Metric Dimension Parameterized by FVS and Other Structural Parameters

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Introduction

Metric Dimension

An invisible immobile target t is hidden at a vertex of a graph G.

Probe a vertex $v \in V(G)$: returned $d(v, t)$.

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Question. How many probes do we need?

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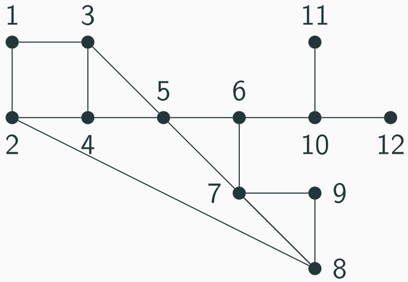

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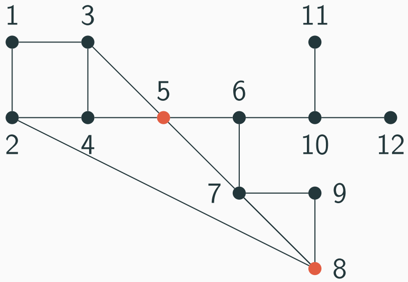

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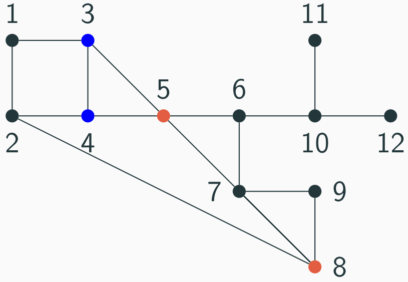

Vertices 3 and 4 are resolved by 8th vertex.

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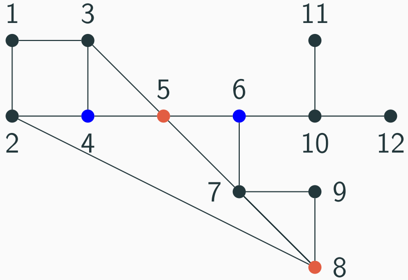
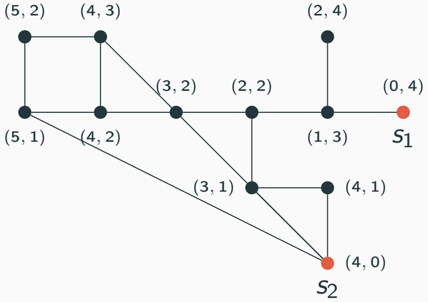


Vertices 4 and 6 are resolved by neither 5^{th} nor 8^{th} vertex.

Metric Dimension

Def. A resolving set is an ordered set $S = \{s_1, s_2, \ldots, s_k\} \subseteq V(G)$ s.t. $\forall\ v, u \in V(G),\ v \neq u$

$$\langle \text{dist}(v, s_1), \ldots, \text{dist}(v, s_k) \rangle \neq \langle \text{dist}(u, s_1), \ldots, \text{dist}(u, s_k) \rangle.$$



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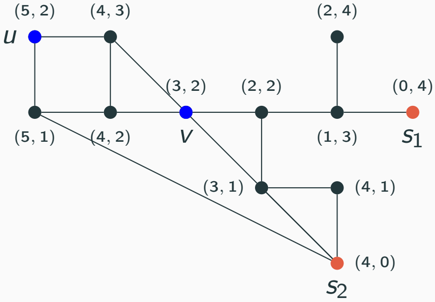

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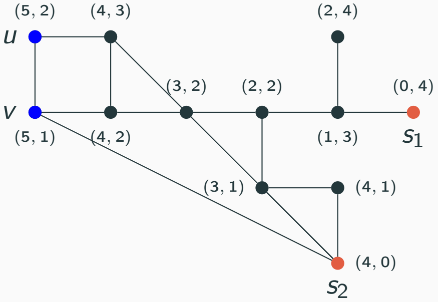


Metric Dimension [Slater, 1975; Harary and Melter, 1976]

Def. Metric dimension (md(G)) is the size of a smallest resolving set of G.

Metric Dimension

Input: an undirected graph $G = (V, E)$, integer k

Question: Is $\mathrm{md}(G) \leq k$?

