)$

- $T(\textsc{InsertQ}) = O(\log n)$
- $T(\textsc{ExtractMinQ}() = O(\log n)$
- $T(\textsc{DecreasePrioQ}() = O(\log n)$

Fibonacci Heap for Q: $T(n, m) = O(m + n \log n)$

- $T(\textsc{InsertQ}) = O(1)$
- $T(\textsc{ExtractMinQ}() = O(\log n)$ amortized
- $T(\textsc{DecreasePrioQ}() = O(1)$ amortized

Algorithm by Ramalingam and Reps

Algorithm supports the operations

- DelEdge(v, w): delete the edge (v, w) from $G \rightarrow$ we get G'
- InsEdge(v, w, c): insert the edge (v, w) with weight c into $G \rightarrow$ we get G'

Remark: Using the operations above, also the operations IncreaseWeight(v, w, c) and DecreaseWeight(v, w, c) can be simulated.

Algorithm supports the queries

- $dist(v)$: output distance between vertices v and t
- $path(v)$: output a shortest path from v to t if there exists one

Basics of the Algorithm

Definition

A subgraph T of G is called shortest-path tree for G with sink t if

- T is a directed tree with root t
- $V(T)$ is a set of vertices that can reach t, and
- for each edge (u, v) in T we have: $dist(u) = dist(v) + c(u, v)$.

Basics of the Algorithm

Definition

- An edge is called SP-edge, if it is part of the shortest (v, t)-path for some $v \in V$.
- Hence, an edge is an SP-edge iff $dist(u) = dist(v) + c(u, v)$.
- Let $SP(G)$ be the subgraph of G induced by the set of all SP-edges.

Observation

- Every shortest path is part of $SP(G)$ and vice versa:
- Every path in $SP(G)$ is a shortest path in G.
- Since all edge weights are positive, $SP(G)$ is a directed acyclic graph.

Operation DeleteEdge(v, w)

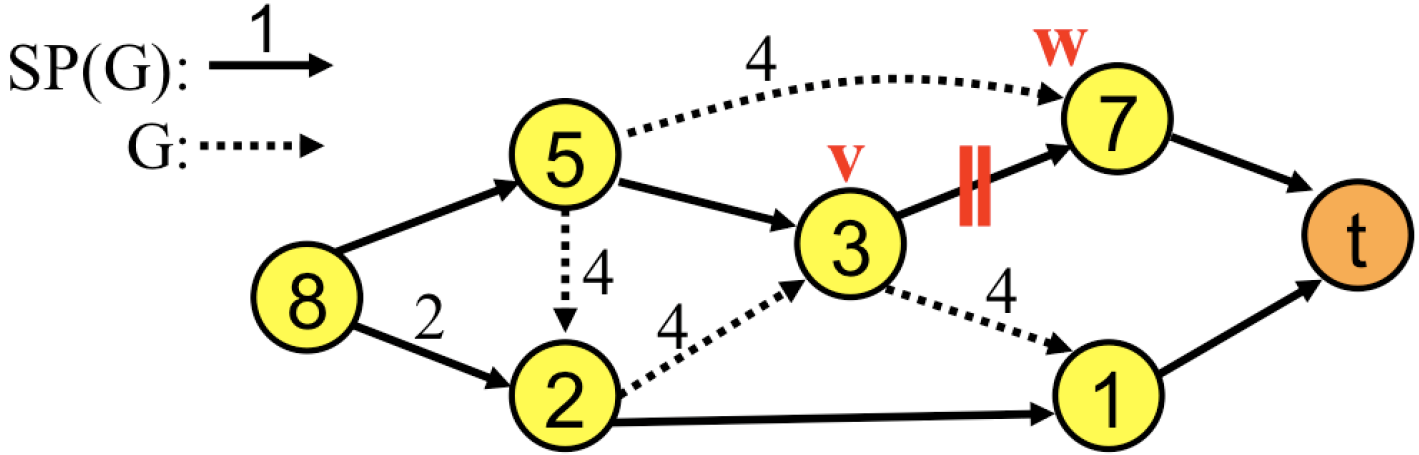
DeleteEdge(v, w) from $G \to G'$

Remark

- A vertex v is affected if v was inserted or it got a new output value.
- Hence: a vertex v in G' is affected if $dist_G(v) \neq dist_{G'}(v)$.

Definition

An SP-edge (x, y) is called affected by DeleteEdge(v, w) if there is no (x, t)-path in G' with length $dist_G(x)$ using the edge (x, y).



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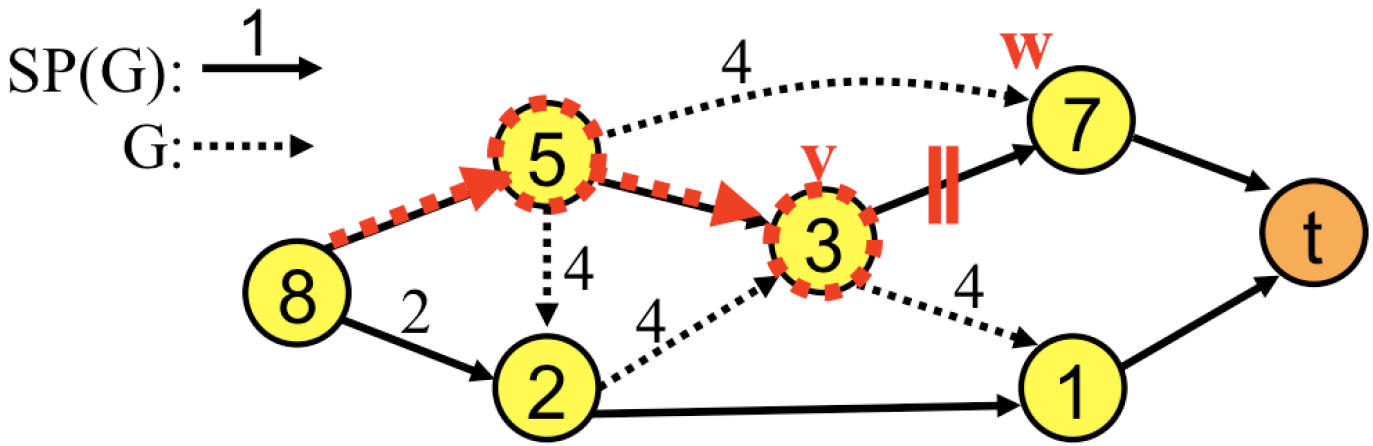


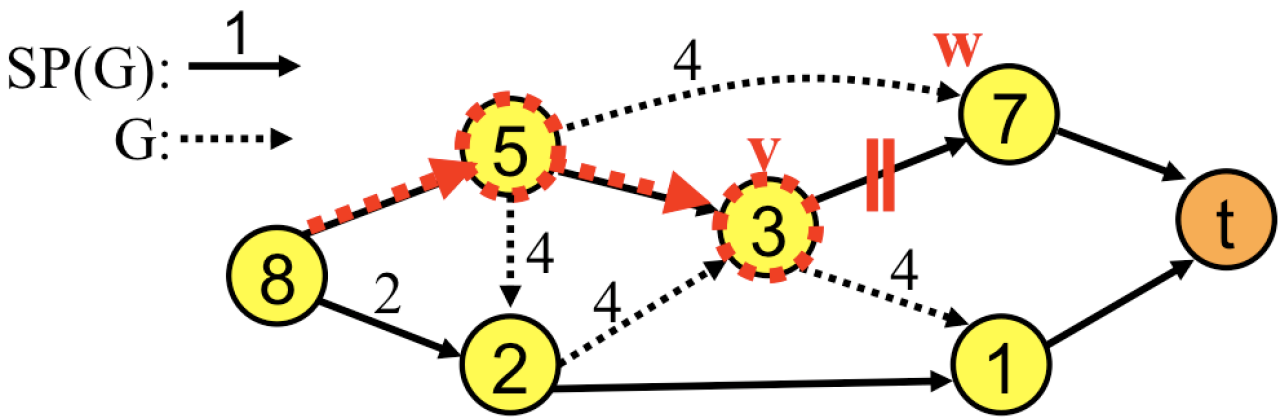
Operation DeleteEdge(v, w)

Definition

An SP-edge (x, y) is called affected by $\textsc{DeleteEdge}(v, w)$ if there is no (x, t)-path in G' with length $dist_G(x)$ using the edge (x, y).

