Graph r-admissibility in theory and practice

LOGALG 2025
Nov 21

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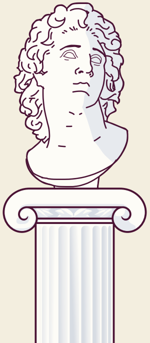
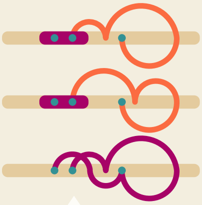


Bounded expansion

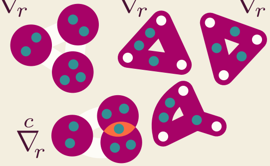
wcol_r

col_r

adm_r



∇_r $\widetilde{\nabla}_r$ $\dot{\widetilde{\nabla}}_r$ $\overset{c}{\nabla}_r$



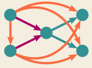
[Zhu09]
[Dvrk13]

Size of r-*reachable sets/ path packings* in ordering

Density of shallow (topological) minors

[NODM12]

$\Delta^-(\vec{G}_r)$



In-degree of r-step (d)tf-augmentation

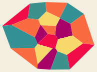
[NODM12]

ν_r

Number of *traces* r-neighbourhoods leave in any subset

[RSVS19]

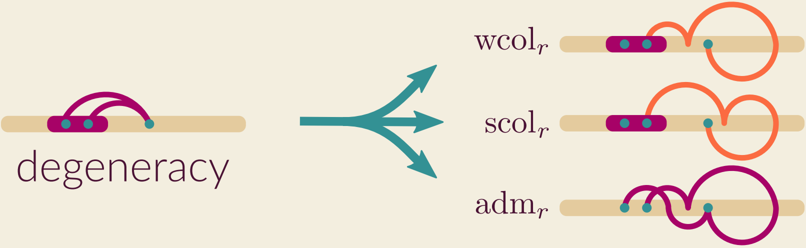
χ_r



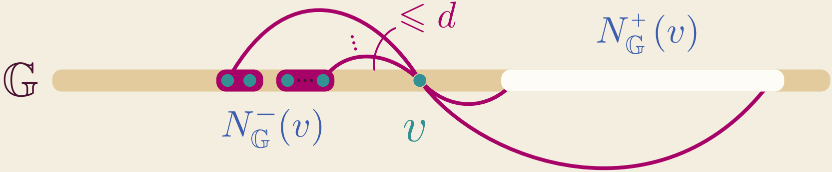
Number of colours in r-treedepth colouring

[NODM12]

Lifting degeneracy



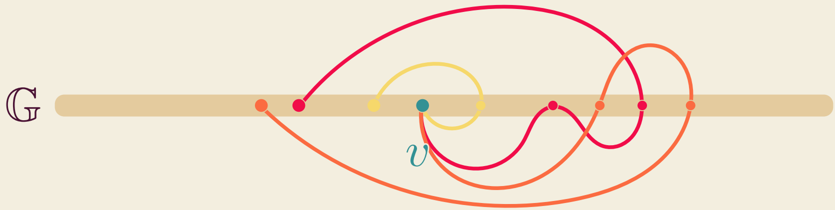
Degeneracy



A graph G is d-degenerate if there exists a linear ordering $\mathbb{G}$ of G such that every vertex has at most d neighbours to its left.

Ordered graph: $\mathbb{G} = (G, <)$

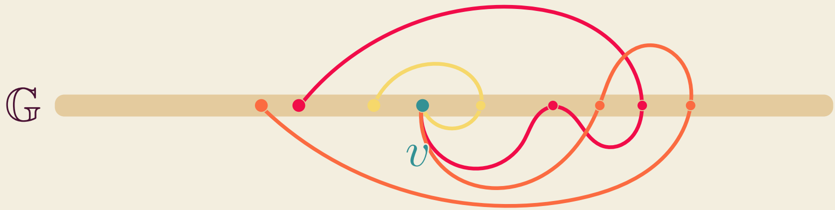
Admissibility



The r-path-packing number $\mathbf{pp}_r(v)$ is the maximum number of paths that

1) All start at v but are otherwise disjoint,
2) have each length at most r, and
3) have only their endpoint left of v.