Overview of what is known

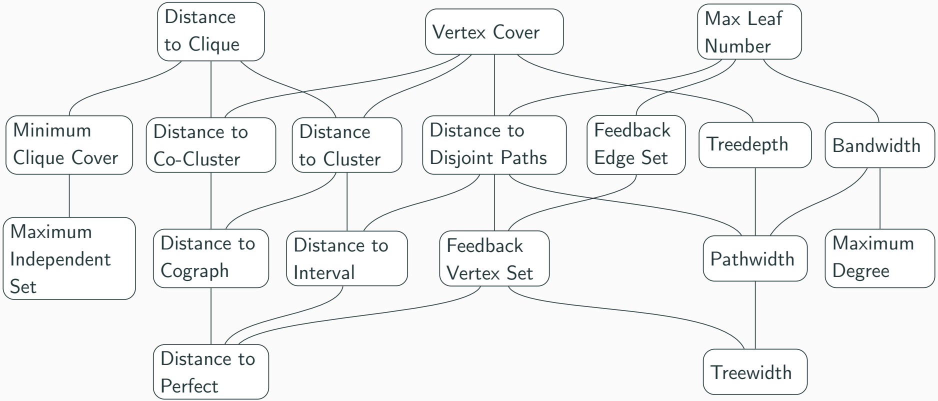
Known results

MetricDimension

NP-complete	Linearly solvable
split graphs	cographs
bipartite	trees
co-bipartite	cactus block graphs
line graphs of bipartite graphs	
planar with bounded degree	Polynomially solvable
interval	outerplanar graphs
permutation graphs of diam 2	

Hasse diagram

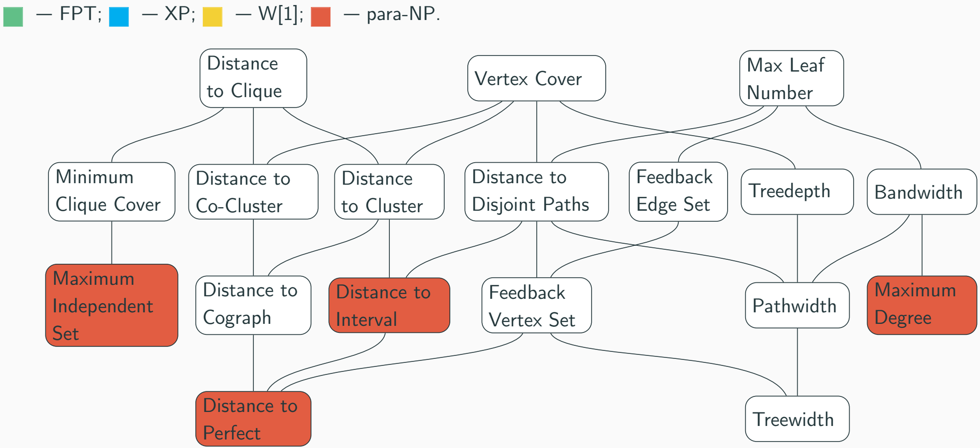
■ — FPT; ■ — XP; ■ — W[1]; ■ — para-NP.



W[2]-hard when parameterized by the natural parameter.

An edge from a lower parameter to a higher parameter indicates that the lower one is upper bounded by a function of the higher one.

Known results



From NP-hard cases that were listed above.

Known results

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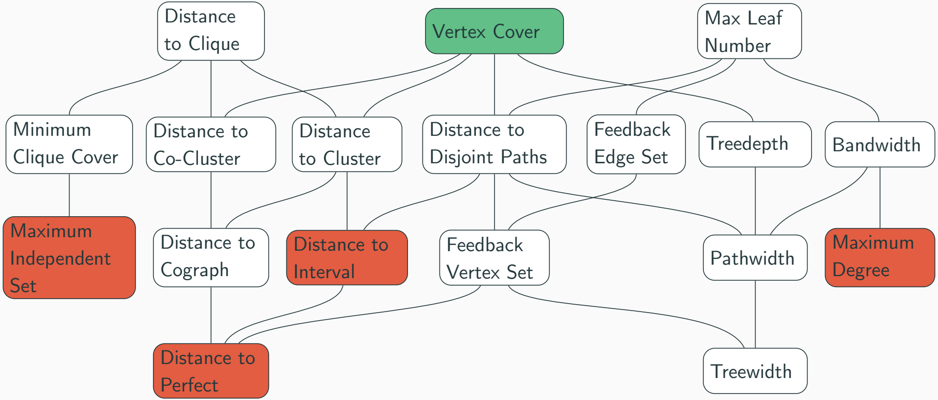

Hartung and Nichterlein, 2013: W[2]-hard for natural parameterization even for bipartite and maxdeg ≤ 3;
FPT when parameterized by the VC;
Stated as an open: on planar graphs; for tree-width parameterization; complexity for FVS.

