Between FO and MSO

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3rd Workshop on Logic, Graphs, and Algorithms,
LoGAlg 2025

Vienna, Austria November 19th, 2025

European Research Council
Established by the European Commission

Supported by ERC project BOBR,
ref. no. 948057.

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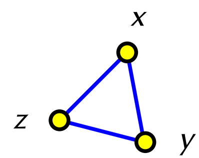

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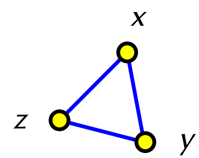

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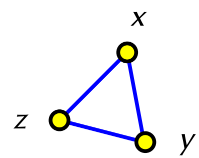

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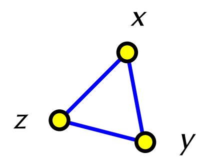

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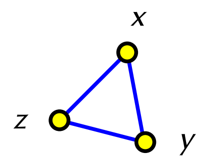

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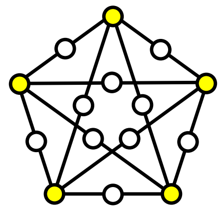

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MSO_2 boils down to MSO_1 on the **1-subdivision** of G.

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Note: This is a **parameterized** problem, with φ being the parameter.

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- FO: brute-force runtime $|G|^{\mathcal{O}(\|\varphi\|)}$, but $f(\varphi) \cdot |G|^c$ unlikely. (AW[$\star$]-hard)
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Q: On what graph classes is model-checking FO / MSO_1 / MSO_2 FPT?

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Idea: **Nowhere denseness** defined by exclusion of **local** obstructions $\rightsquigarrow$

Exploit **local decompositions** and **locality** of FO.

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Theorem

If $\mathscr{C}$ is **subgraph-closed** and **not nowhere dense**,

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Formally: There is a 1-dimensional $\mathcal{L}$-interpretation
from (colorings of $\mathscr{C}$) onto $\mathscr{D}$.

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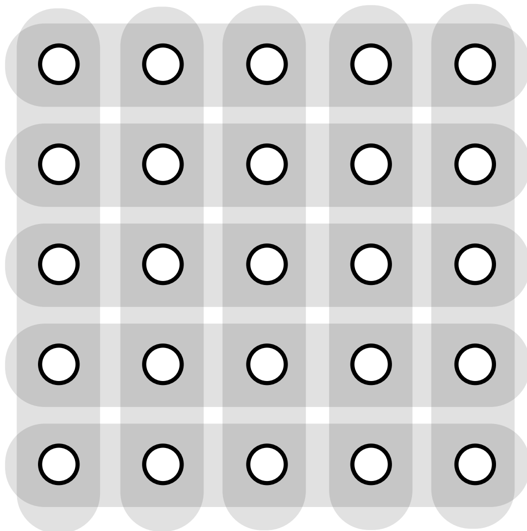

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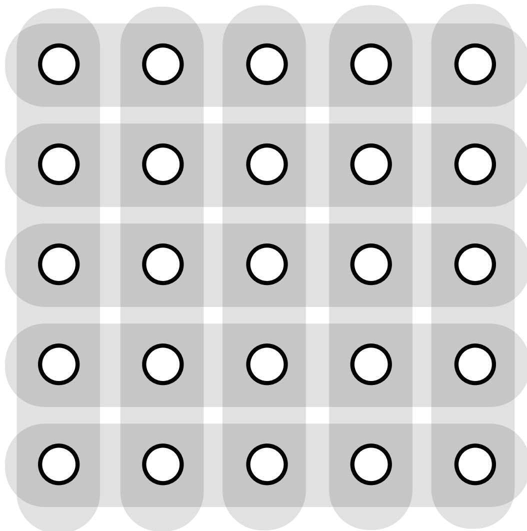

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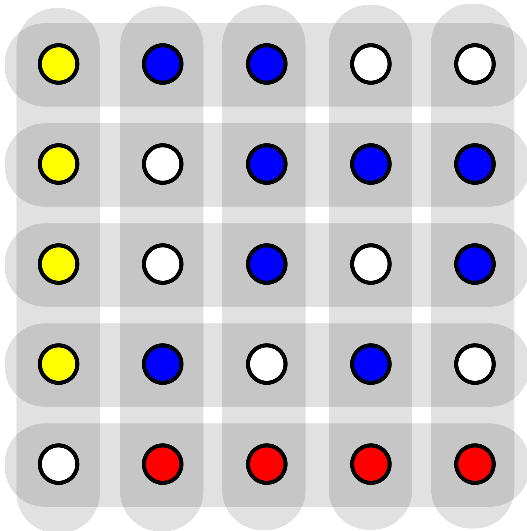

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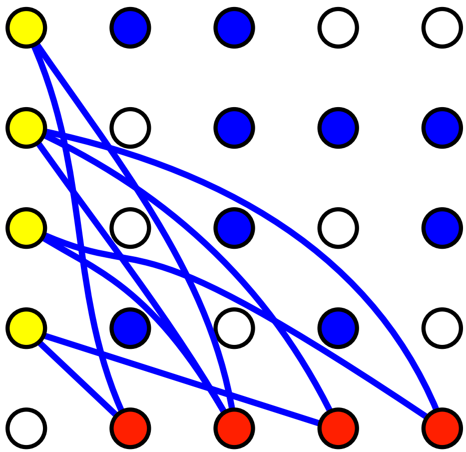

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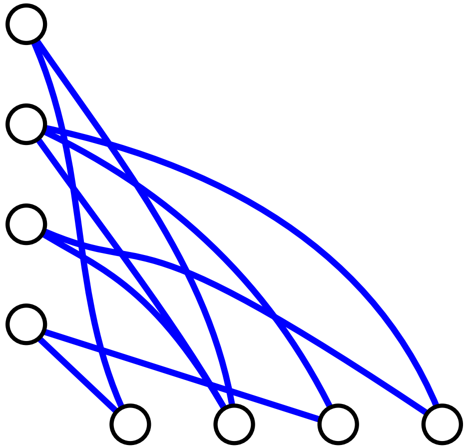

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Obs: $\mathcal{L}$-transductions closed under composition (for reasonable $\mathcal{L}$) $\Rightarrow$

$\mathcal{L}$-transducibility is a **quasi-order** on graph classes.

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$\Leftarrow$: MSO transductions preserve bounded **cliquewidth**. $\square$

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Suppose $\mathscr{C}$ is **subgraph-closed**. Then

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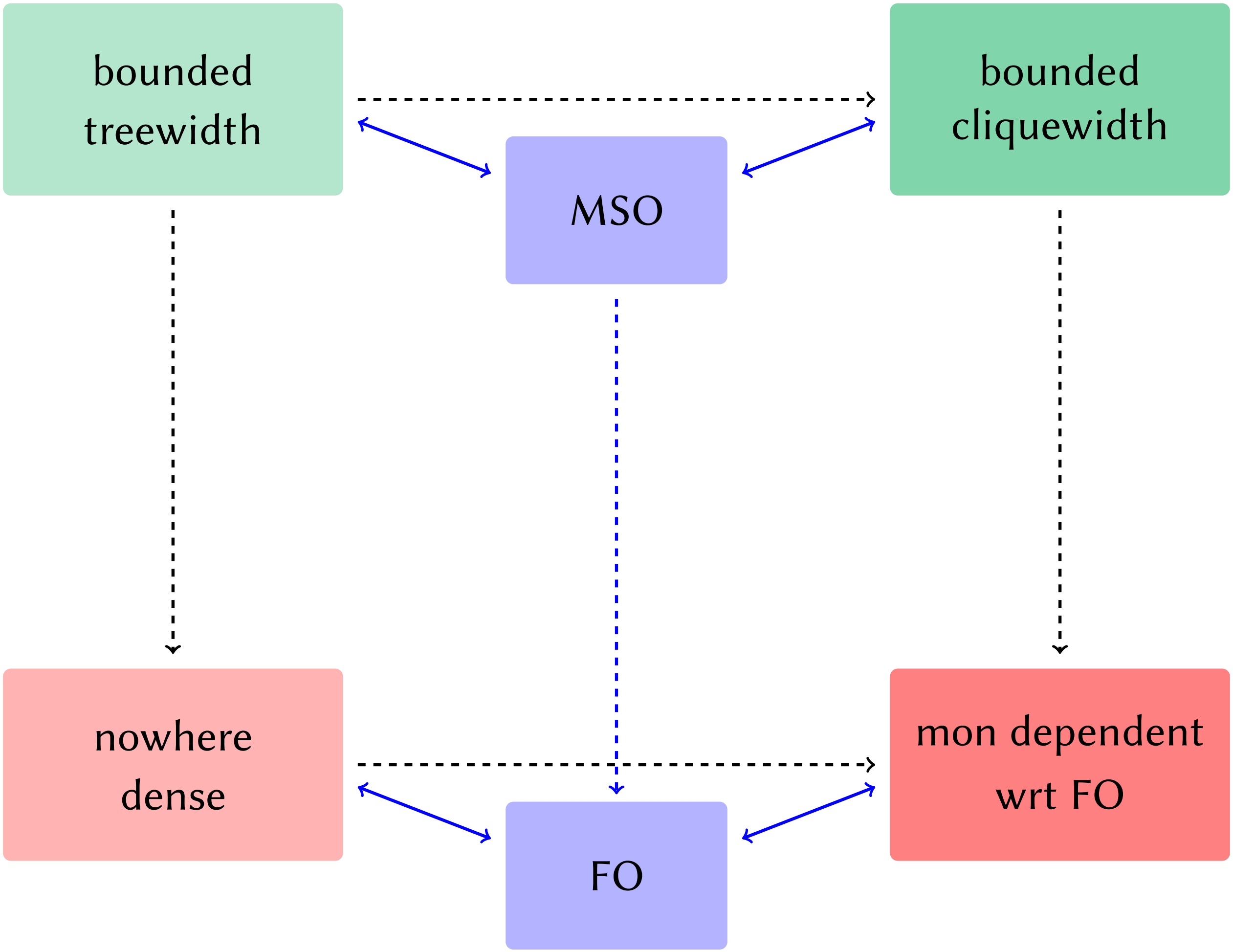
Theorem [Dreier, Mählmann, Toruńczyk '24]

If $\mathscr{C}$ is **hereditary** and **not mon dependent**, then model-checking FO on $\mathscr{C}$ is AW[$\star$]-hard.

Big picture



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Now: A thread of results exemplifying this.