Admissibility



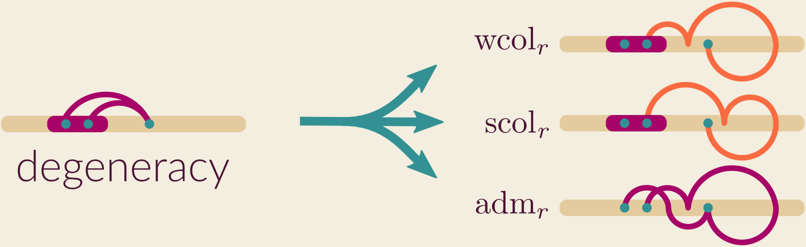
The *r-admissibility* $\mathrm{adm}_r(\mathbb{G})$ of an ordered graph $\mathbb{G}$ is

$$\mathrm{adm}_r(\mathbb{G}) = \max_{v \in \mathbb{G}} \mathrm{pp}^r(v)$$

The r-admissibility of a graph G is the minimum value over all its orderings:

$$\mathrm{adm}_r(G) = \min_{\mathbb{G} \in \pi(G)} \mathrm{adm}_r(\mathbb{G})$$

Bounded expansion



For every ordered graph $\mathbb{G}$ and $r \geq 1$ it holds that

$$\mathrm{adm}_r(\mathbb{G}) \leq \mathrm{scol}_r(\mathbb{G}) \leq \mathrm{wcol}_r(\mathbb{G}) \leq (r^2 \mathrm{adm}_r(\mathbb{G}))^r$$

[Dvrk13]
[Dvrk22]

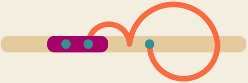
A graph class has *bounded expansion* iff it is adm_r / scol_r / wcol_r-bounded.

[Zhu09]

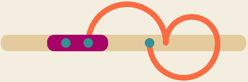
Algorithmic uses

Application | Computation

wcol_r



scol_r

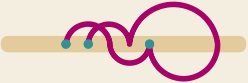


Application: Quite natural for algorithm design
once you get used to it

Computation: NP-hard for $r = 2$

[BKLS25]

adm_r



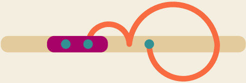
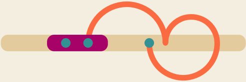
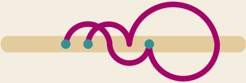
Application: Direct use?

Computation: Poly-time for $r \leq 3$

Greedy algorithm for ordering
with an NP-complete subproblem for $r \geq 4$

[IPS82] [Dvrk13]

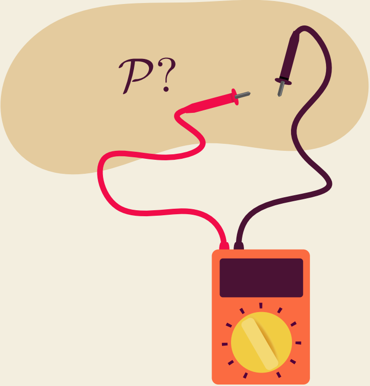
Algorithmic uses

	Application	Computation
wcol_r  scol_r 	Quite natural for algorithm design once you get used to it	NP-hard for $r = 2$ [BKLS25]
adm_r 	Direct use?	Poly-time for $r \leq 3$ Greedy algorithm for ordering with an NP-complete subproblem for $r \geq 4$ [IPS82] [Dvrk13]

Part I

Uses in theory, uses *in* Theory

Property testing



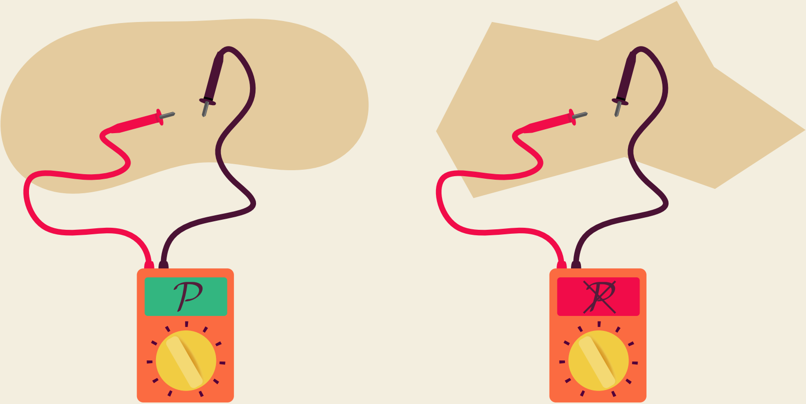
We want to check whether the input has some property $\mathcal{P}$ in sublinear time