Known results

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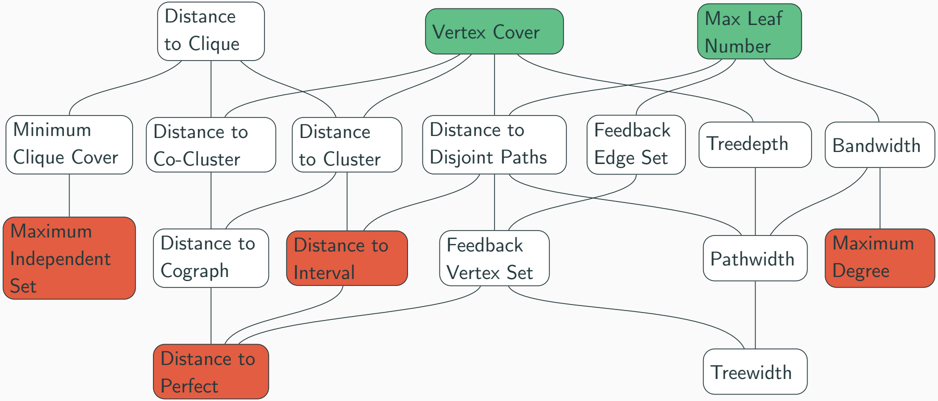

Eppstein, 2015: FPT when parameterized by the max leaf number;

Known results

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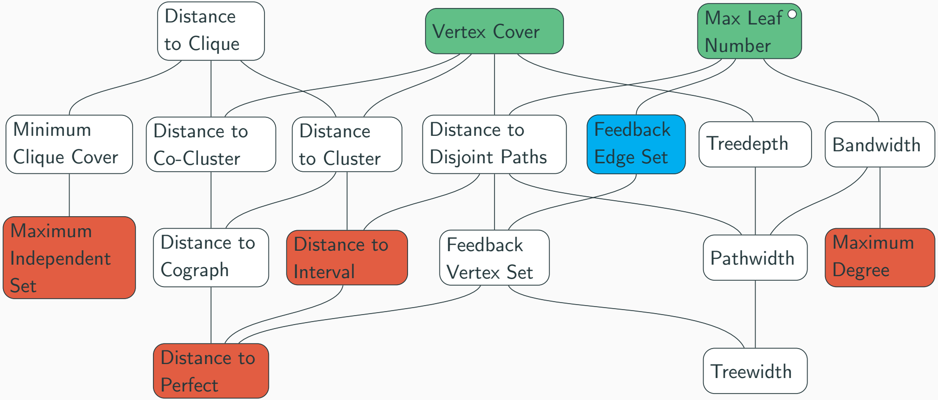
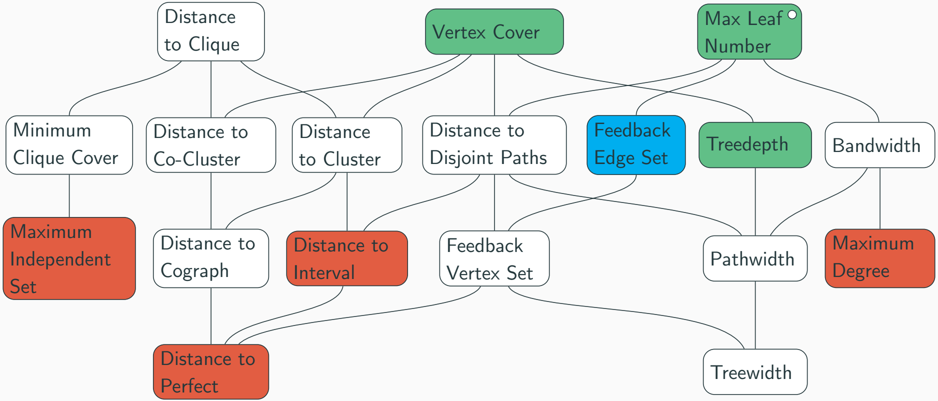


Epstein, et al, 2015: XP when parameterized by the feedback edge set;
White circle means that METRICDIMENSION admits a polynomial size kernel under the parameter marked.

Known results

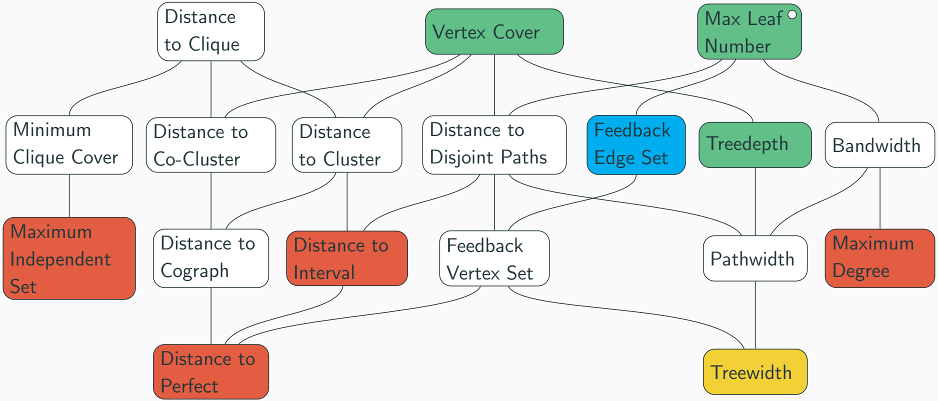
■ — FPT; ■ — XP; ■ — W[1]; ■ — para-NP.