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Idea: There is no FO formula $\varphi(x, y)$ such that

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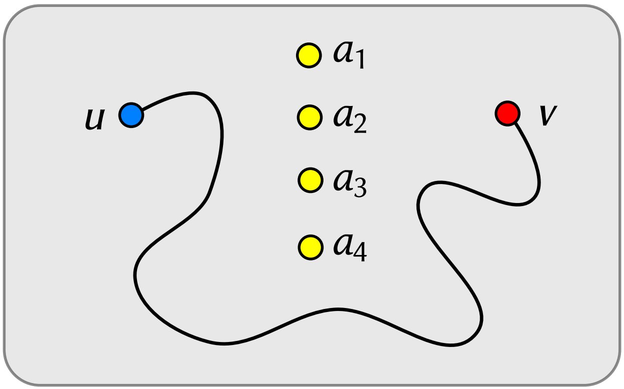

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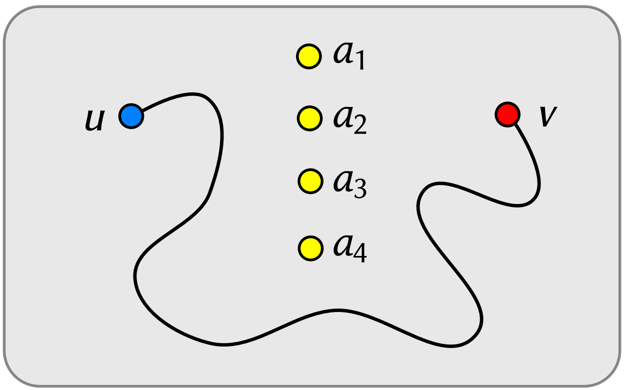

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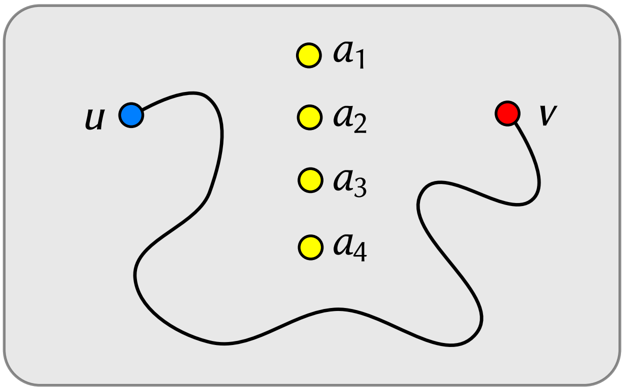

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Idea: Speak about small-size separators.

Logic FO + conn: **motivation**

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Subgraph-closed classes **tamed** wrt $\mathsf{FO} + \mathsf{conn}$ =

Classes excluding a **topological minor**

Tameness of FO + conn

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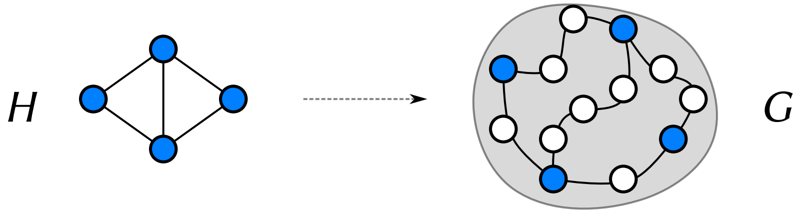

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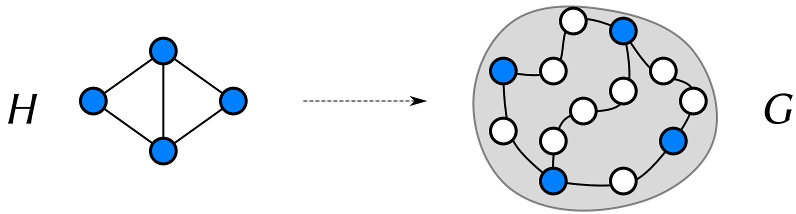

Theorem [P, Schirrmacher,Siebertz, Toruńczyk, Vigny '22]

Suppose $\mathscr{C}$ can excludes some fixed graph H as a topological minor.

Then model-checking FO + conn on $\mathscr{C}$ in time $\mathcal{O}_\varphi(|G|)$.

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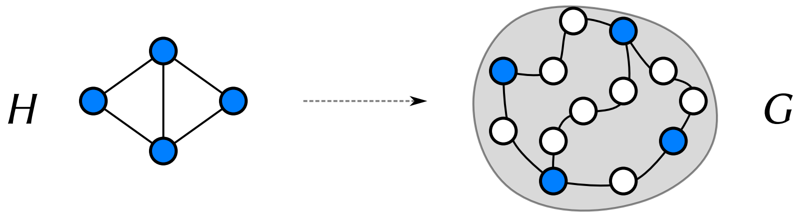

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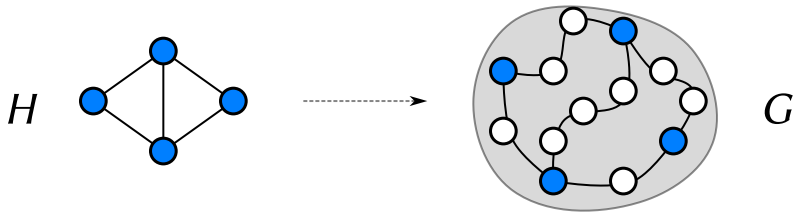

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Note: $\Rightarrow$ is easy, because FO + conn can shorten long paths.

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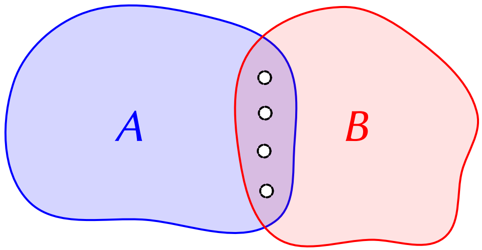
Order of (A, B) is $|A \cap B|$.



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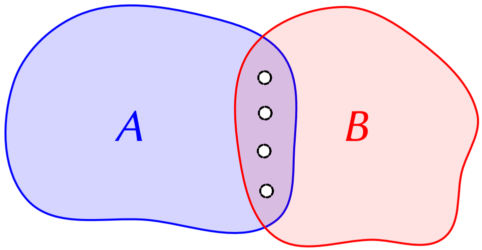

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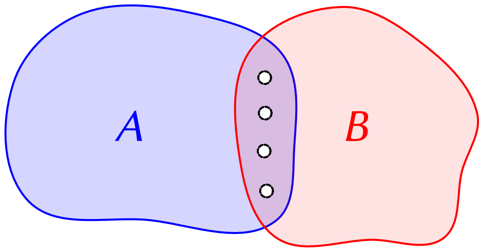

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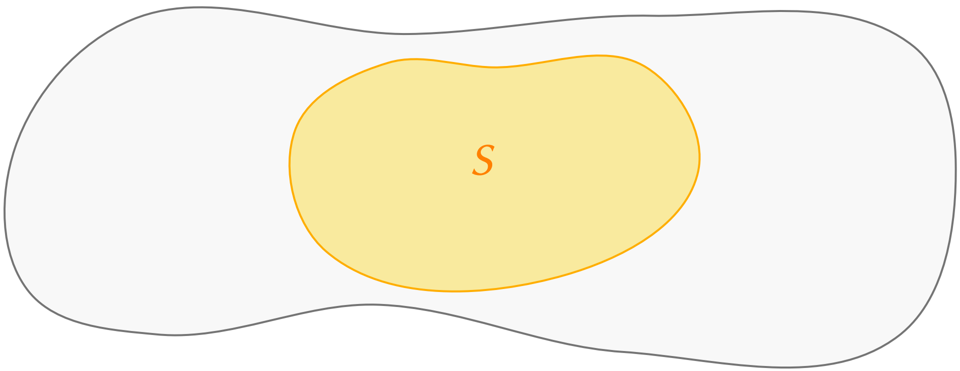
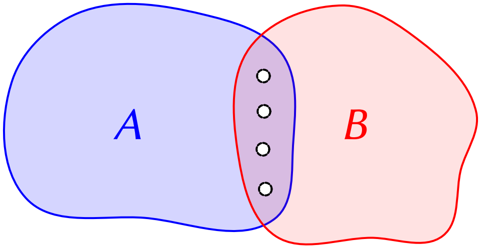


Unbreakability

Separation of G is (A, B) with $A \cup B = V(G)$ and $E(A - B, B - A) = \emptyset$.

Order of (A, B) is $|A \cap B|$.



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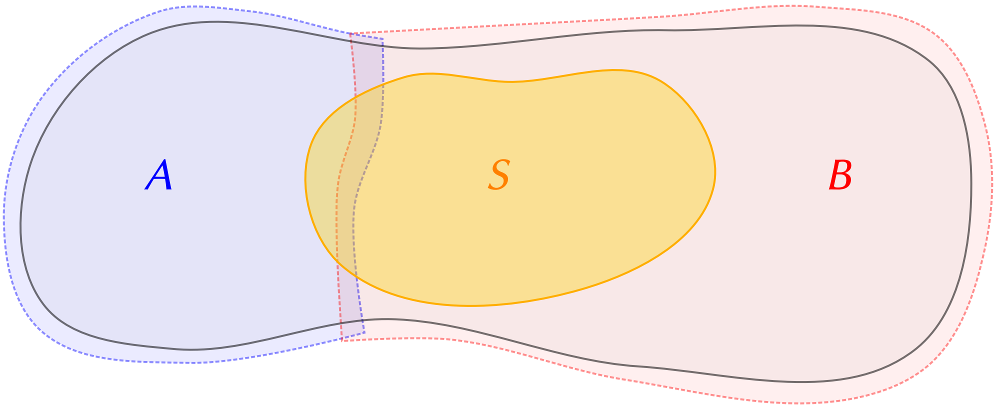

Unbreakability and FO + conn

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Obs: If G is (q,k)-**unbreakable** for some $q, k \in \mathbb{N}$, (i.e. $S = V(G)$)

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If u, v disconnected in $G - \{a_1, \ldots, a_k\}$,

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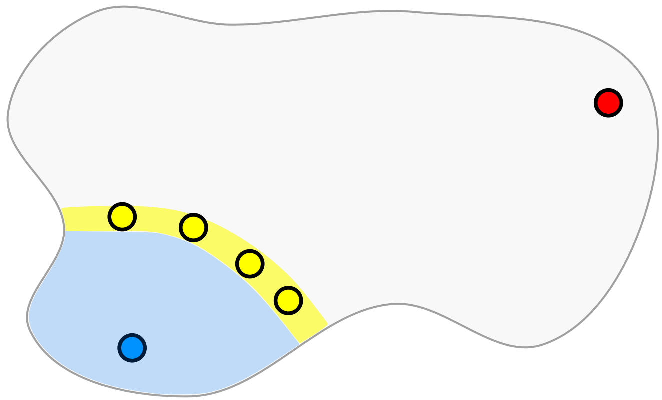

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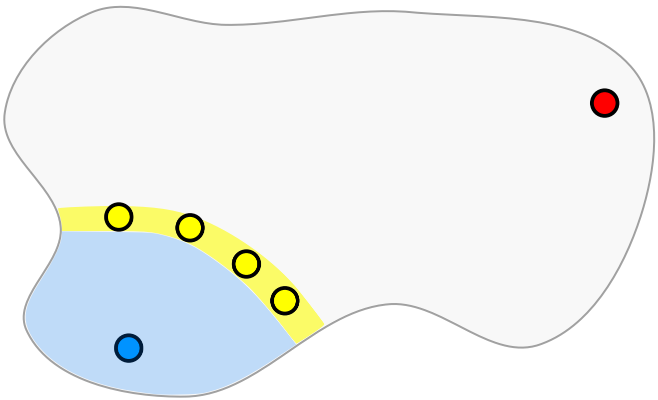


Hence, $\neg\mathrm{conn}_k(u, v, a_1, \ldots, a_k)$ is equivalent to:

There is X with $|X| \leqslant q$, $|X \cap \{u, v\}| = 1$, and $N(X) \subseteq \{a_1, \ldots, a_k\}$.