A *property tester* is an algorithm that does that (with some complications)

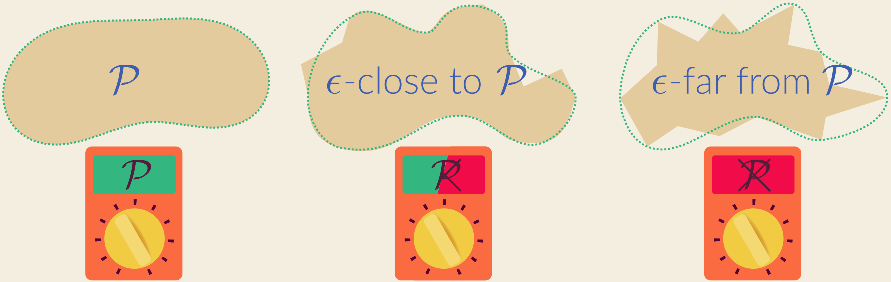
Since we cannot read the input in full, the tester consults an *oracle* to read certain parts of the input

Property testing: some complications

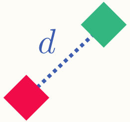
We are happy if the tester works in cases where an instance has a property $\mathcal{P}$, or when it ‘clearly’ does not.



Property testing: some complications

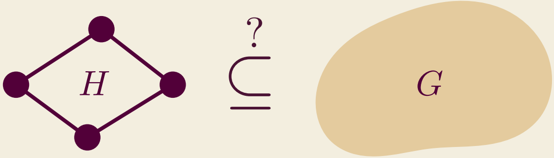


The tester* must accept instance that are in $\mathcal{P}$.
It must reject instances that are ϵ-far from $\mathcal{P}$ with probability at least 2/3.

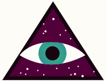


*This is a one-sided tester. Two-sided testers are allowed to err in both cases.

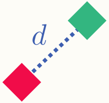
H-freeness in r-admissible graphs



$\mathrm{adm}_r(G) \leq p$



1. Return random vertex of G
2. Given a vertex, return a random neighbour



A graph is ϵ-far from being H-free if we must delete more than ϵpn edges to remove all subgraphs isomorphic to H.

H-freeness: results

- H-freeness with $\mathrm{diam}(H) \leq 2$ is testable in graphs with bounded 2-admissibility (C_4, C_5)
- C_6 and C_7-freeness is testable in graphs with bounded 3-admissibility

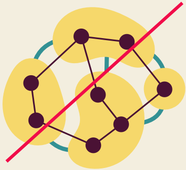
C_r-freeness is not testable in graphs of bounded $(\lfloor r/2 \rfloor - 1)$-admissibility.

Results on H-freeness testing in graphs of bounded r-admissibility [AGLR25b]
C. Awofeso, P. Greaves, O. Lachish, FR (STACS '25)

- C_{2r} and C_{2r+1}-freeness is testable in graphs with bounded r-admissibility

Testing C_k-Freeness in Bounded Admissibility Graphs [AGLLR25]
C. Awofeso, P. Greaves, O. Lachish, A. Levi, FR (ICALP '25)

H-freeness: results



H-freeness is testable in minor-closed classes

Properties testable on minor-closed classes with one-sided error are precisely those that can be defined by a finite set of forbidden subgraphs.

A Characterization of Graph Properties Testable for General Planar Graphs with one-Sided Error (It's all About Forbidden Subgraphs)
Artur Czumaj, Christian Sohler (STOC'19)

[CS19]

H-freeness: results

H-freeness is testable in graphs with bounded $|H|$-admissibility.

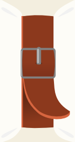
Properties testable on bounded expansion classes with one-sided error are precisely those that can be defined by a finite set of forbidden subgraphs.

A sufficient condition for characterizing the one-sided testable properties of families of graphs in the Random Neighbour Oracle Model [AGLLR]
Christine Awofeso, Patrick Greaves, Oded Lachish, Amit Levi, FR (Under review)

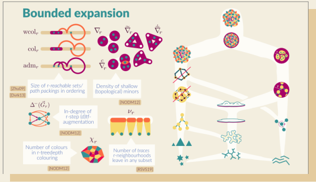
Why admissibility?

Using adm_r / scol_r / wcol_r gives tighter bounds than e.g. low treedepth colourings

The property testing community cares a lot about lower bounds, so tightness matters!