Gima et al, 2021: FPT when parameterized by the treedepth;

Known results

— FPT; — XP; — W[1]; — para-NP.



Bonnet and Purohit, 2019: W[1]-hard when parameterized by the tw;

Known results

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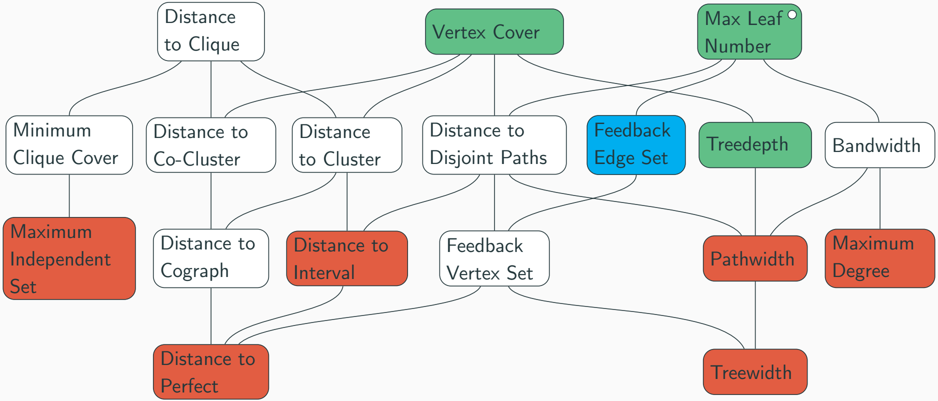
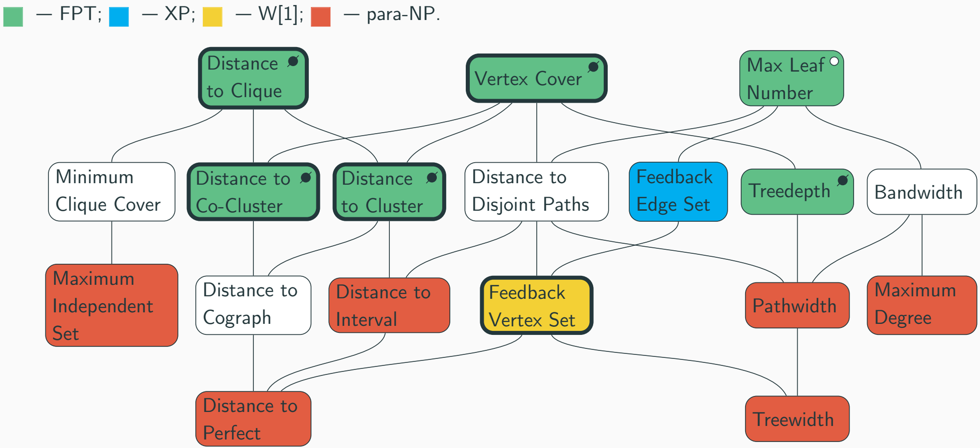


Li and Pilipczuk, 2021: NP-hard in graphs of pw ≤ 24;

Our results



Black circle means that MetricDimension **does not admit a polynomial size kernel under the parameter marked.**

Observation

Def. Any two vertices $u, v \in V(G)$ are true twins if $N[u] = N[v]$, and are false twins if $N(u) = N(v)$.

Observation.

For any (true or false) twins $u, v \in V(G)$, for any resolving set S of a graph G, $S \cap \{u, v\} \neq \varnothing$.

[No] Polynomial Kernels

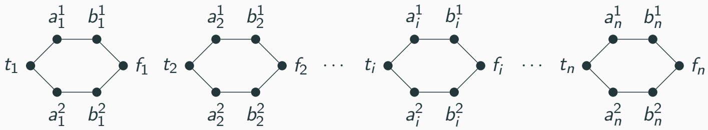
No Poly Kernel, VC

Theorem

METRIC DIMENSION parameterized by the minimum size of a vertex cover of the graph does not admit a polynomial kernel unless $\mathrm{NP} \subseteq co\mathrm{NP}/poly$.

Reduction from SAT parameterized by the number of variables.