Unbreakable decomposition

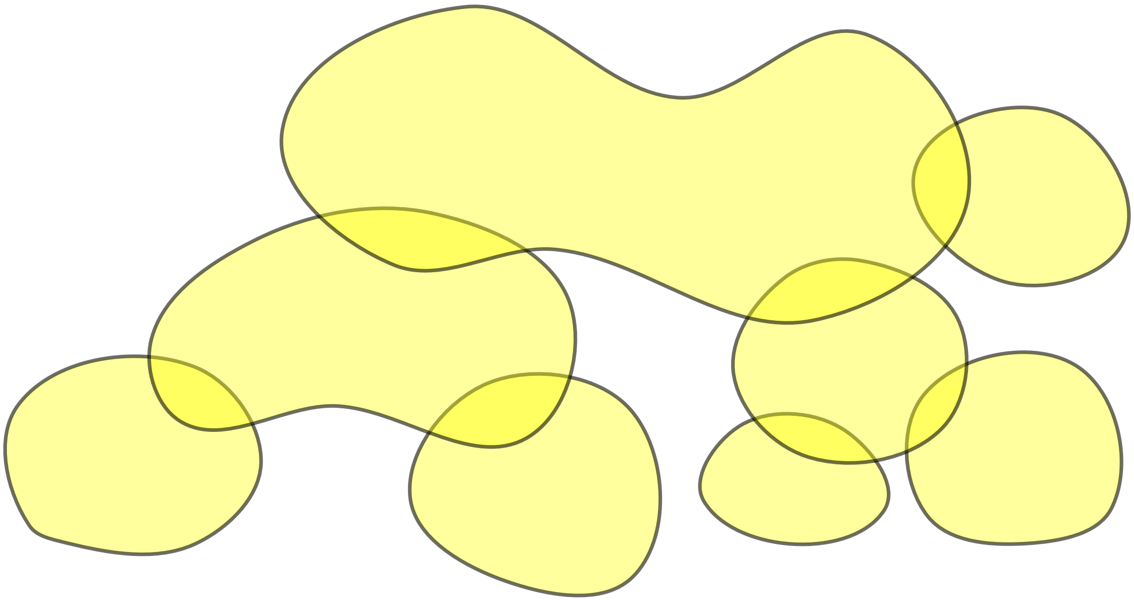
Theorem

[Cygan, Lokshtanov, Pilipczuk, P, Saurabh '14]
[Cygan, Komosa, Lokshtanov, Pilipczuk, P, Saurabh, Wahlström '18]
[Korhonen '24]

For every $k \in \mathbb{N}$, every graph G has a tree decomposition with

adhesion $\leqslant k$ and (k, k)-**unbreakable bags**.

Moreover, such a tree decomposition can be found in time $\mathcal{O}_k(\|G\|)$.



Model-checking $\mathrm{FO} + \mathrm{conn}$

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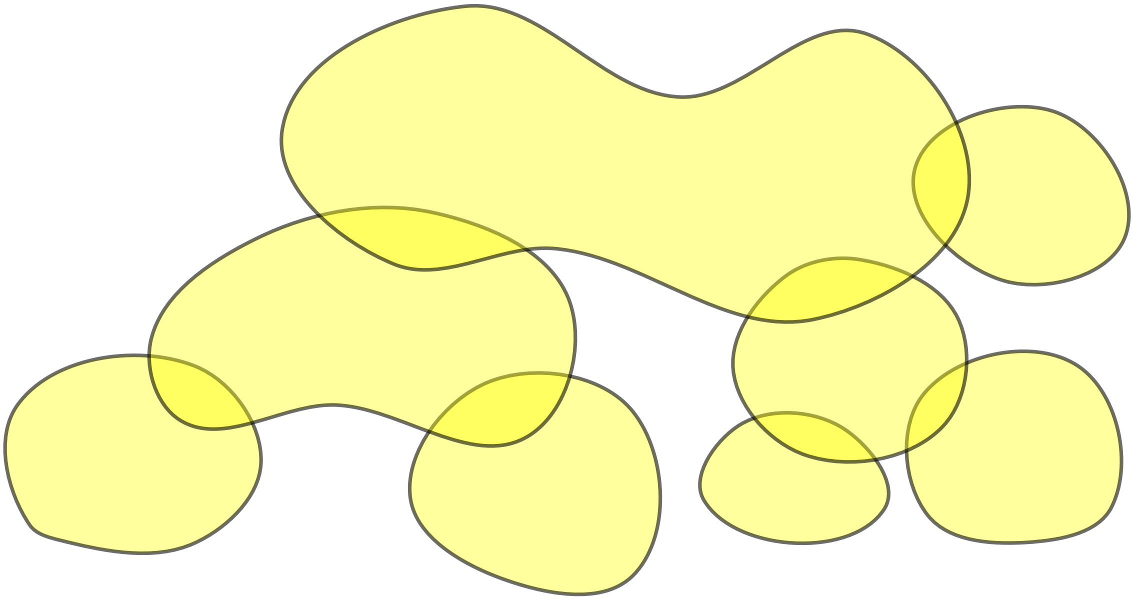

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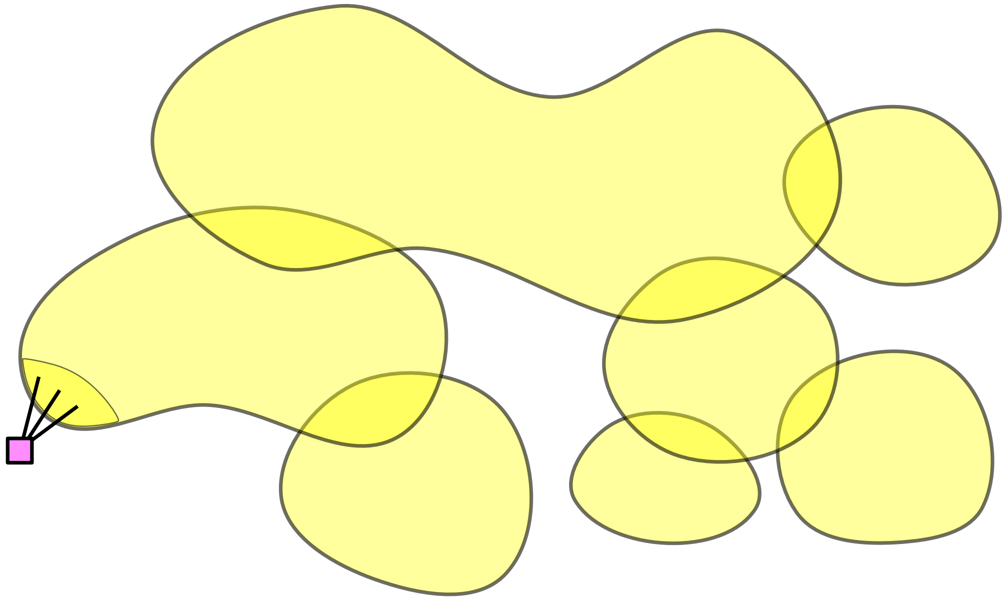

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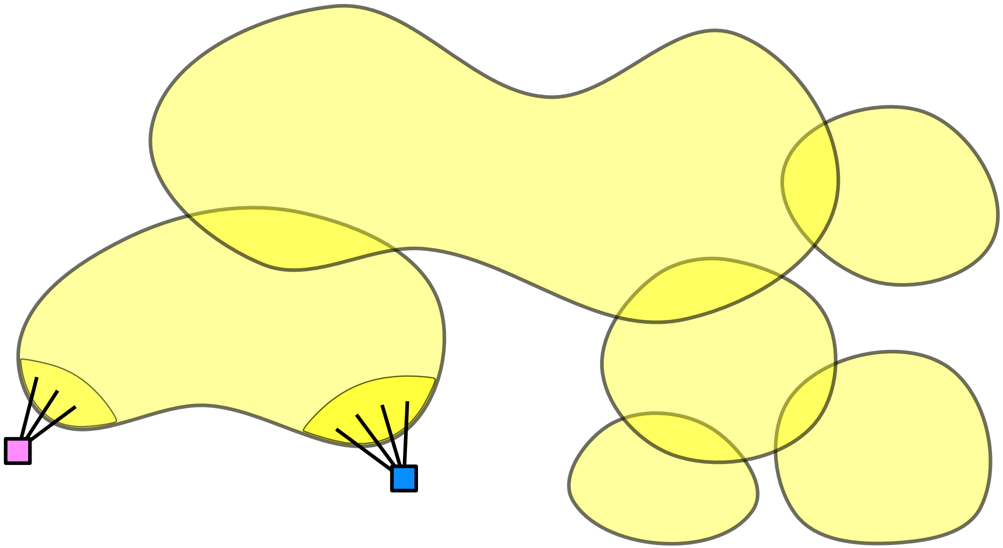

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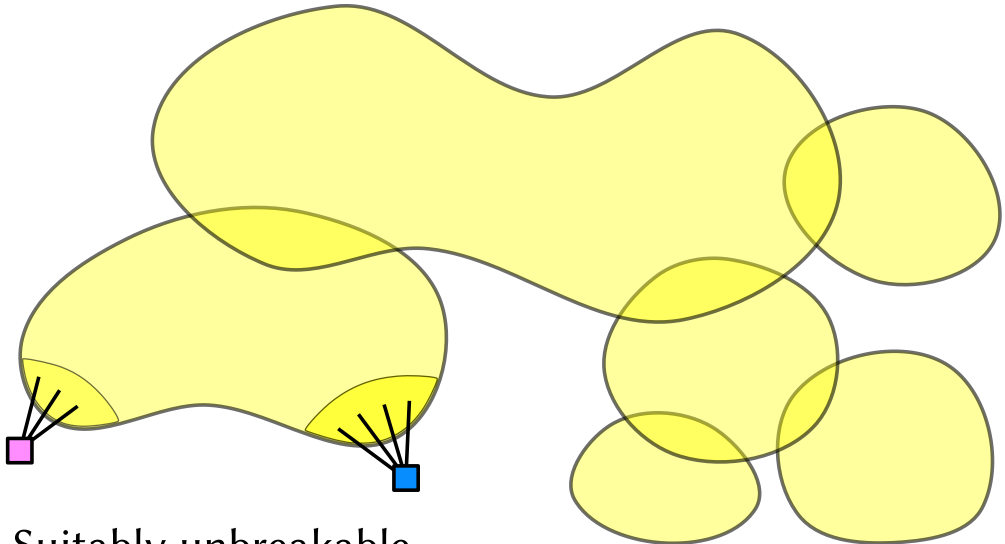

Suitably unbreakable
K_h-top-minor-free for $h = h(t, k)$.