When writing papers for a different community, we cannot bring in ***all*** of our tools!



Admissibility is easier to motivate than weak/strong colouring numbers and sits at the 'bottom'.

Part II

Computing admissibility

Computing admissibility

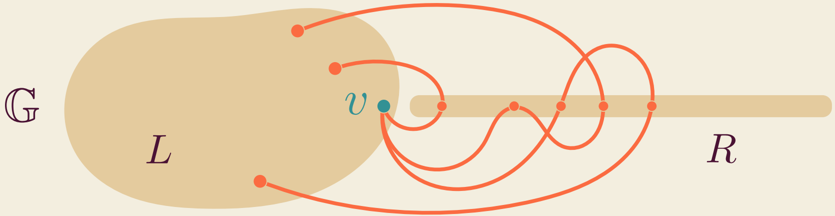
r-admissibility can be computed in linear fpt-time in bounded expansion classes

In classes with bounded f(r)-admissibility, for some horrible f(r)

[Dvrak13] [DKT13]

1. Can we compute r-admissibility in linear fpt-time in classes with ‘only’ bounded r-admissibility?
2. Can we design a *practical* algorithm for small values of r?

Computing admissibility

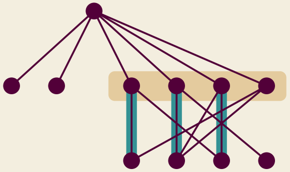


If for some partition L, R we can determine the maximum r-path-packing for vertices in L, then we are done!

If we move a vertex v from L to R, only vertices in $S^r(v)$ are affected!

Computing admissibility

2-admissibility



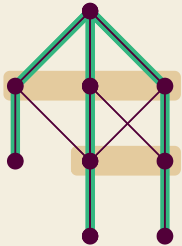
It suffices to store a matching between $N(v) \cap R$ and $S^2(v)$ and be prudent about updates.

$O(p^4 n)$ time

$O(m + p^2)$ space

[AGLR25a]

3-admissibility



Store and maintain 3-path packings with nice properties

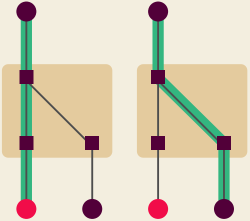
$O(p^7 n)$ time

$O(p^3 m)$ space

[AGLR26a]

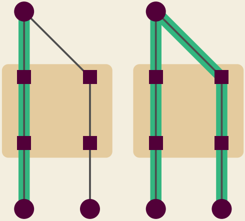
Computing 3-admissibility: careful escalation