No Poly Kernel, VC



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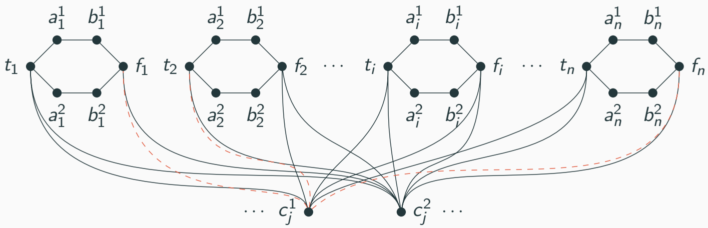
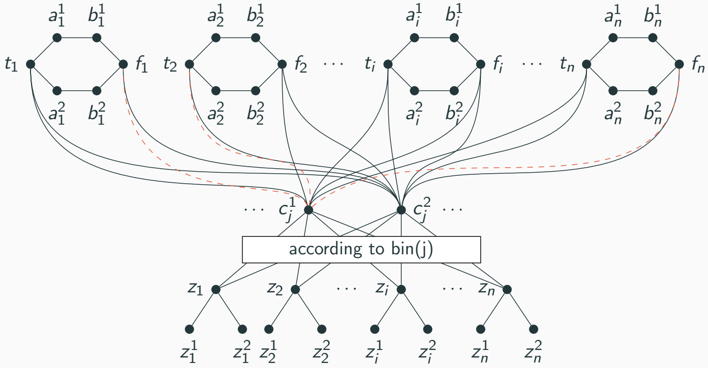


$$C_j = (\overline{x_1} \vee x_2 \vee \overline{x_n})$$

No Poly Kernel, VC



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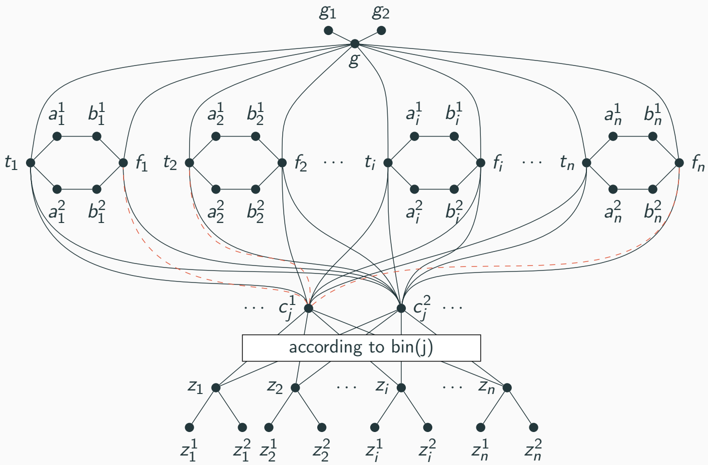
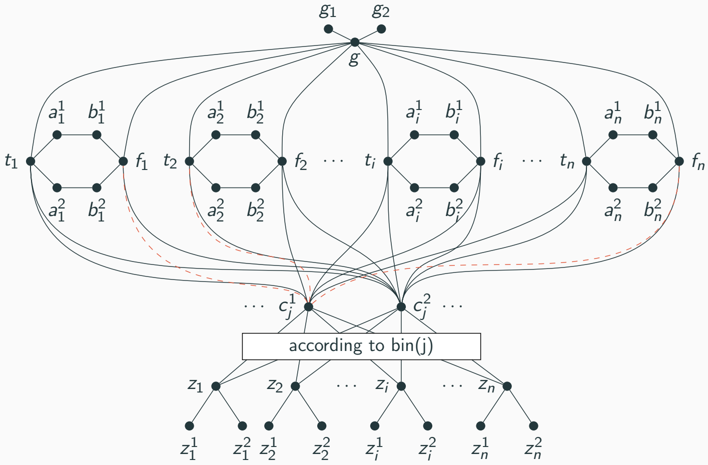


No Poly Kernel, VC

$\exists$ a satisfying assignment iff $\mathrm{md}(G) = 2n + 1$.