Observations in $SP(G)$

- (x, y) is affected $\Leftrightarrow$ y is affected
- vertex $x \neq v$ is affected $\Leftrightarrow$ all outgoing edges of x are affected
- v itself is affected $\Leftrightarrow$ (v, w) is the only outgoing SP-edge from v



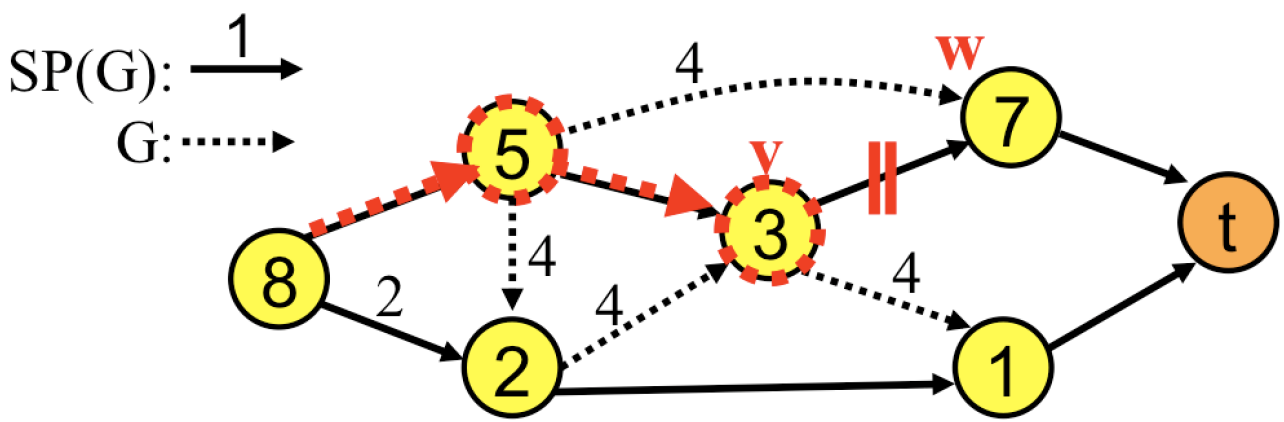
Algorithmic Idea for DeleteEdge(v, w)

Phase 1:

- Find the set of all affected edges and vertices
- Delete the affected edges from $SP(G)$

Phase 2:

- Compute the new $dist()$ values for the affected vertices
- Update $SP(G)$



Algorithmic Idea for Phase 1 of DeleteEdge(v, w)

Observation

The set of affected vertices corresponds to the set of vertices u for which there is no (u, t)-path in $SP(G) \setminus (v, w)$.

Algorithmic Idea of Phase 1:

- Let $H = SP(G) \setminus (v, w)$.
- While there exists a vertex u in H with $outdeg(v) = 0$:
  - Delete u from H and add u to the set of affected vertices.

Remarks

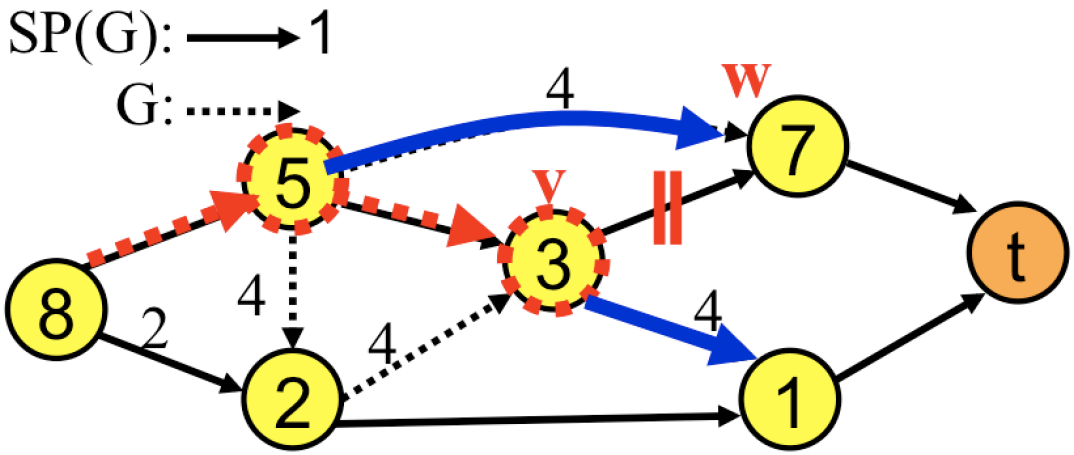
- The algorithm realizes Topological Sort
- Please note that BFS is not working!
- $SP(G)$ is maintained either as graph with adjacency lists or implicitly (test if $dist(u) = dist(v) + c(u, v)$.

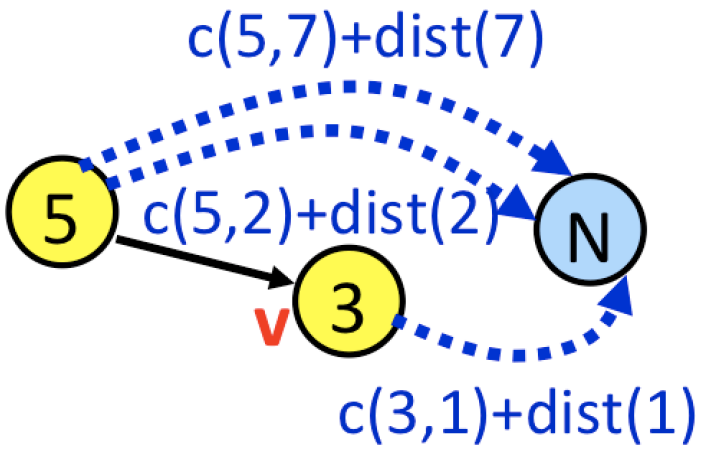
Algorithmic Idea for Phase 2 of DeleteEdge(v, w)

- Let N be the set of non-affected vertices.
- **Objective:** Compute $dist(x)$ for all affected vertices in $A = V \setminus N$.

Algorithmic Idea of Phase 2:

- Let $H = SP(G) \setminus (v, w)$.
- Think of contracting N to a vertex t' and substitute the edges (x, y) with $x \in A$, $y \in N$ by (x, t') with weights $x(x, y) + dist(y)$
- Run the Dijkstra algorithm on the new graph





Algorithm for Phase 2 of DeleteEdge(v, w)

Algorithm for Phase 2:

- We do not really contract N, since this would need too much time

Initialization of Dijkstra

- For all affected vertices a: calculate
  $dist(a) = \min(\{c(a,b) + dist(b) \mid (a,b) \in E \text{ and } b \text{ not affected } \}, M)$
- If $dist(a) \neq M$ then $\textsc{InsertQ}(a, dist(a))$

Main Loop of Dijkstra after $a \leftarrow$ ExtractMinQ():

- For each $b \in succ(a)$: If $dist(a) == c(a,b) + dist(b)$, then insert (a,b) to $SP(G)$
- For each $b \in pred(a)$: If $dist(b) > c(b,a) + dist(a)$, then $dist(b) = c(b,a) + dist(a)$; Only then call:
- $\textsc{DecreasePrioQ}(b, dist(b))$ resp. $\textsc{InsertQ}(b, dist(b))$

Algorithm Analysis of DeleteEdge(v, w)

Correctness of Algorithm for Phase 2:

- Obviously, affected vertex set will be computed correctly
- All affected vertices that are able to reach t will be inserted into Q during the Dijkstra run.
- Their distance will be set correctly due to the correctness of Dijkstra
- Non-affected vertices will never be inserted into Q
- The SP-graph will be maintained correctly.