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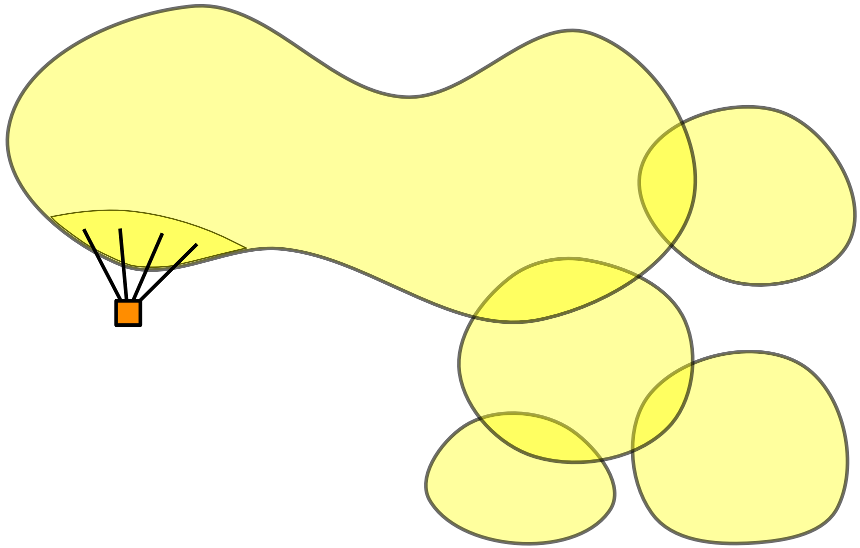

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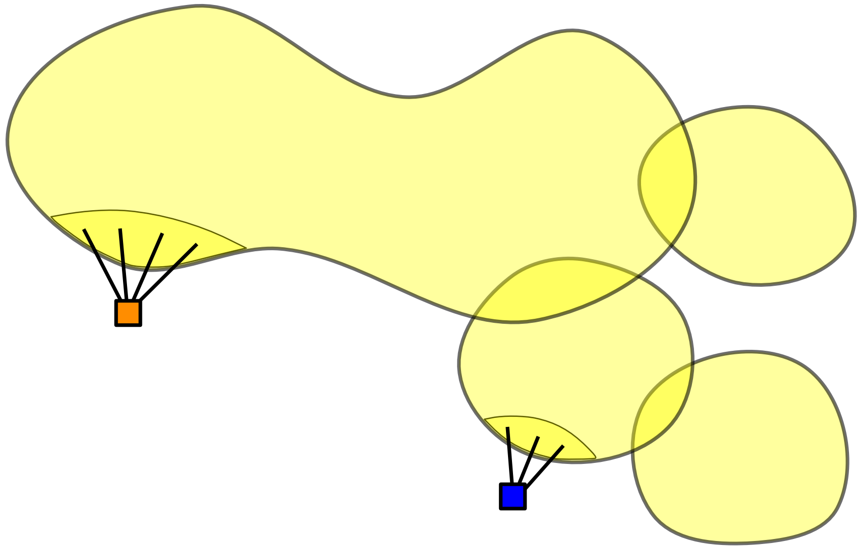

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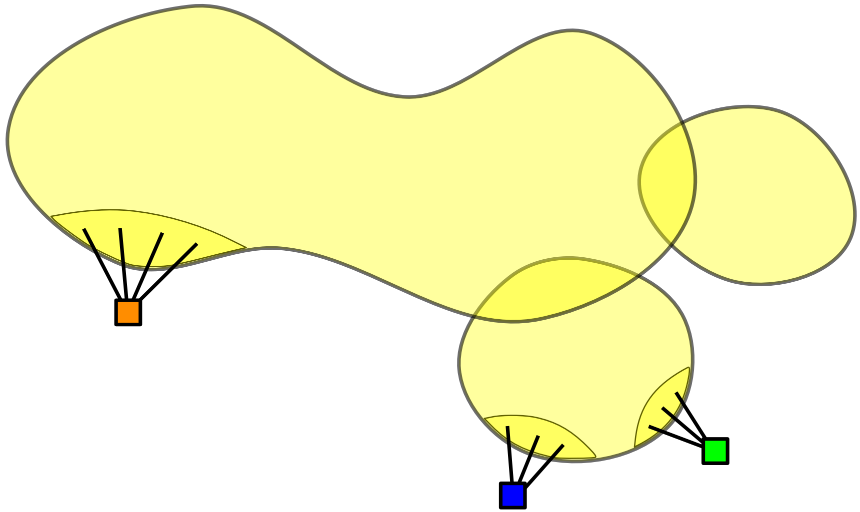

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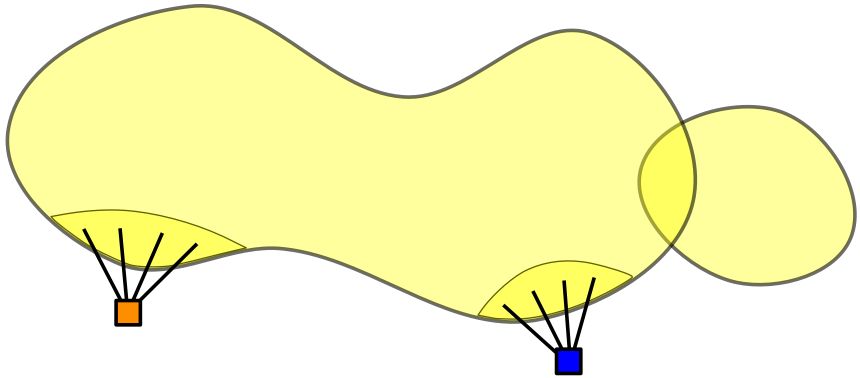

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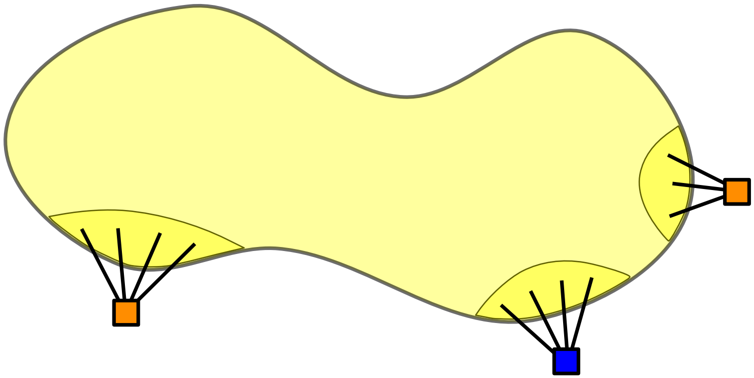

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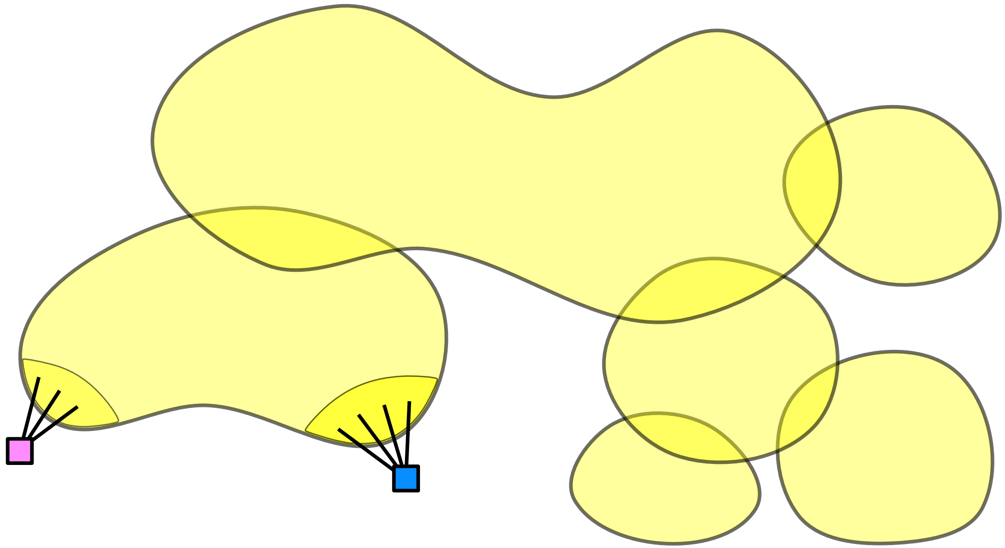

Top-minor-free: Stable under stellations and efficient FO model-checking.

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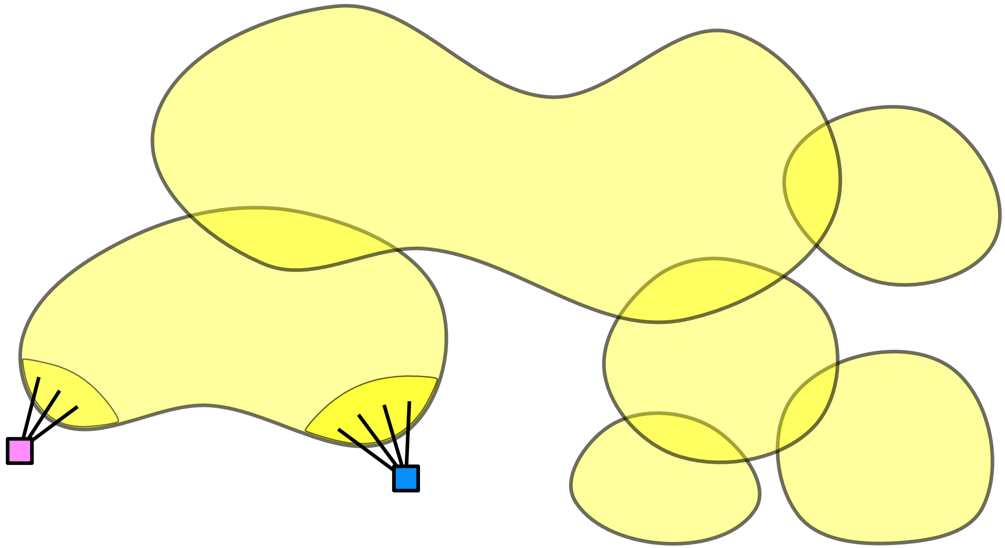

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General scheme for **local-to-global** lifts in **linear time**.

Beyond FO + conn

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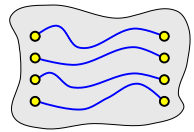

FO + dp: FO with **disjoint paths** predicates.

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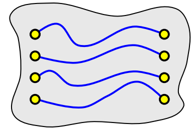

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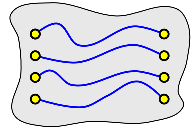

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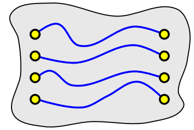

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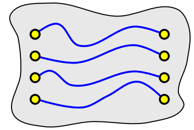

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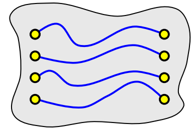

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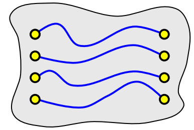

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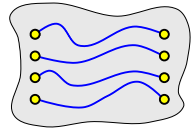

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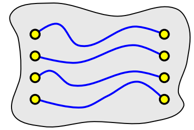

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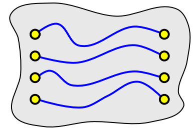

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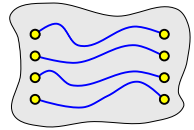

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Quantification over sets of small **cutrank**.