1



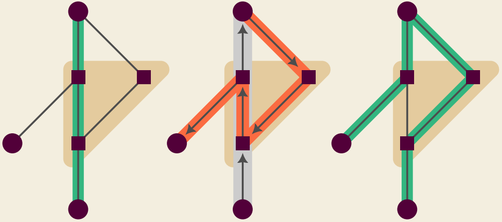
Simple rerouting

2



Find disjoint paths

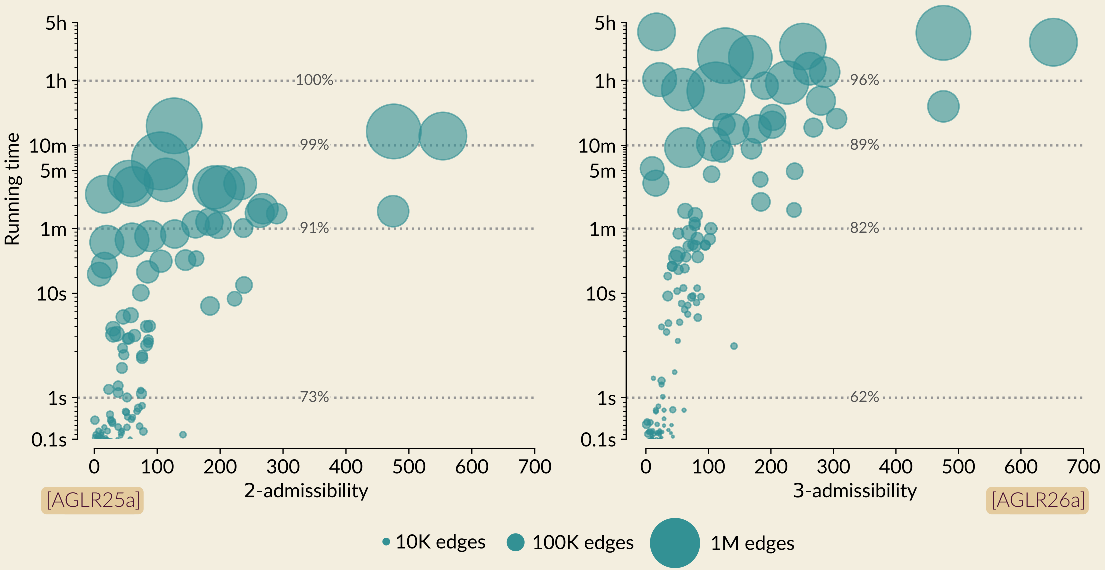
3



Find maximum packing

[AGLR26a]

Experiments: running time



[AGLR25a]

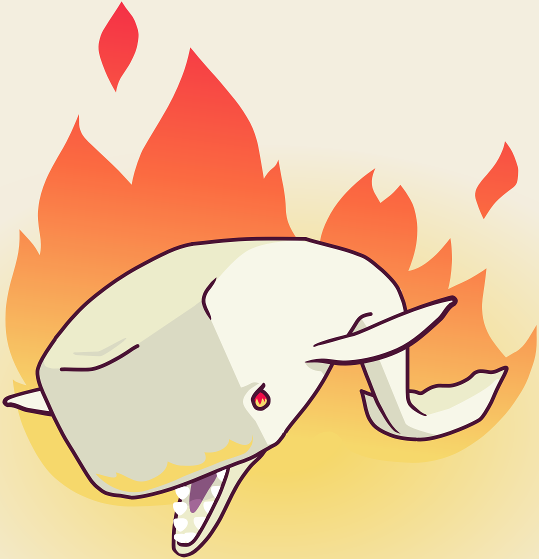
[AGLR26a]

Part III

In practice

The big question

The big question

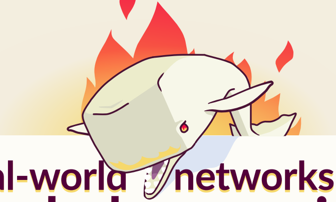


The big question

***Do* real-world networks have**

‘bounded expansion’?

What we know



Do real-world networks have ‘bounded expansion’?

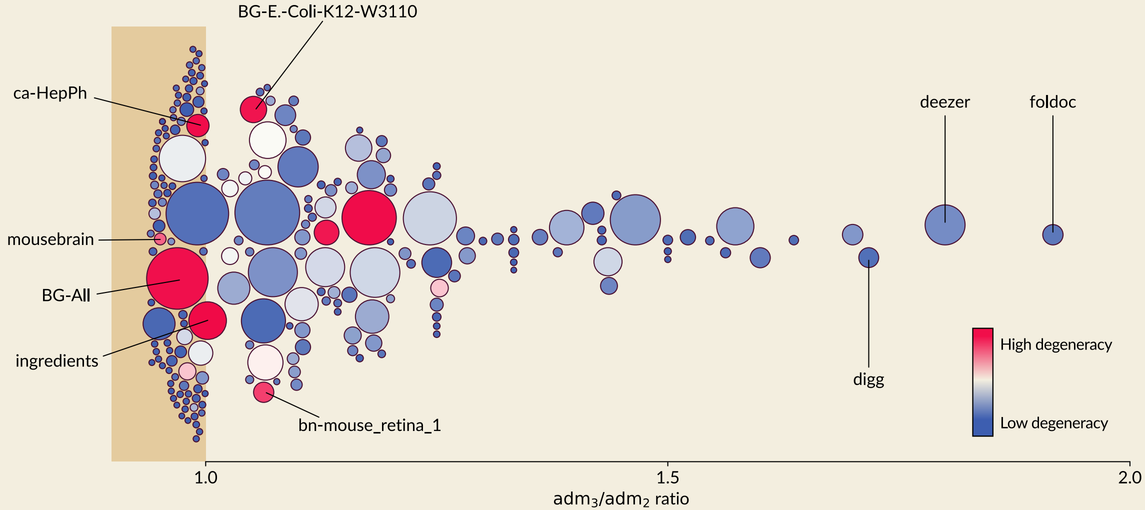
[FGLRVS15] [DRRSSS19]
- Many random graph models predict B.E.
  Chung-Lu & configuration model, random intersection model, block model, bounded degree+noise

On many networks, with larger r,

- [Rdl16] dtf-augmentations grow quickly
- [OS17] low-treedepth colourings grow *very* quickly
- [NPRRS18] scol and wcol grow pretty quickly

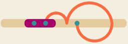
- It just makes sense!