Algorithm Analysis of DeleteEdge(v, w)

Running time for Phase 1:

- Number of iterations: δ
- One iteration for vertex u needs $O(|pred(u)|)$
- Total running time: $O(\epsilon)$

Running time for Phase 2 (Fibonacci Heaps):

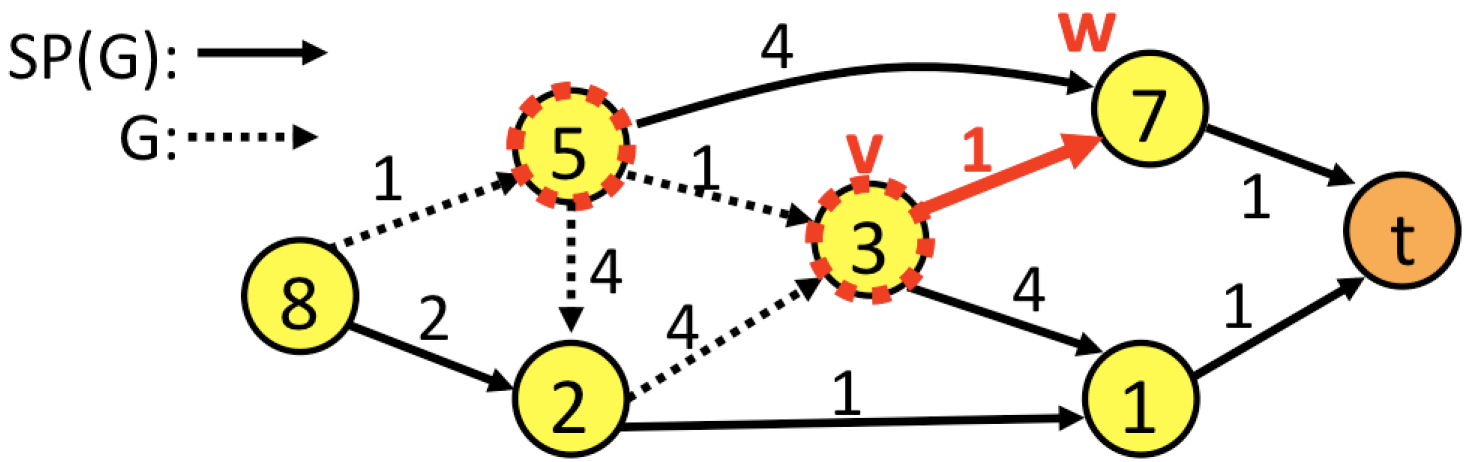
- $T(\text{INSERTQ}) = O(1)$
- $T(\text{DECREASEPRIOQ}() = O(1)$ amortized
- $T(\text{EXTRACTMINQ}() = O(\log p)$ amortized with p is the number of elements in the heap
- Number of iterations: at most δ
- Total running time: $O(\epsilon + \delta \log \delta)$

Operation InsertEdge(v, w)

InsertEdge(v, w) from $G \to G'$

Remark

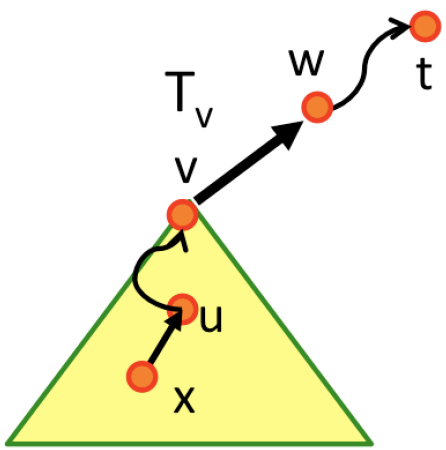
- If a vertex u is affected then all the new shortest (u, t)-paths in G' are of the form:
- (shortest (u, v)-path, edge (v, w), shortest (w, t)-path)
- u is affected $\Leftrightarrow dist_{G'}(u, v) + c(v, w) + dist_G(w) < dist_G(u)$, where $dist_{G'}(u, v)$ is the length of the shortest path from u to v in G', which is equal to $dist_G(u, v)$ for this case.



Operation InsertEdge(v, w)

Consider the new shortest path tree T_v for vertex v:

- Let $x \neq v$ be an arbitrary vertex and (x, u) an edge in T_v.
- If x is affected, then also u must be affected, because otherwise:
- If there is a shortest (u, t)-path P not using the edge (v, w), then the path $((x, u), P)$ would also be a shortest (x, t) path and x would not be affected.
- $\Rightarrow$ The set of affected vertices builds a connected subtree of T_v with root v



Algorithm for InsertEdge(v, w)

Algorithmic Idea for InsertEdge(v, w)

- Use Dijkstra with priority $dist(x) - dist(v)$ for all affected vertices x
- Since $dist(v)$ is a constant for all vertices during the run, we can instead calculate with the distances $dist(x)$

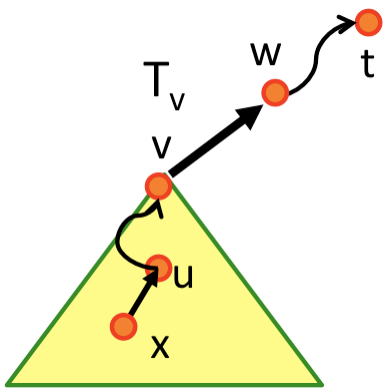
Initialization of Dijkstra

- If v is affected, then $\textsc{InsertQ}(v, dist_{G'}(v))$
- Otherwise: If there exists a shortest path in G' using the edge (v, w) then update $SP(G)$.

Algorithm for InsertEdge(v, w)

Main Loop of Dijkstra after $u \leftarrow$ ExtractMinQ():

- Delete all outgoing edges from u in $SP(G)$
- For each $y \in succ(u)$: If $dist(u) == c(u, y) + dist(y)$, then insert (u, y) to $SP(G)$
- For each $x \in pred(u)$: If $dist(x) > c(x, u) + dist(u)$, then $dist(x) = c(x, u) + dist(u)$; Only then call:
- DecreasePrioQ$(x, dist(x))$ resp. InsertQ$(x, dist(x))$
- Otherwise If $dist(x) == c(x, u) + dist(u)$ then insert (x, u) into $SP(G)$



Algorithm Analysis of InsertEdge(v, w)

Correctness

- If v is affected, then Dijkstra will start at v with the new *dist* values
- All paths of the affected vertices use vertex v
- Since the set of affected vertices is connected, all affected vertices will be inserted into Q and edge scanning will be executed.
- Distances will be calculated correctly (Dijkstra)

Running time

- $T(\text{InsertQ}) = O(1)$
- $T(\text{DecreasePrioQ}() = O(1)$ amortized
- $T(\text{ExtractMinQ}() = O(\log p)$ amortized with p is the number of elements in the heap
- Number of iterations: at most δ
- Total running time: $O(\epsilon + \delta \log \delta)$

Conclusion

Theorem

The algorithm by Ramalingam and Reps provides a bounded dynamic algorithm for the single-source shortest path problem with running time $O(\epsilon + \delta \log \delta)$ *for each update operation* DeleteEdge(v, w) *and* InsertEdge(v, w, c).

Experimental Comparison: Test Setup

Experimental comparison by Demetrescu and Italiano 2006

Random graphs

- Randomly generated graphs with random update sequence

Real-world graphs

- US Road networks: `ftp://edcftp.cs.usgs.gov`
  - 148–952 vertices, 434–2896 edges, weights: $\leq 200\,000$ (distances)
- Internet graphs (AS): `http://www.routeviews.org`
  - 500–3000 vertices, 2406–13 734 védges, weights: $\leq 20\,000$

Bottleneck graphs

- Artificially generated bottleneck graphs: two bipartite components that are connected via one single edge which is updated

Experimental Comparison: Test Setup

- AMD Athlon – 1.8 GHz, 256KB L2 cache, 512MB RAM
- AMD Athlon – 2.1 GHz, 256KB L2 cache, 768MB RAM

- Intel Xeon – 500 MHz, 512KB L2 cache, 512MB RAM
- Intel Pentium 4 – 2.0 GHz, 512KB L2 cache, 2GB RAM