Cutrank

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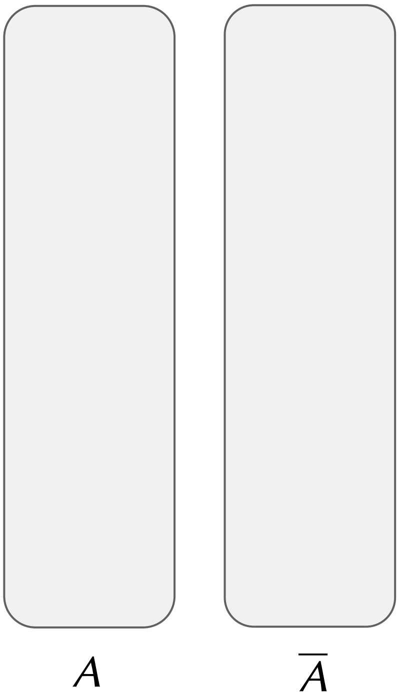

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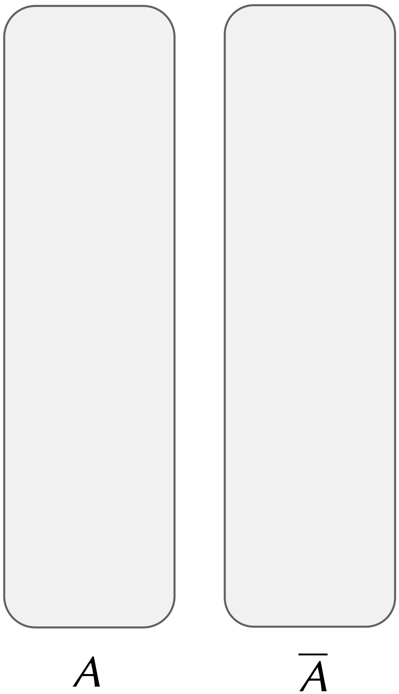

$\mathrm{cutrank}(A) := $ rank over $\mathbb{F}_2$ of $\mathrm{Adj}[A, \overline{A}]$.

$$A \quad \underset{\overline{A}}{\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}$$

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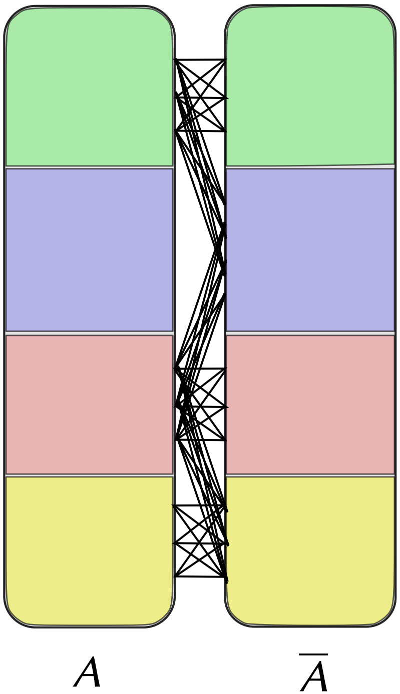

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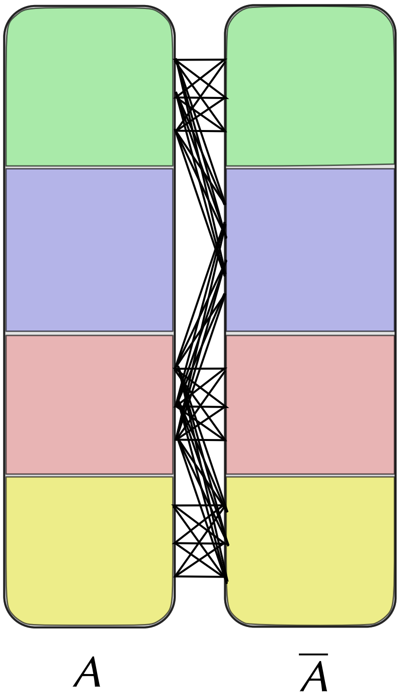

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$$\mathrm{cutrank}(A) \leqslant \mathrm{diversity}(A) \leqslant 2 \cdot 2^{\mathrm{cutrank}(A)}$$

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Treewidth: hierarchical decomposition by **separations** of order $\leqslant k$.

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Obs: Bounding **diversity** yields the same logic. $\exists_{X:\ \mathrm{diversity}(X) \leqslant r}\ \psi$

Low rank MSO

Treewidth: hierarchical decomposition by **separations** of order $\leqslant k$.

Rankwidth: hierarchical decomposition by cuts of **cutrank** $\leqslant k$.

Cliquewidth: hierarchical decomposition by cuts of **diversity** $\leqslant k$.

Note: Cutrank and diversity are **symmetric**, cutrank is **submodular**.

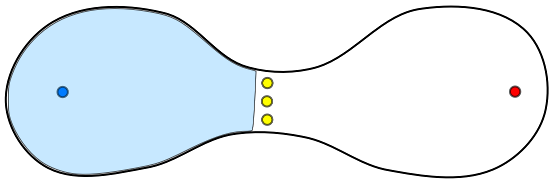
Low rank MSO

MSO_1 where every set quantification is guarded by a bound on **cutrank**.

$$\varphi = \exists_{X\colon \mathrm{cutrank}(X)\leqslant r}\ \psi$$

Obs: Bounding **diversity** yields the same logic. $\exists_{X\colon \mathrm{diversity}(X)\leqslant r}\ \psi$

Obs: FO + conn $\subseteq$ low rank MSO.



$\mathrm{cutrank}(\mathrm{Reach}(u, G - \{a_1, \ldots, a_k\})) \leqslant k$

Low rank MSO: complexity

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joint work with

M. Bojańczyk, W. Przybyszewski, M. Sokołowski, and G. Stamoulis

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Suppose $\mathscr{C}$ is **weakly sparse** (excludes $K_{t,t}$ as a subgraph for some $t \in \mathbb{N}$).

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Corollary

Model-checking low rank MSO is in XP.

Low rank MSO on weakly sparse classes

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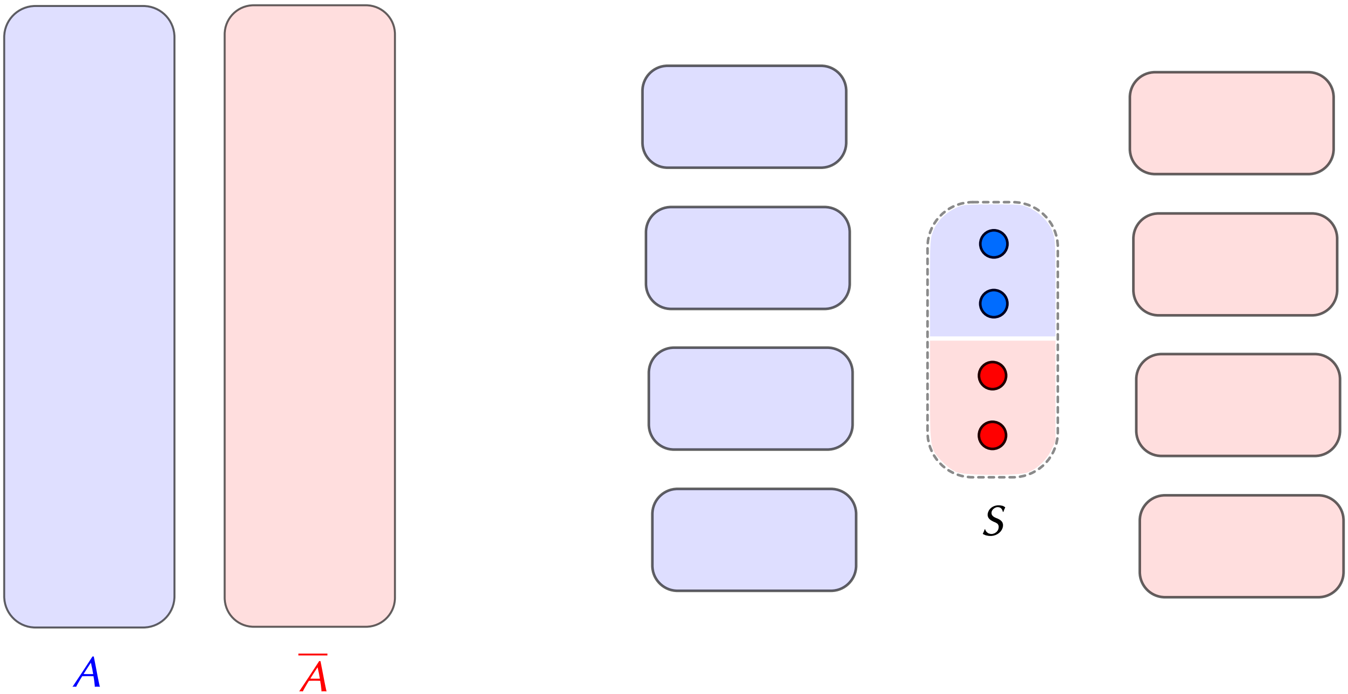
Lemma

Let G be $K_{t,t}$-free and diversity$(A) \leqslant k$. Then there is S with $|S| \leqslant kt$ s.t. every component of $G - S$ is entirely in A or entirely in $\overline{A}$.



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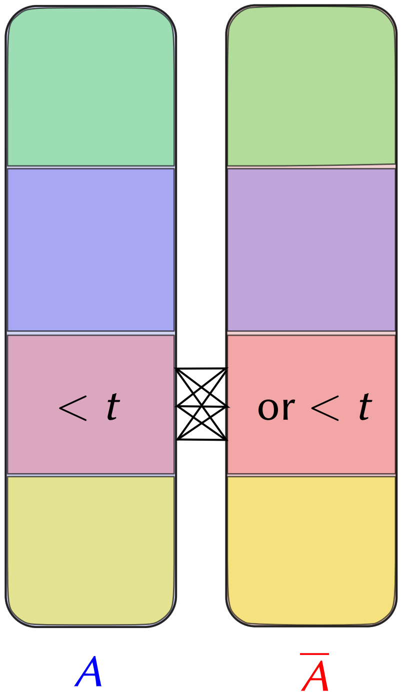

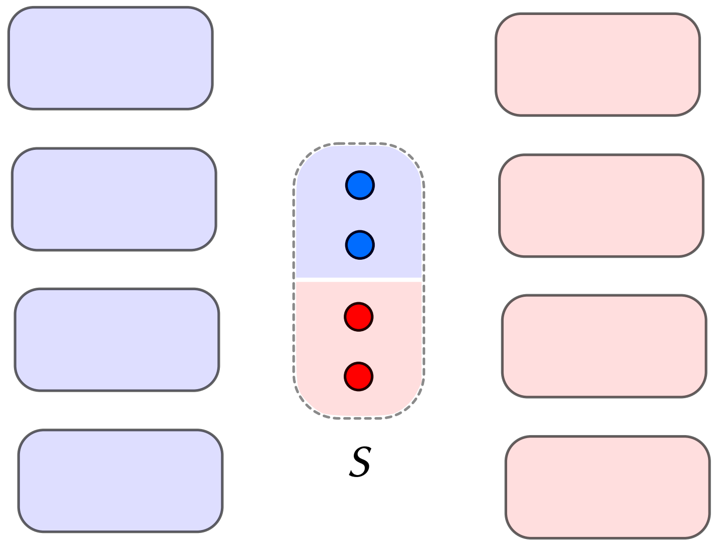

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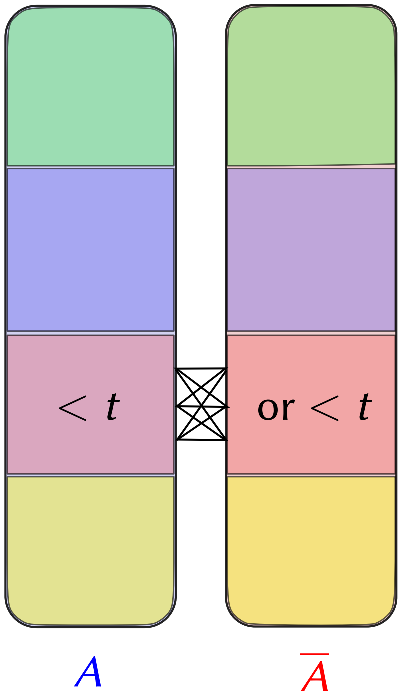