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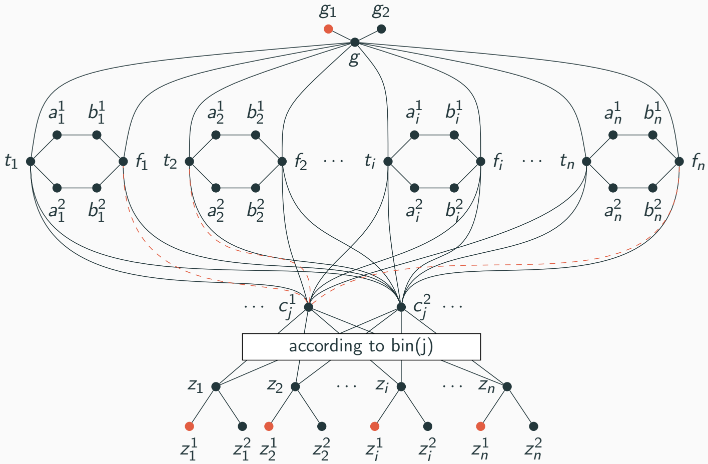
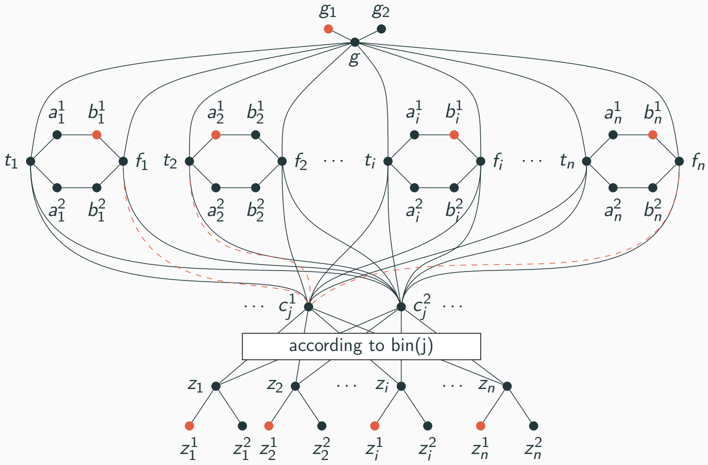


No Poly Kernel, VC

∃ a satisfying assignment iff $\mathrm{md}(G) = 2n + 1$.



No Poly Kernel, Dist. to clique

By making the vertices of $\{C_j \mid j \in [m]\}$ into a clique, the distance to clique of the resulting graph is at most $9n + 3$.

Then, for this modified G:

Theorem

Metric Dimension parameterized by the distance to clique does not admit a polynomial kernel unless $\mathrm{NP} \subseteq \mathrm{coNP}/poly$.

W[1]-hardness, FVS

$W[1]$-hardness

NAE-Integer-3-Sat, W[1]-hard param. by the number of variables

Input: a set X of variables, a set C of clauses, and an integer d.

- Each variable $x \in X$ takes a value in $\{1, \dots, d\}$;
- Each clause is of the form $(x \leq a_x, y \leq a_y, z \leq a_z)$, $a_x, a_y, a_z \in [d]$;
- A clause is satisfied if not all three inequalities are true and not all are false.

Question: Does a satisfying assignment of the variables exist?

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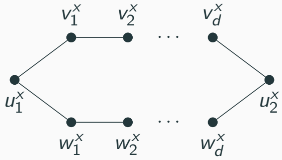
Theorem

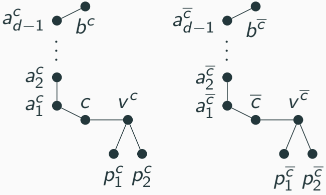
Metric Dimension param. by the feedback vertex set number is $W[1]$-hard.

$W[1]$-hardness

The variable gadget G_x:

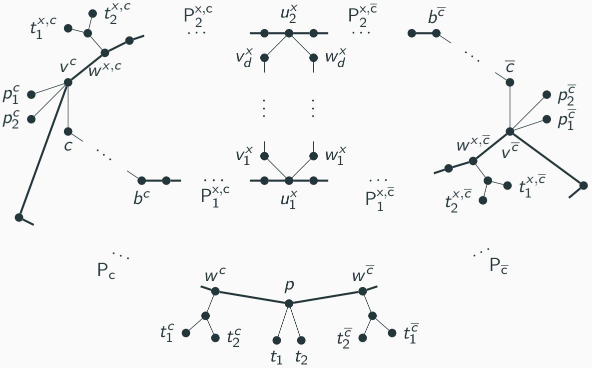


The clause gadget G_c: a disjoint union of H_c and $H_{\bar{c}}$



$W[1]$-hardness

Complete construction:

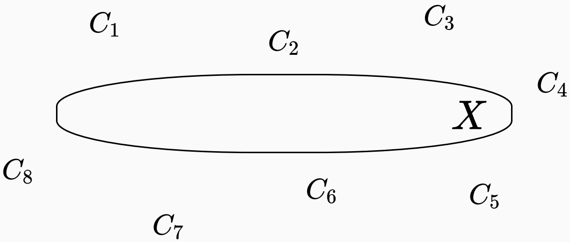


(X, C, d) is satisfiable iff (G, k) is a yes-instance for $k = |X| + 10|C| + 1$.

FPT, the dist to cluster

FPT, the dist to cluster

Def. The distance to $\mathcal{F}$ of graph G is the size of minimum set $X \subseteq V(G)$ such that $G - X \in \mathcal{F}$.