- Sun UltraSPARC IIi – 440 MHz, 2MB L2 cache, 256MB RAM

- IBM Power 4 – 1.1 GHz, 1.4MB L2 cache, 32MB L3 cache, 64GB RAM

Experimental Comparison: Test Setup

- D-RRL: dynamic SSSP by Ramalingam and Reps

- D-KIN: dynamic All-Pairs Shortest Path (APSP) by V. King (1999)
  (This was the first fully dynamic algorithm for SP)

- D-LHP: dynamic APSP-LHP by Demetrescu and Italiano (2004)
  (Locally historical shortest path, $O(n^2 \log^3 n)$ time per update amortized)
- S-LSP: static APSP-LSP by Demetrescu und Italiano (2004)

- S-DIJ: static SSSP by Dijkstra (as base line)

The same data structures for all C-implementations (Heaps, dynamic arrays, Hash maps, graph data structure)

Evaluated Algorithms

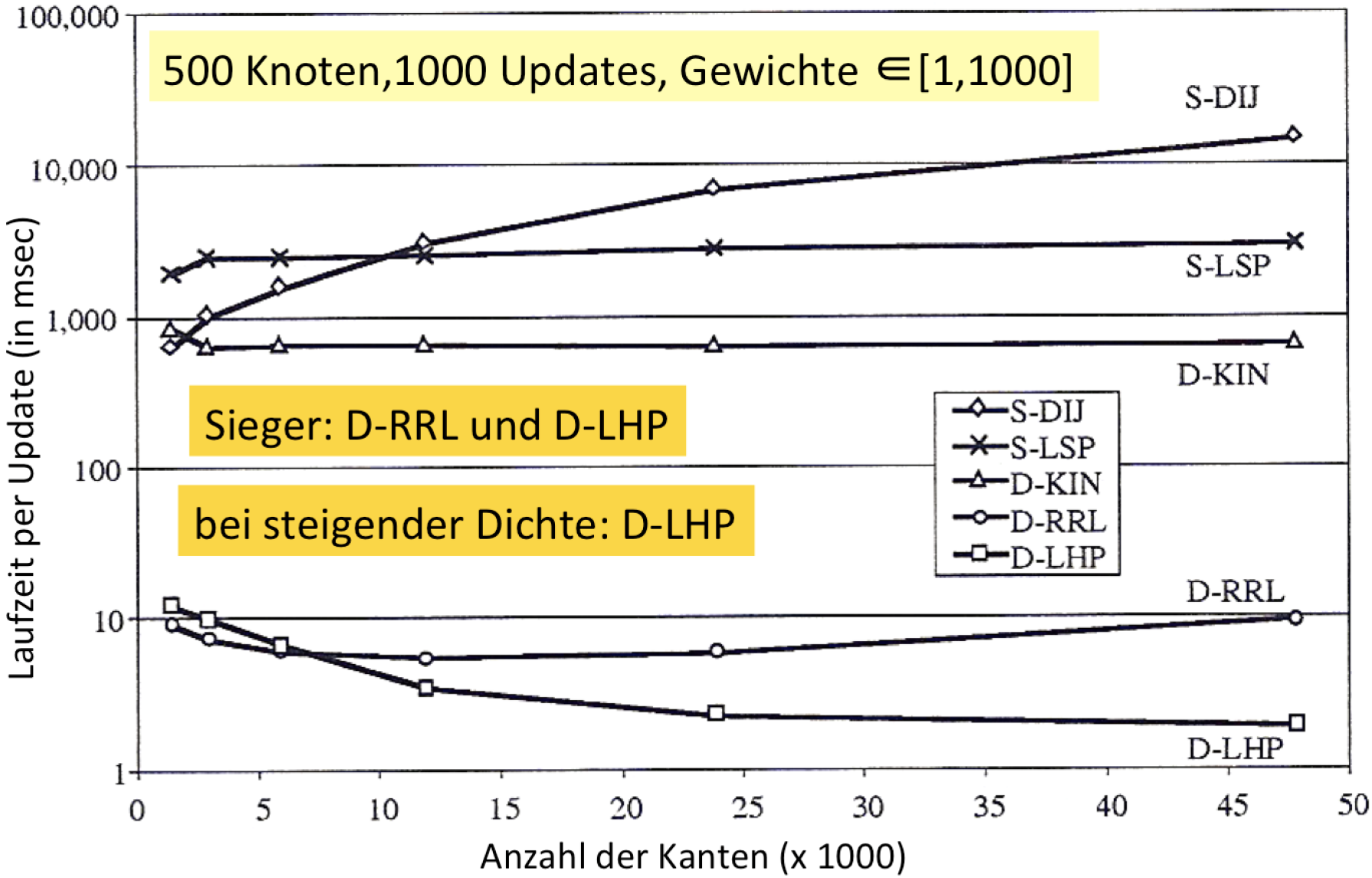
Algorithm	Update time	Query time	Space requirement
S-DIJ	$O(mn + n^2 \log n)$	$O(1)$	$O(n^2)$
S-LSP	$O(\|LSP\| + n^2 \log n)$	$O(1)$	$O(n^2)$
D-RRL	$O(mn + n^2 \log n)$	$O(1)$	$O(n^2)$
D-KIN	$O(n^{2.5}\sqrt{C \log n})$	$O(1)$	$O(n^{2.5}\sqrt{C \log n})$
D-LHP	$O(n^2 \log^3 n)$	$O(1)$	$O(mn \log n)$

C denotes the maximal weight of an edge.

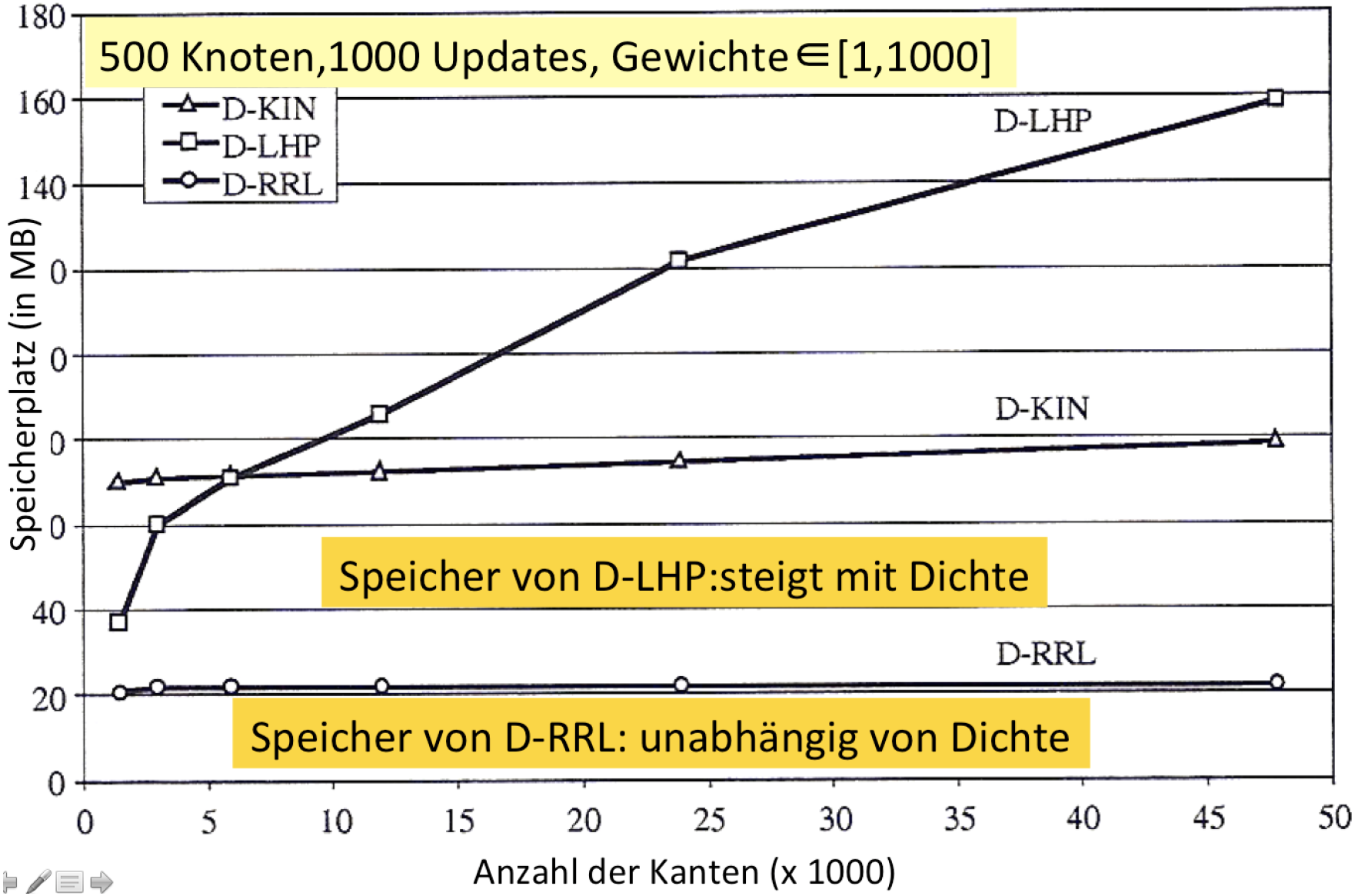
$|LSP|$ is the number of locally shortest paths (relaxing shortest paths by extending it to one more edge) in the graph which could be $O(mn)$ in the worst case.