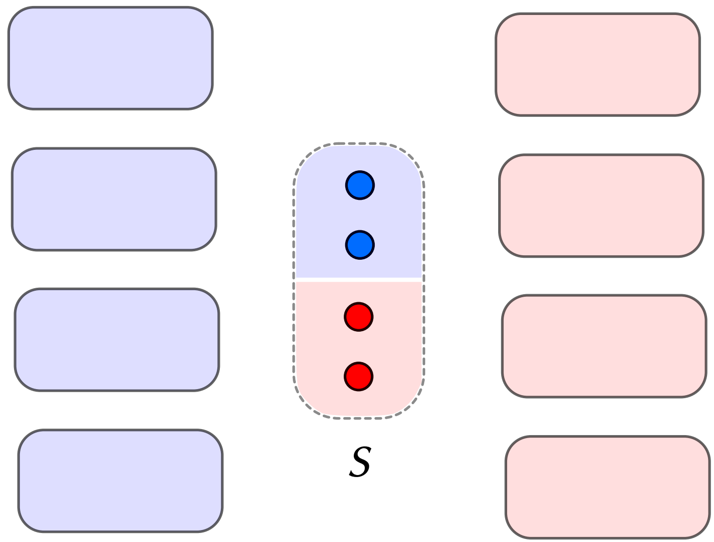

$S :=$ Union of types of size $< t$.

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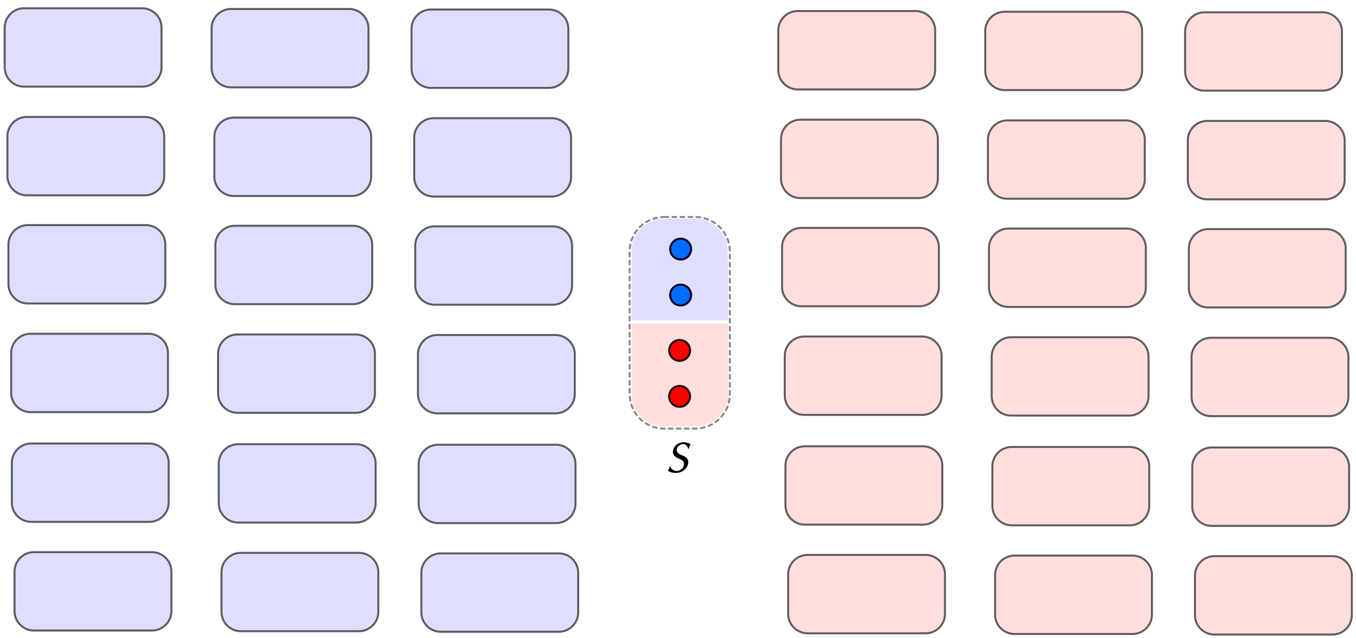

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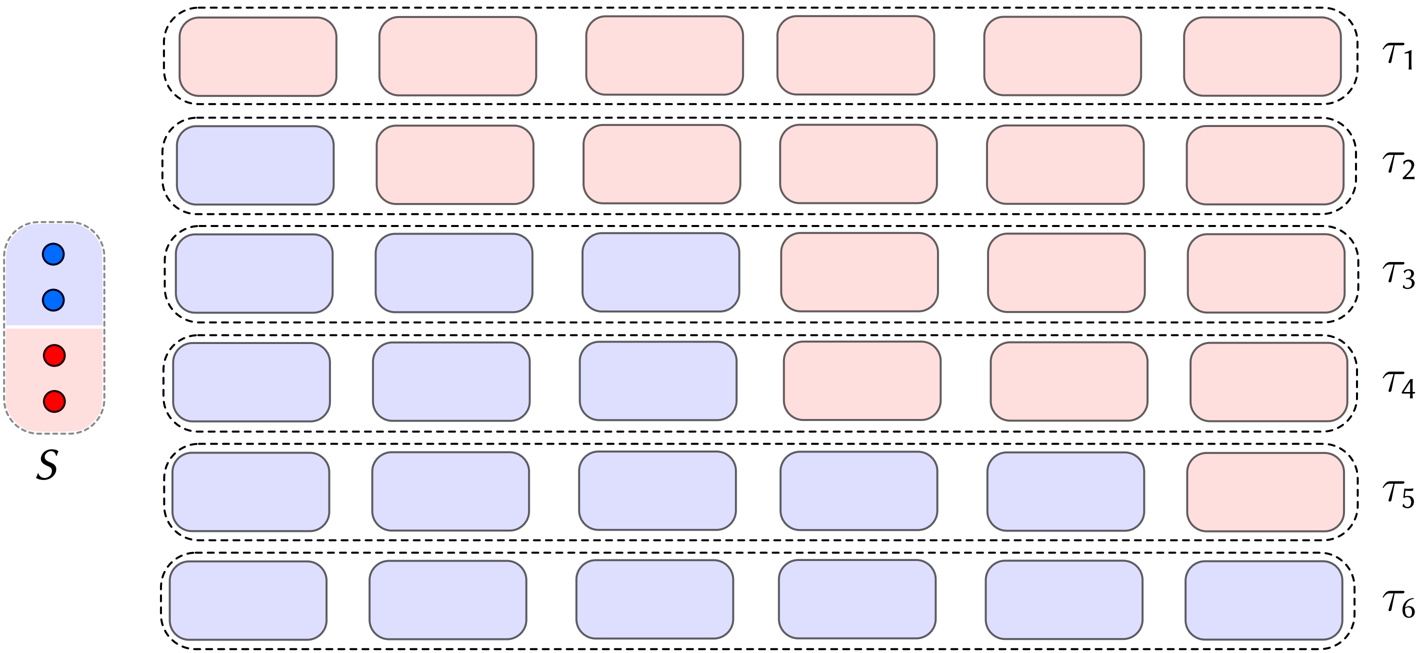

Classify comps wrt high enough FO + conn type.

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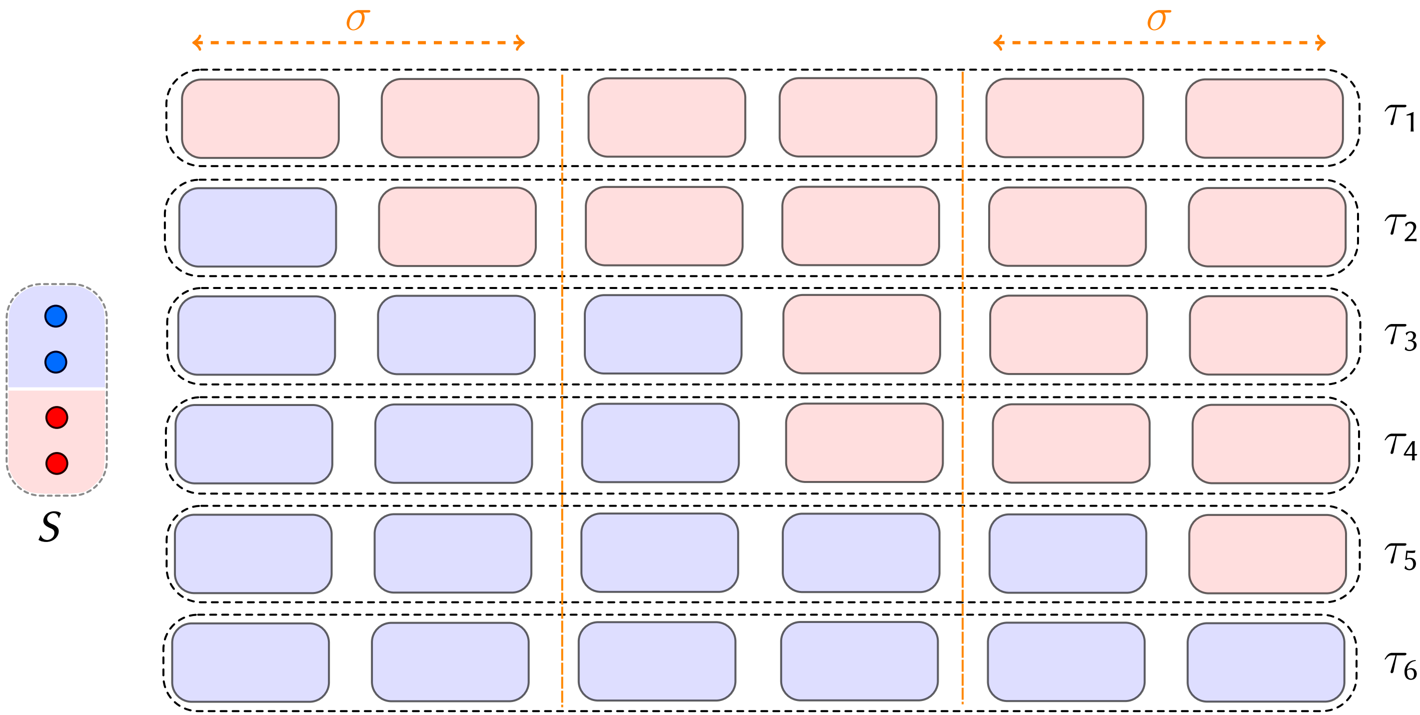

Obs: Can move comps between A and $\overline{A}$ within a type, subject to a margin σ.