FPT, the dist to cluster

Theorem

METRIC DIMENSION is FPT parameterized by the distance to cluster.

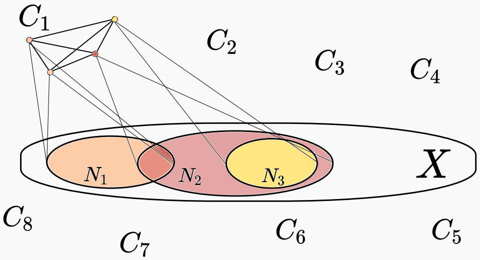
Red. Rule 1. If there exist $u, v, w \in V(G)$ s.t. u, v, w are true (or false) twins, then remove u from G and decrease k by one.

So, $\forall\ C \in G \setminus X$, at most 2 of its vertices have the same neighborhood in X. Thus, $|C| \leq 2^{|X|+1}$.

FPT, the dist to cluster

Def. For every clique C of $G - X$, define the signature $sign(C)$ of C

$$sign(C) = \{N(u) \cap X : u \in C\}.$$

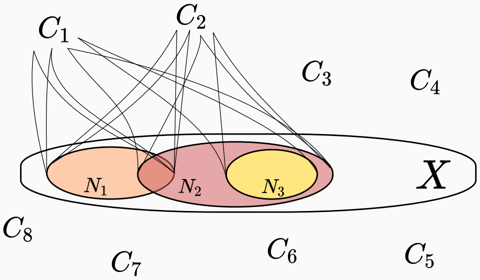


$$sign(C_1) = \{N_1, N_1, N_2, N_3\}.$$

FPT, the dist to cluster

Def. For any two cliques $C_1, C_2 \in G - X$, let $C_1 \sim C_2$, if and only if

$$sign(C_1) = sign(C_2).$$



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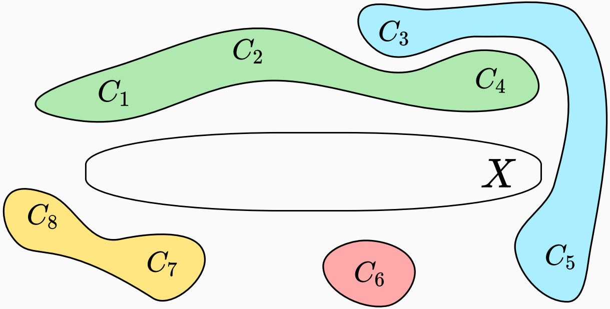


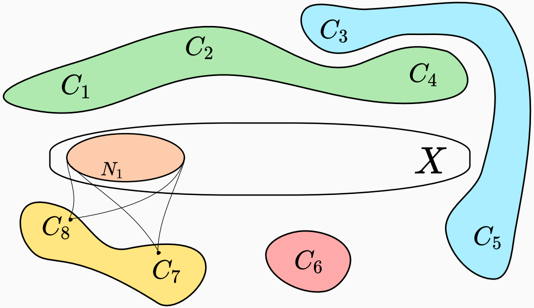
Thus, there are at most $2^{2^{|X|+1}}$ equivalence classes.

FPT, the dist to cluster

$\mathcal{C}$: an equivalence class of $\sim$.

C_7, C_8: cliques from the same $\mathcal{C}$.

Def. Two vertices $u \in C_7$ and $v \in C_8$ are clones if $N(u) \cap X = N(v) \cap X$.



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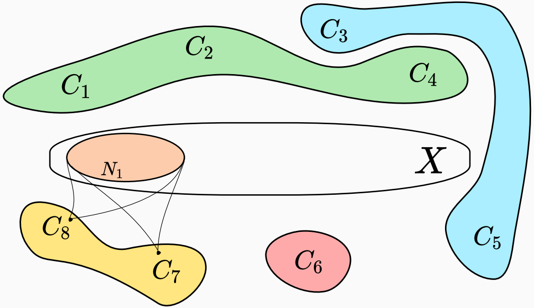


Claim. Let $u \in C_7$ and $v \in C_8$ be clones. Then, for any resolving set S of G,

$$S \cap \big(V(C_7) \cup V(C_8)\big) \neq \varnothing.$$

FPT, the dist to cluster

Red. Rule 2. If there exists $\mathcal{C}$ such that

$$|\mathcal{C}| \geq 2^{|X|+2} + |X| + 2,$$

remove a clique $C \in \mathcal{C}$ from G and reduce k by $\max\{1, t(\mathcal{C})\}$.