The authors have also suggested an incremental algorithm (IncreaseOnly) which needs update time $O(n^2 \log n)$ only.

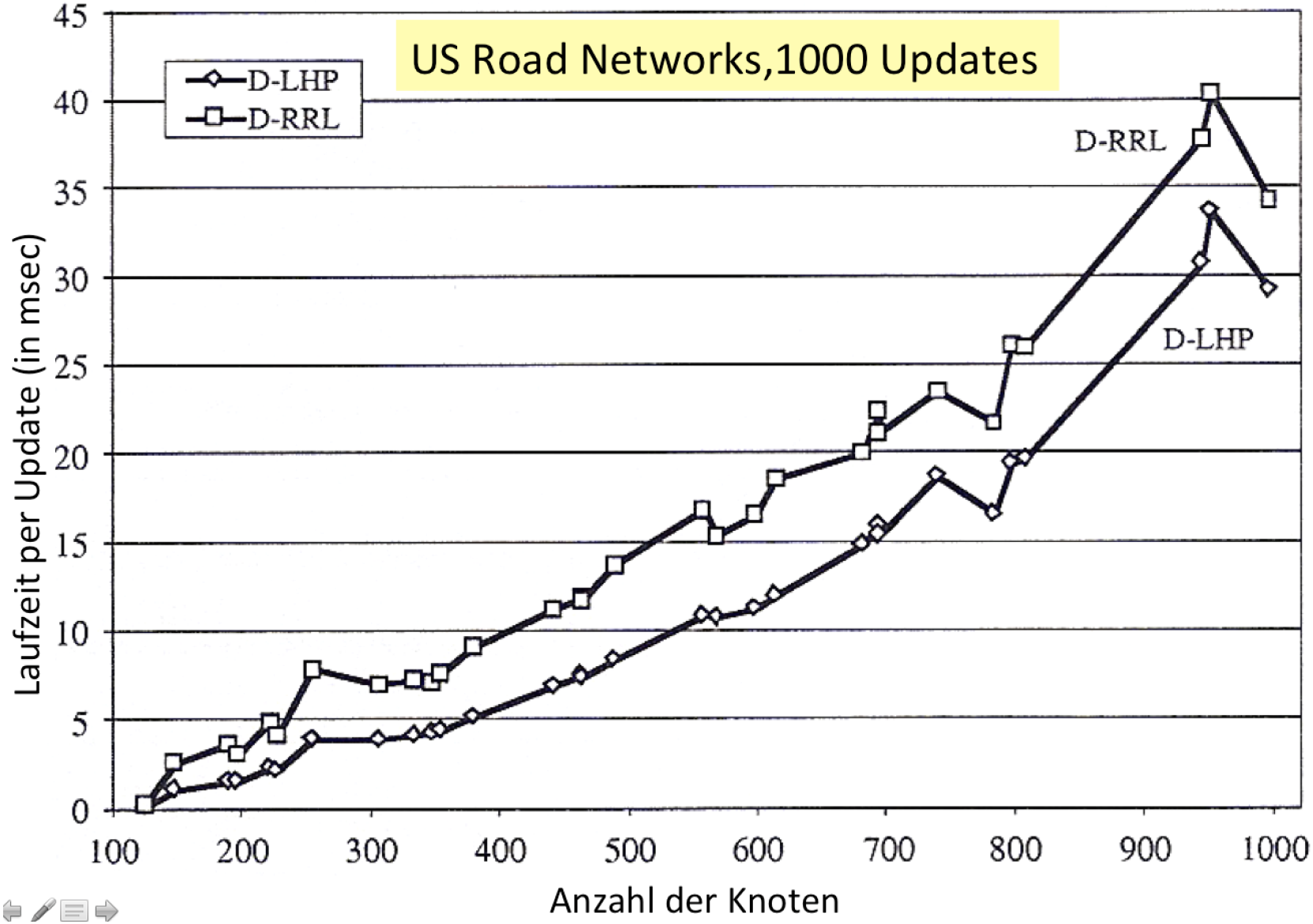
Random Graphs: Running Time per Update



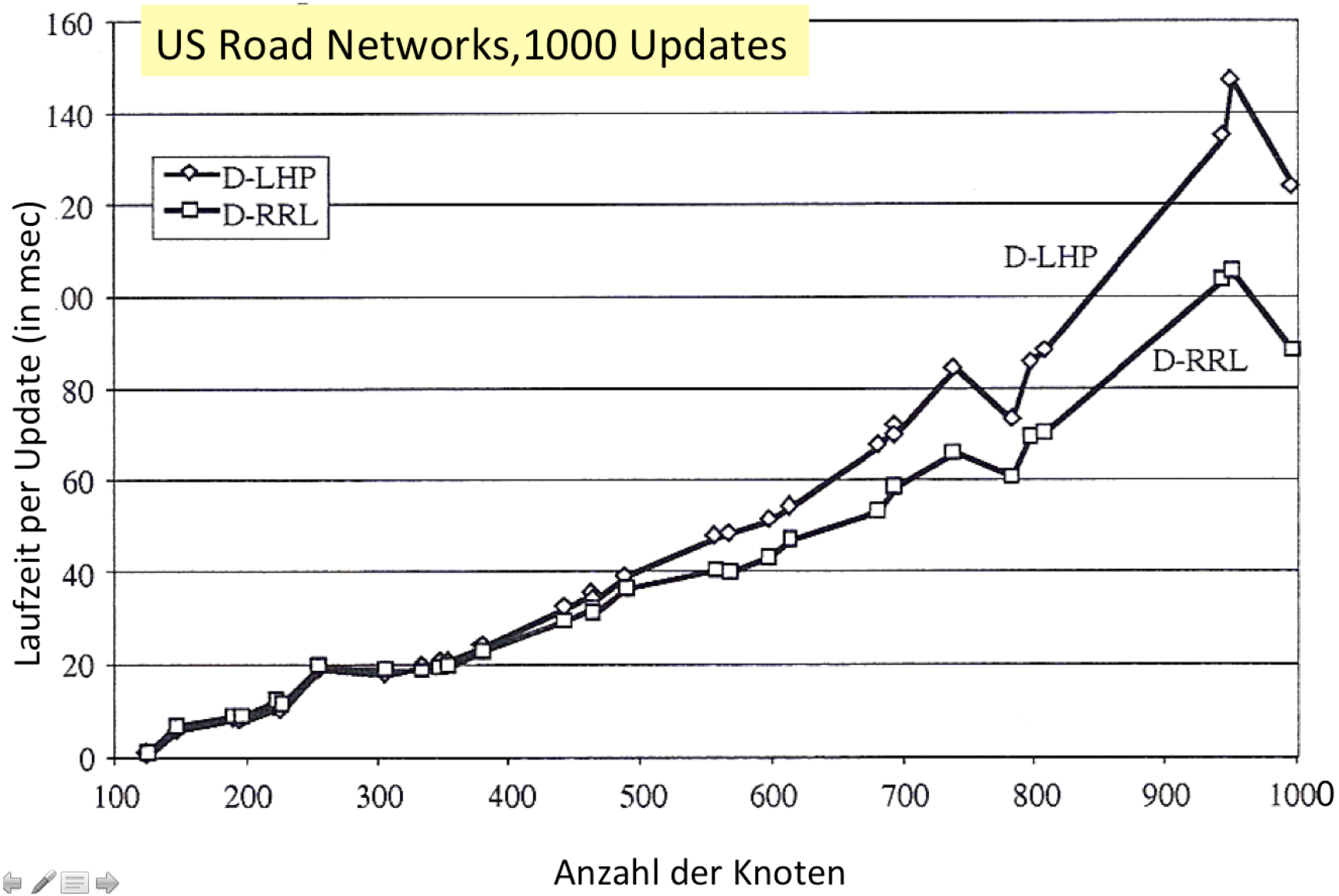
Random Graphs: Space (in MB)