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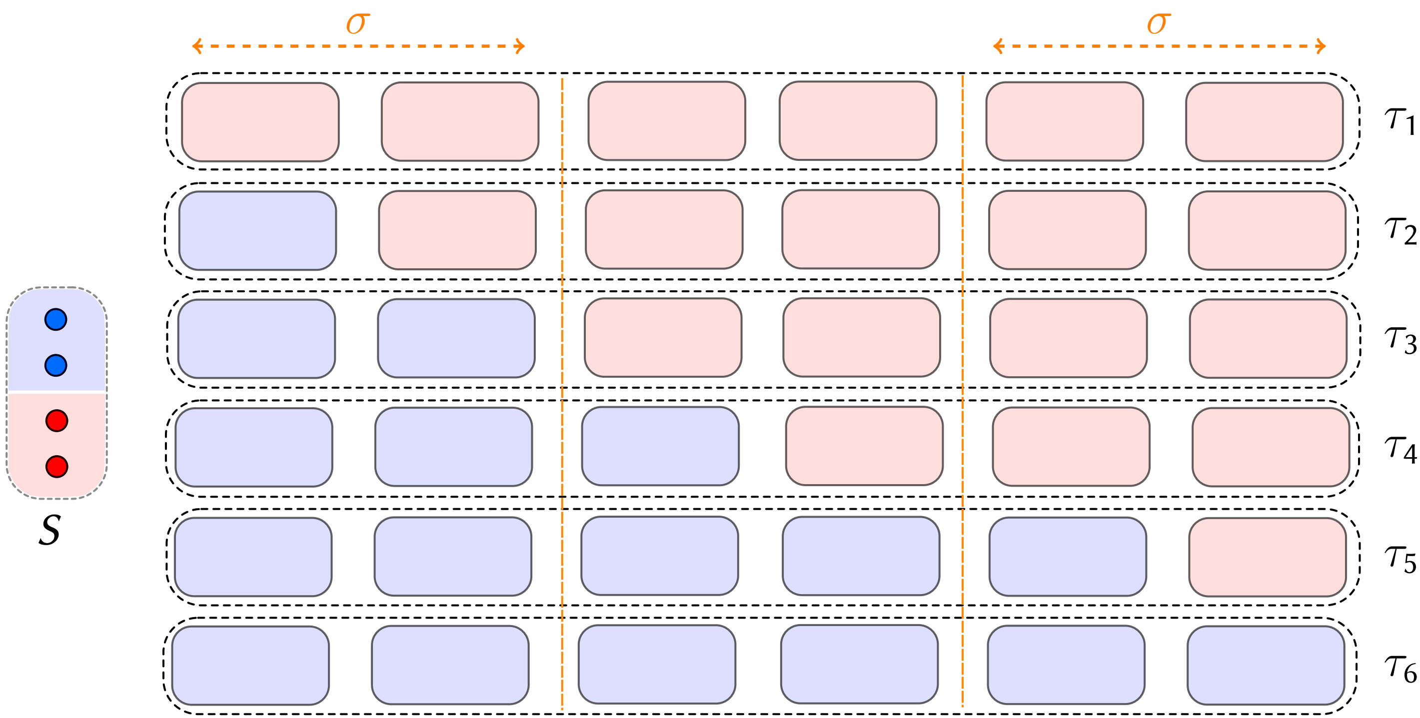

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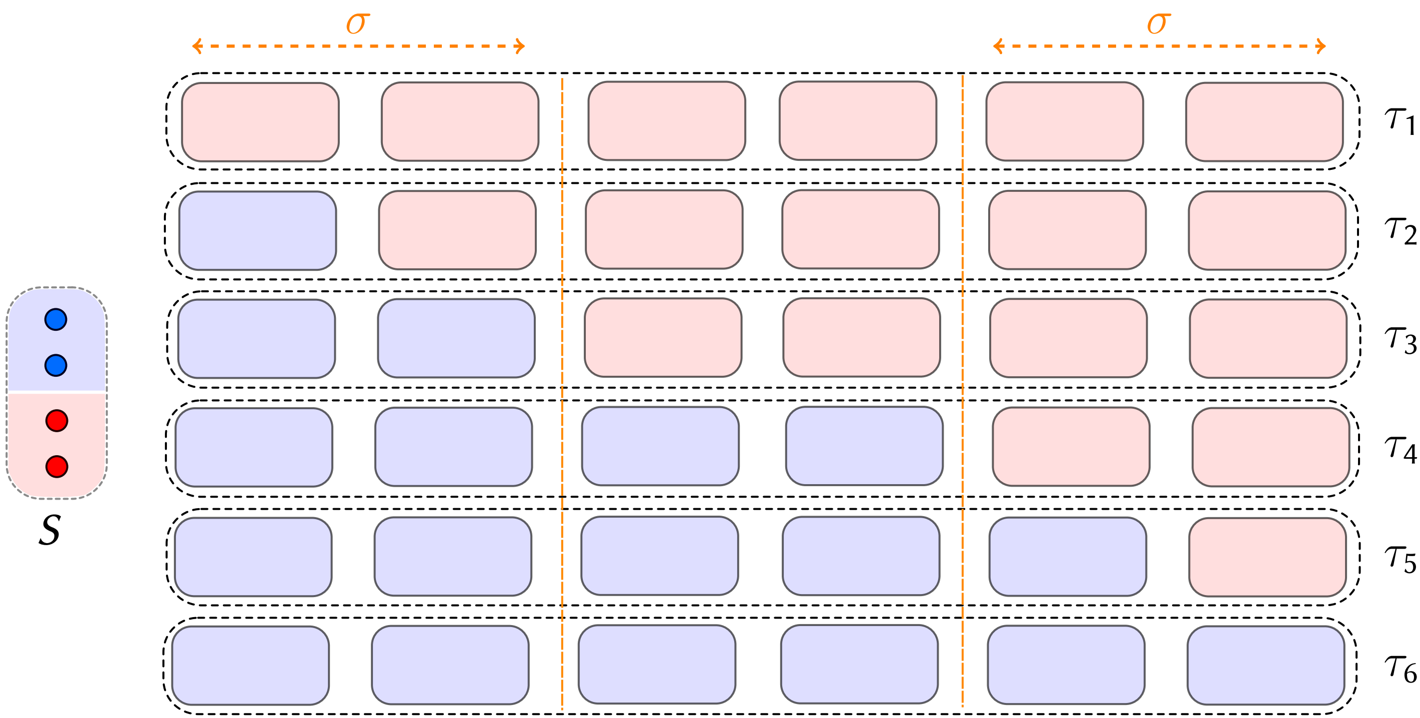

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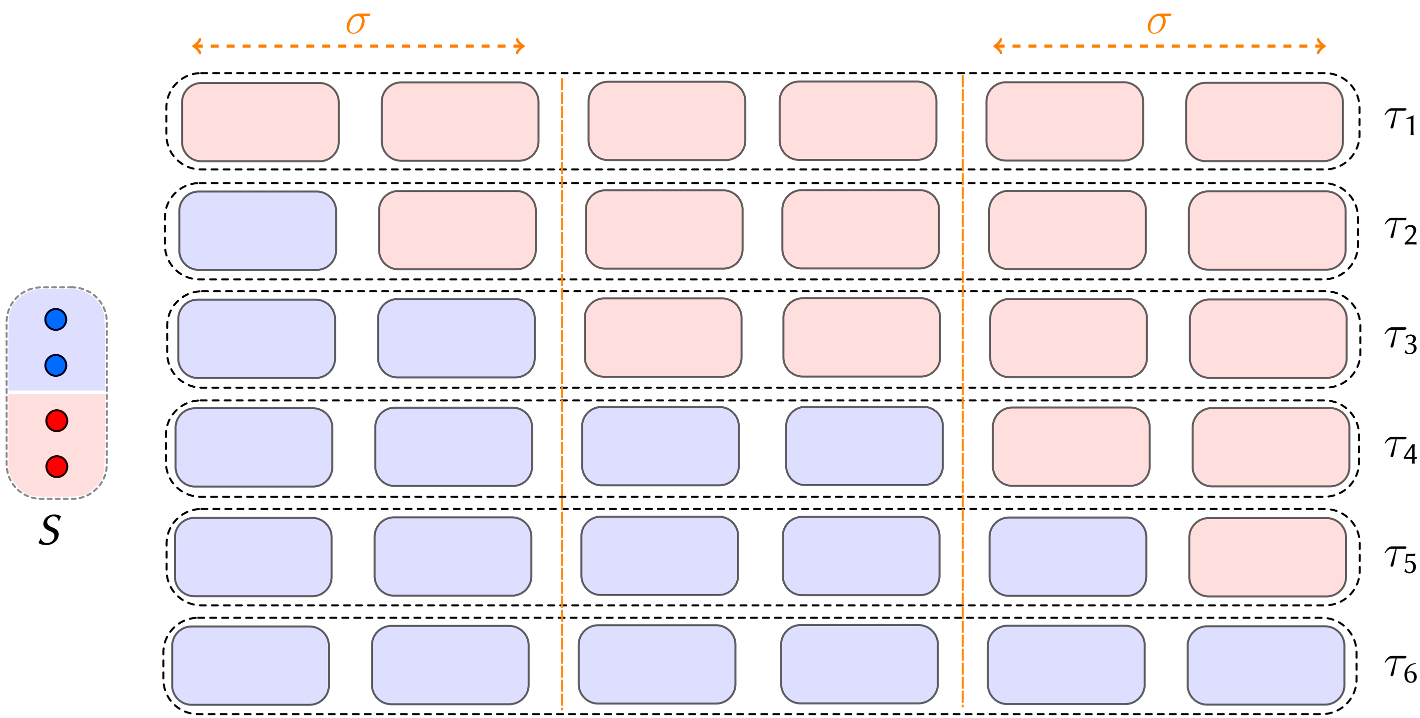