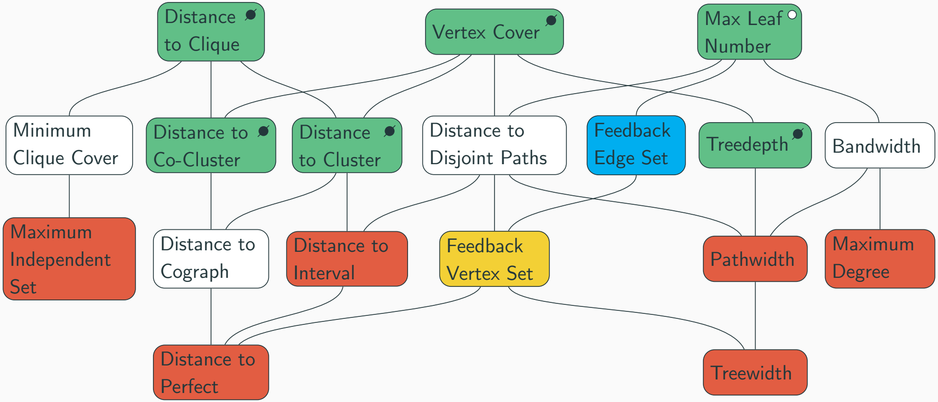
Thus, $|V(G)| \leq 2^{2^{|X|+1}} \cdot (2^{|X|+2} + |X| + 1) \cdot 2^{|X|+1} + |X|$:

- $2^{2^{|X|+1}}$ equivalence classes;
- each $\mathcal{C}$ contains at most $2^{|X|+2} + |X| + 1$ cliques;
- for each clique $C \in G - X$, $|V(C)| \leq 2^{|X|+1}$.

Further directions

Further

■ — FPT; ■ — XP; ■ — W[1]; ■ — para-NP.



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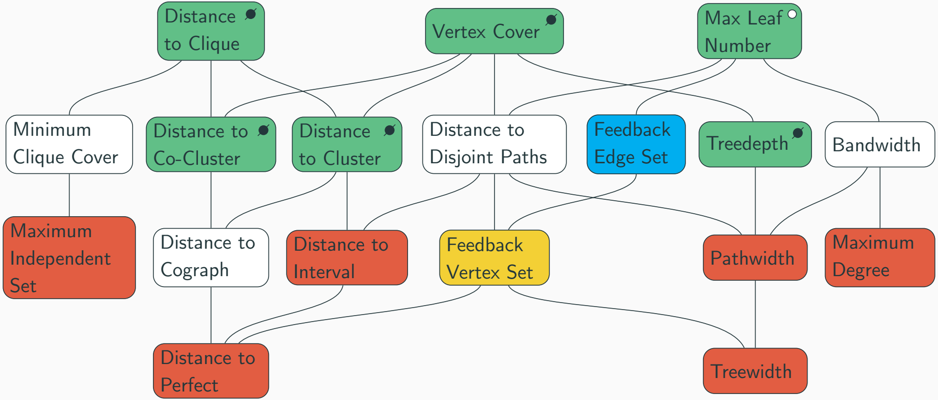
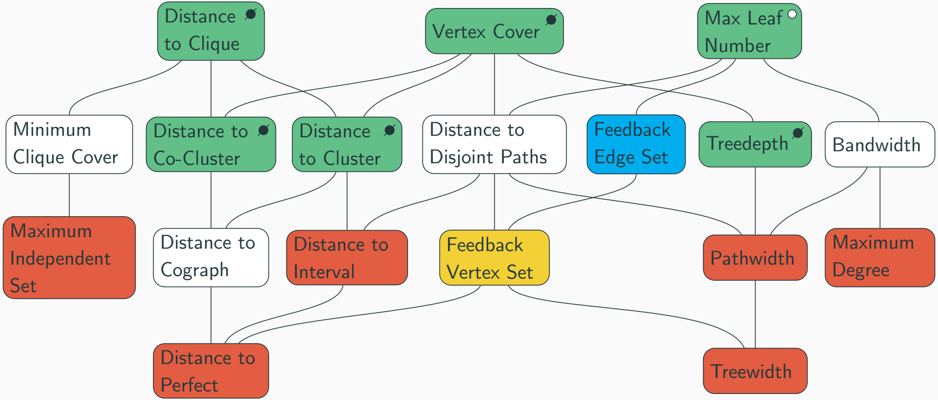


? FPT with the feedback edge set.

Further

■ — FPT; ■ — XP; ■ — W[1]; ■ — para-NP.



? Parameterization with the distance to cograph.

Further

- Structural parameterization by:
  - the feedback edge set;
  - the distance to cograph;
  - dist to disjoint paths;
  - bandwidth;
  - the fvs + solution-size.

Thanks for attention!

Further directions

- Structural parameterization by:
  - the feedback edge set;
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  - dist to disjoint paths;
  - bandwidth;
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Contents