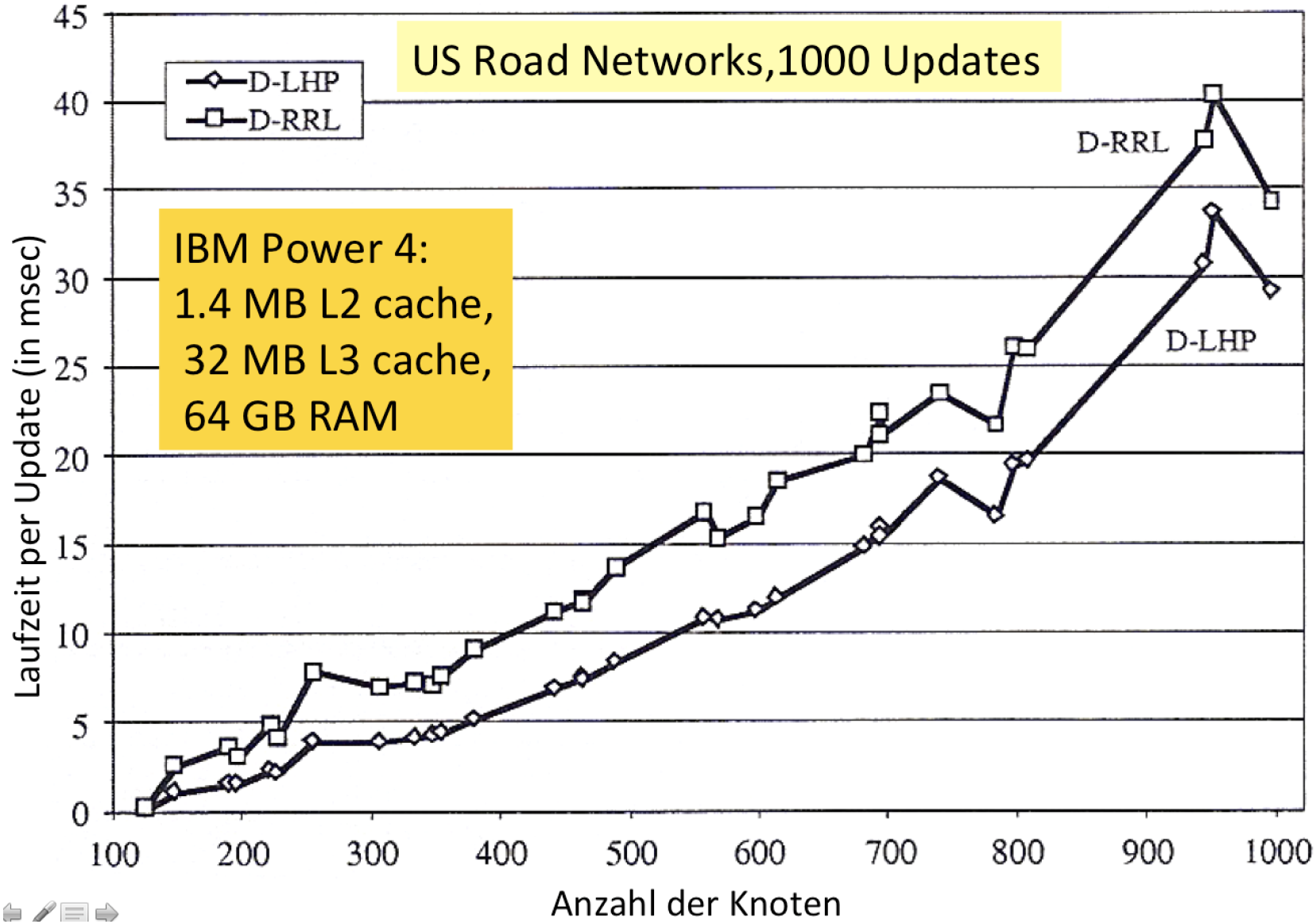
Real World Graphs: Running Time (IBM)



Real World Graphs: Running Time (Sun)



Real World Graphs: Running Time (IBM)