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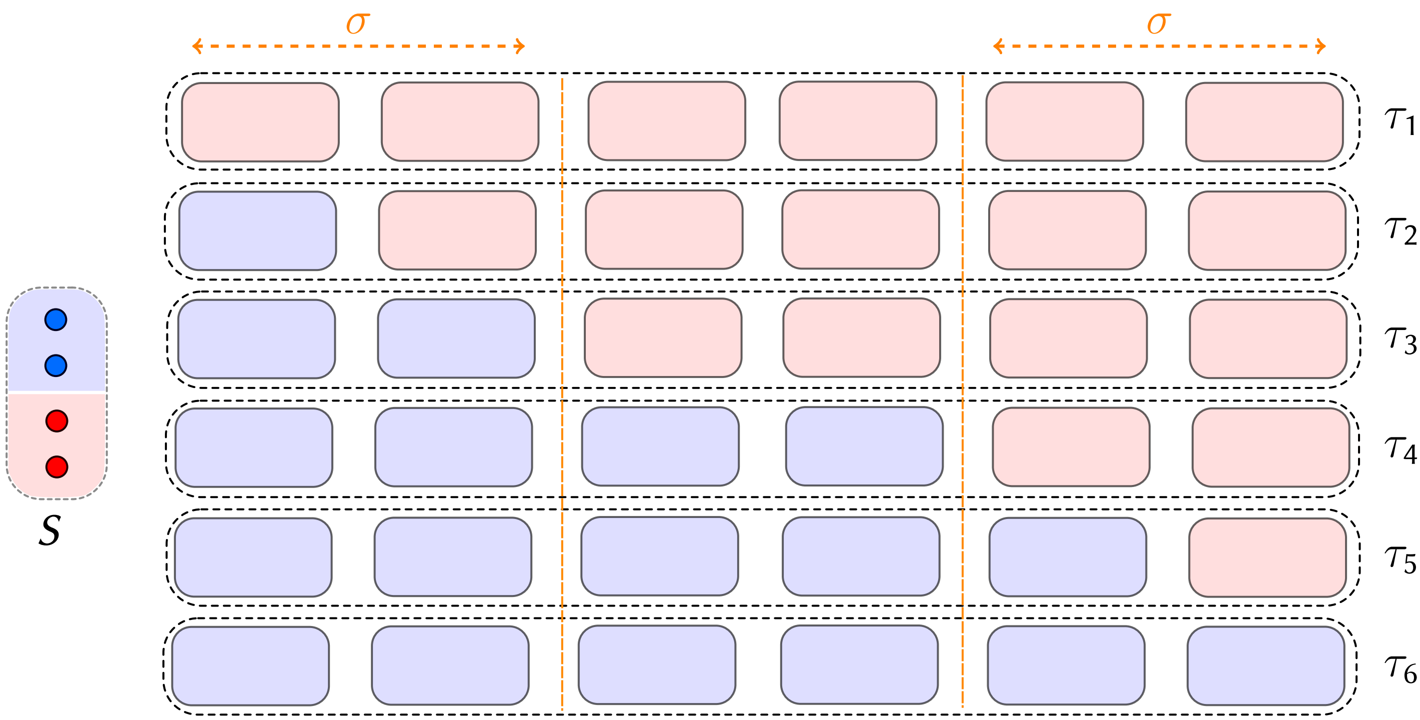

Ergo: May assume that for each τ, A takes or co-takes $\leqslant \sigma$ components.

Such A can be quantified over in FO + conn.

Flip-connectivity logic

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Experience shows that lifting from **sparse** to **dense** is facilitated by:

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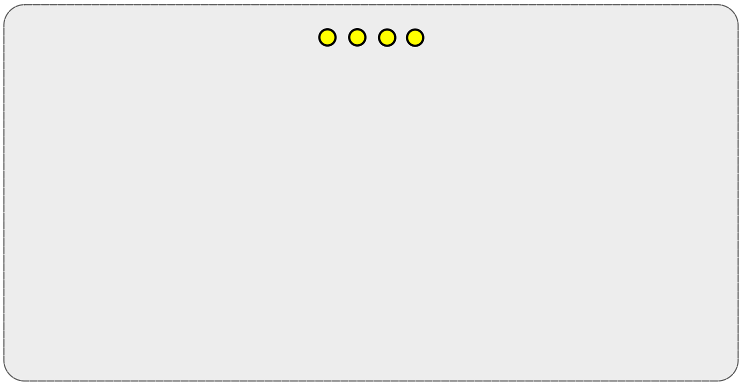

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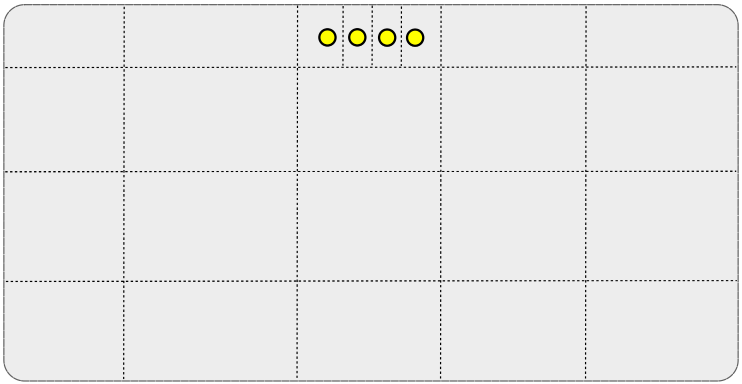

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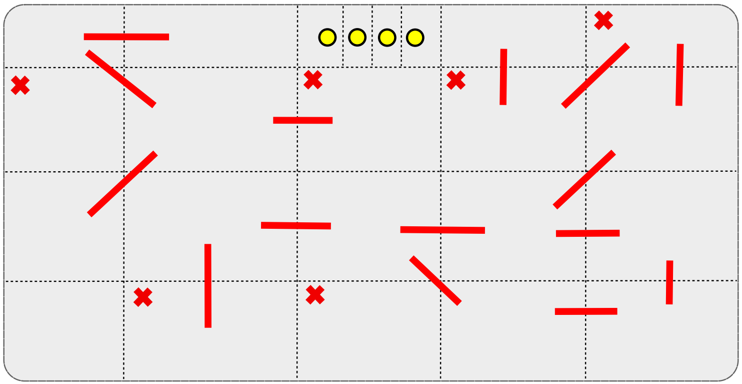

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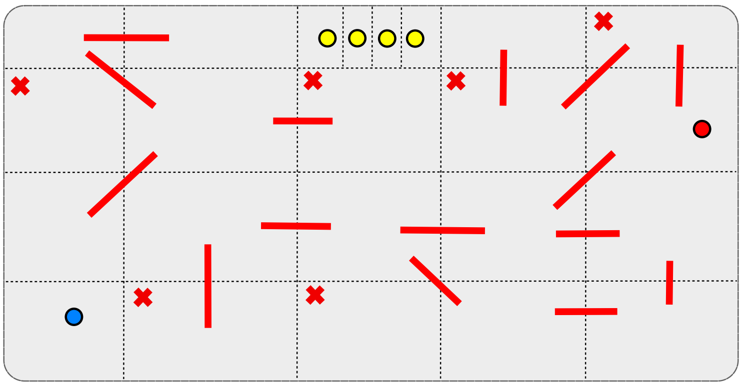

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$\mathrm{flipconn}_\Pi(u, v, a_1, \ldots, a_k)$:= Are u, v connected in $G \oplus \Pi$?

Flip-connectivity and low rank MSO

Theorem

Suppose $\mathscr{C}$ has **bounded VC dimension**.

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Suppose G has **VC dimension** d and $\mathrm{cutrank}(A) \leq k$.

Then there is a flip $G \oplus \Pi$ definable from $f(k, d)$ vertices such that

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Note: **Lemma** fails without the bound on **VC dimension**,

and so does the **equivalence**.

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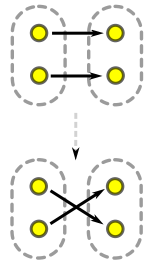

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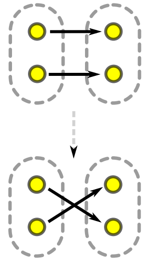

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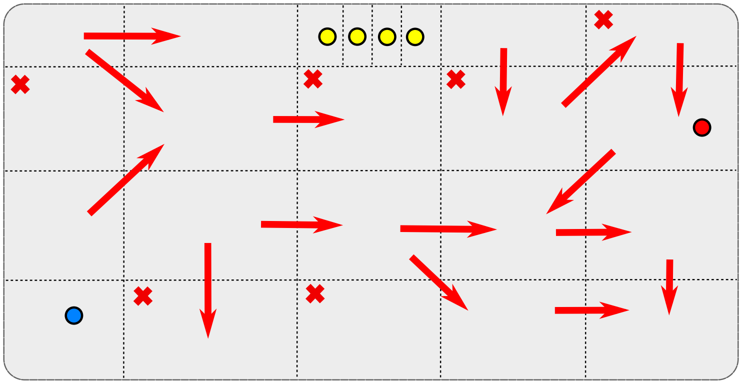

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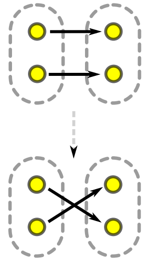

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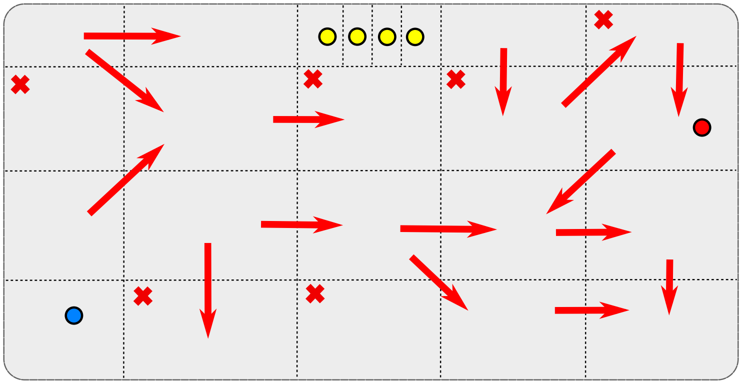

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Low rank MSO coincides with FO + flipreach on **all undirected graphs**.

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$$\mathcal{S}_k = \bigcup_{(X^+,X^-,\mathcal{X})\in\mathcal{F}} \text{Span}(X^+, X^-, \mathcal{X}).$$

Moreover, $|\mathcal{F}| \leqslant |G|^{f(k)}$ and

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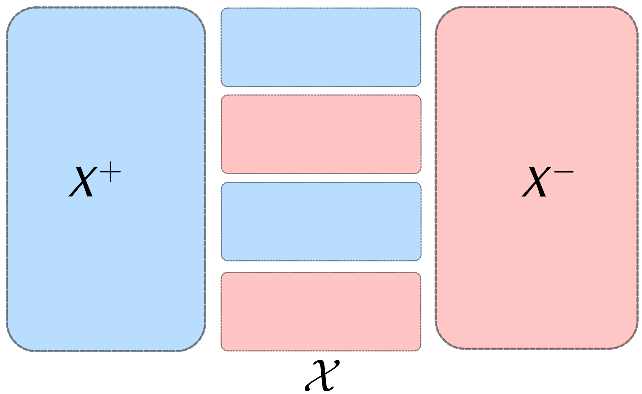

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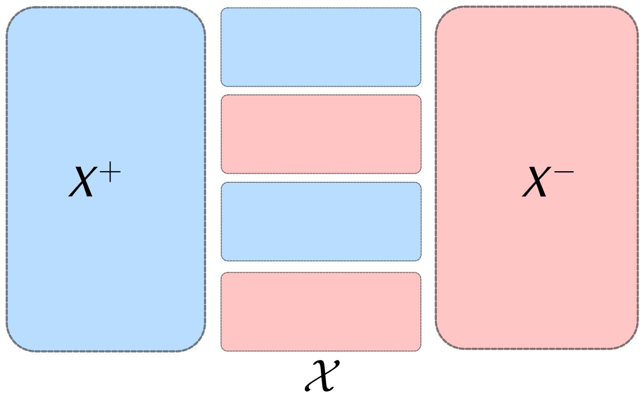

Compact XP representation of sets of cutrank $\leqslant k$.

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Model-checking low rank MSO is FPT

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Thanks for your attention!