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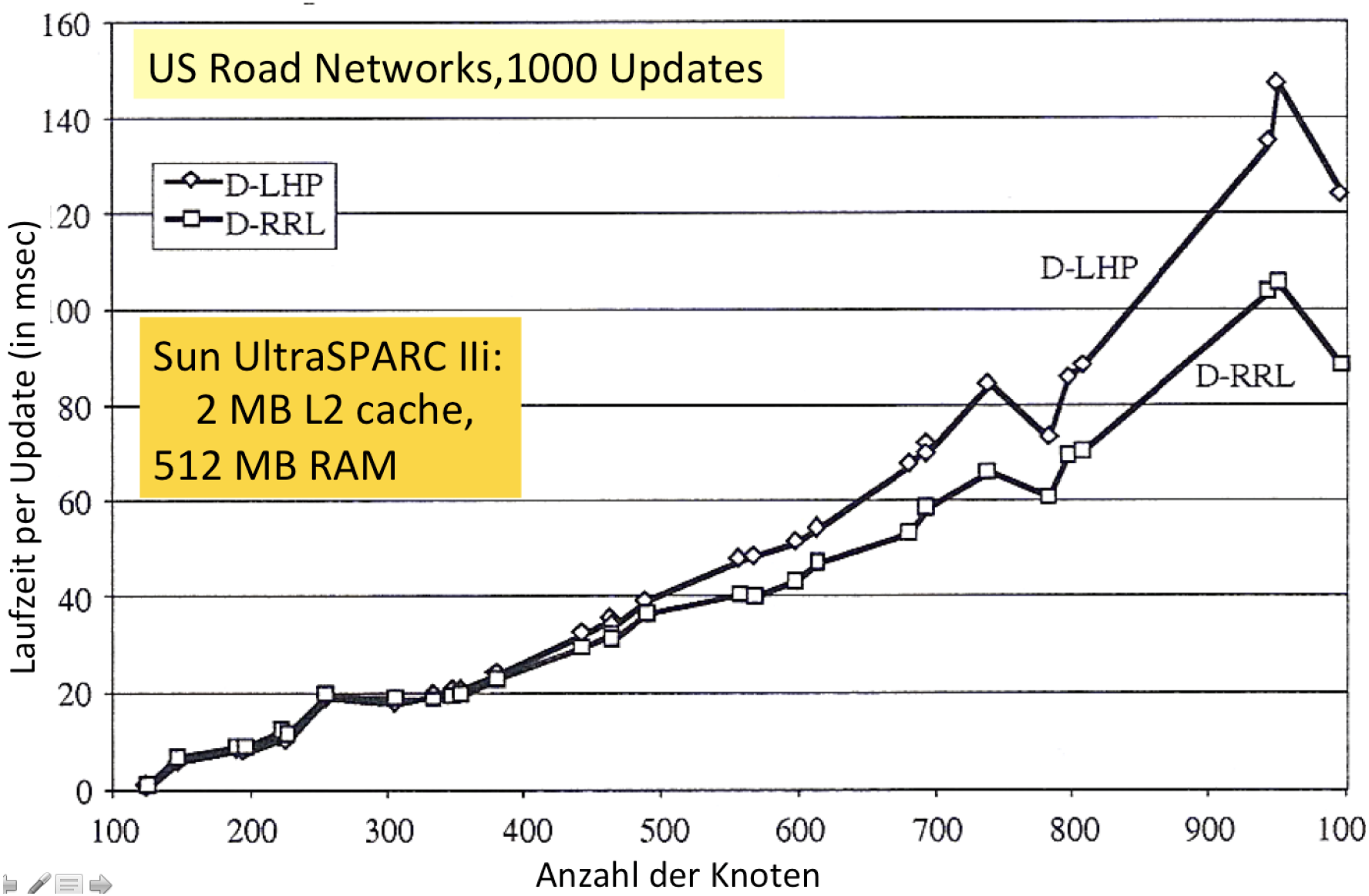
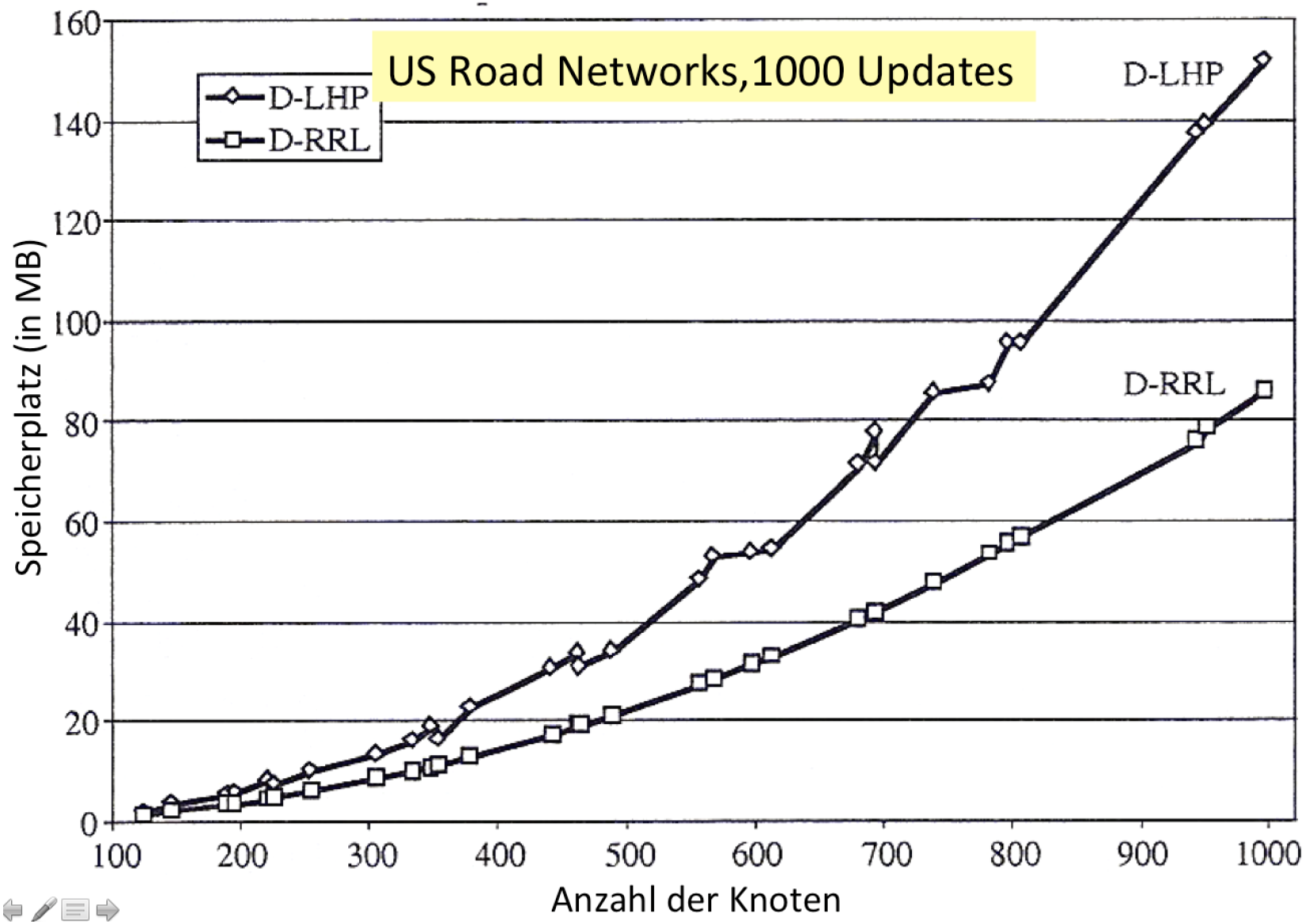


Real World Graphs: Space (in MB)



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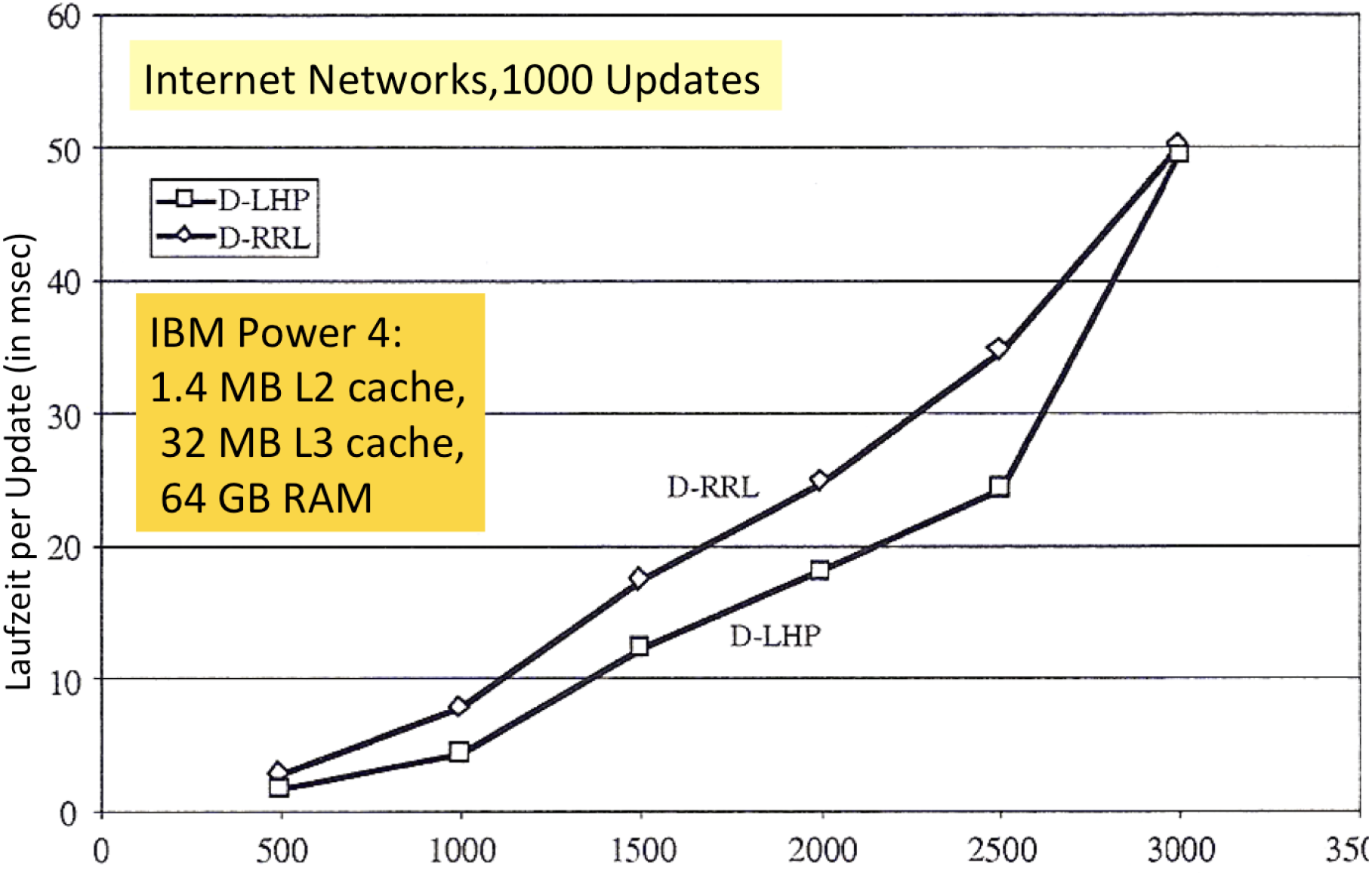
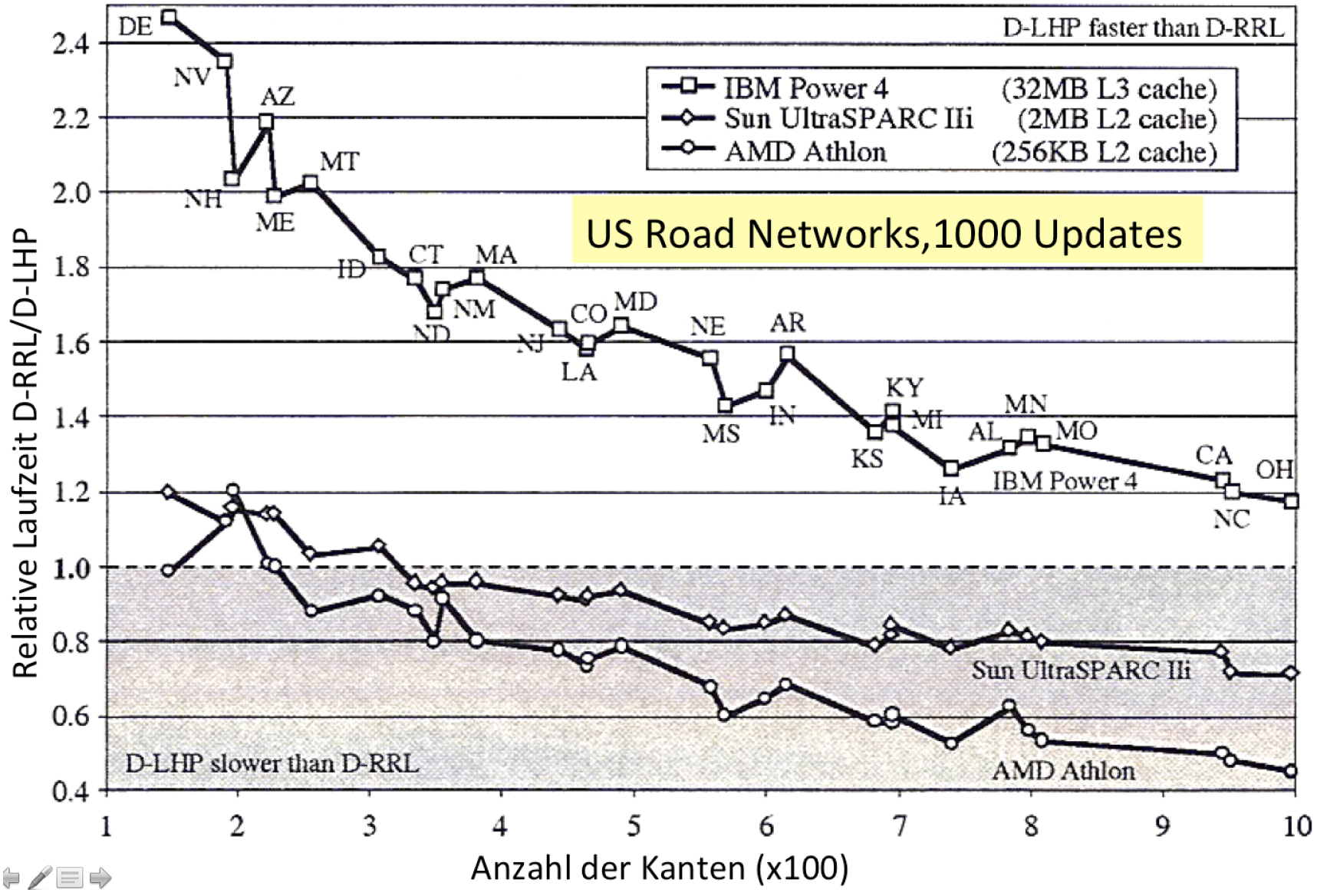
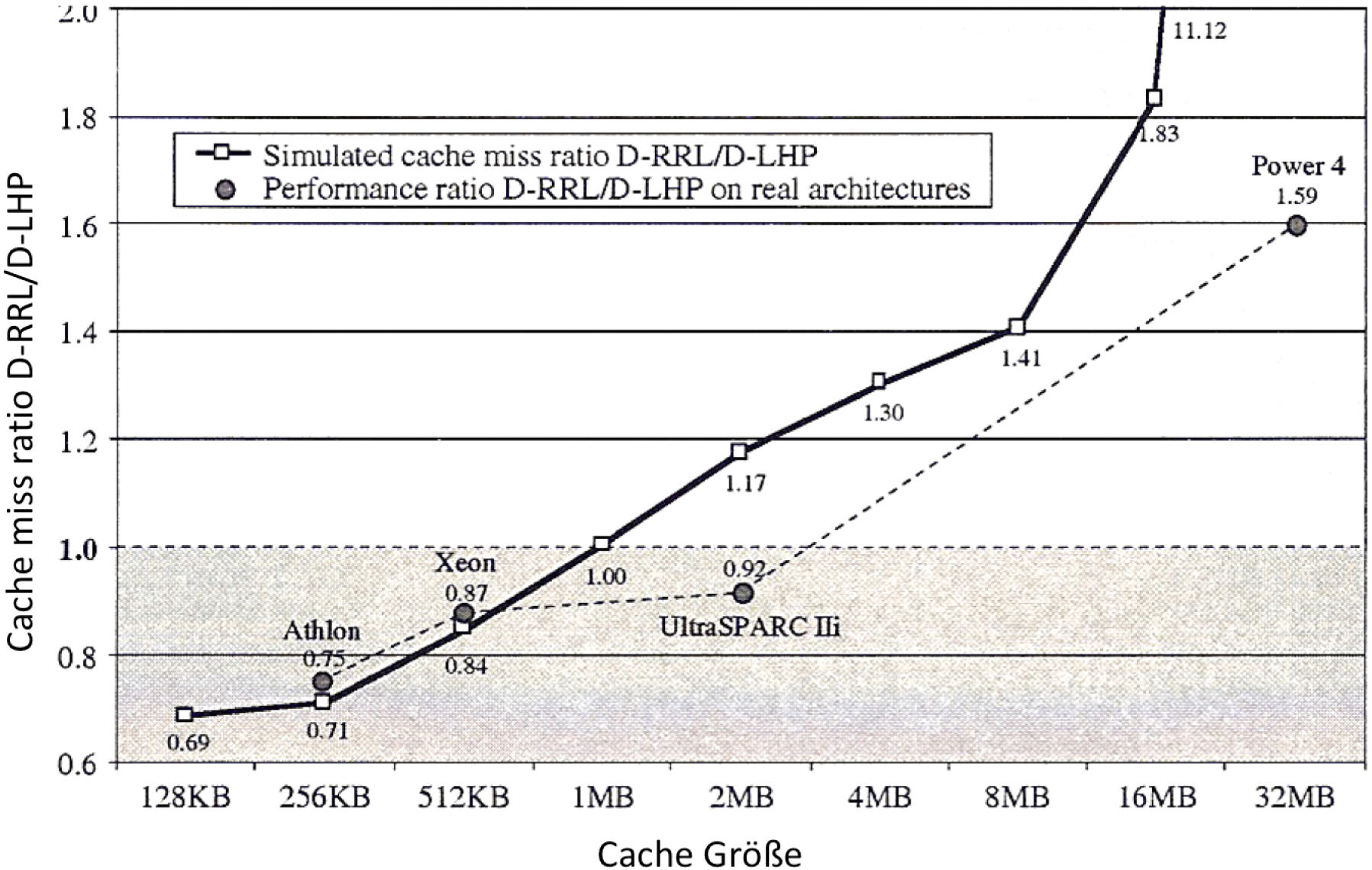


Relative Running Time on various Platforms



Cache Miss Ratio D-RRL / D-LHP



Relative Running Time on Bottleneck Graphs