The true value of admissibility



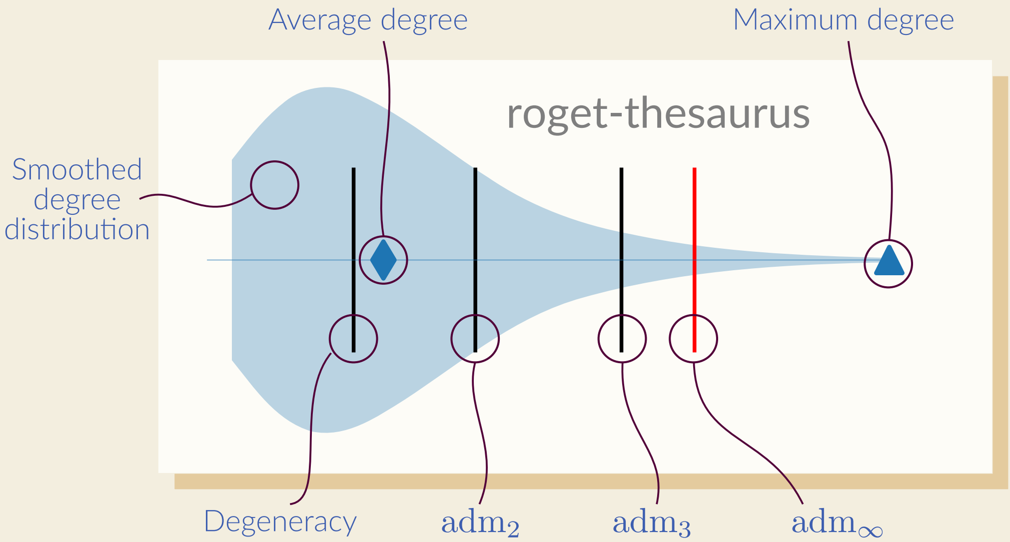
But four is hard?

Algorithmic uses

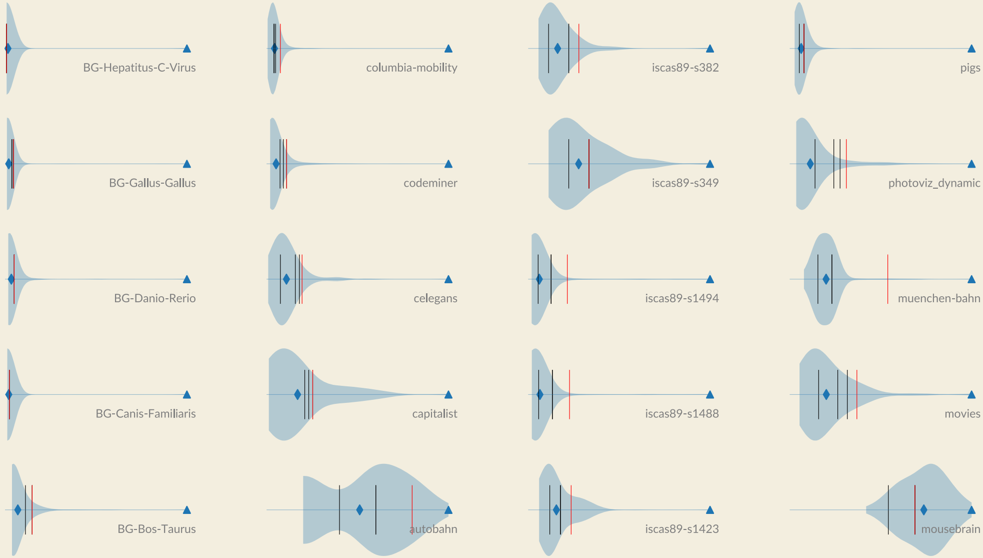
	Application	Computation
wcol_r  scol_r 	Quite natural for algorithm design once you get used to it	NP-hard for $r = 2$ [BKLS25]
adm_r 	Direct use?	Poly-time for $r \leq 3$ Greedy algorithm for ordering

r-admissibility can be computed in polynomial time for $r \in \{1, 2, 3, \infty\}$.

Real-world admissibility

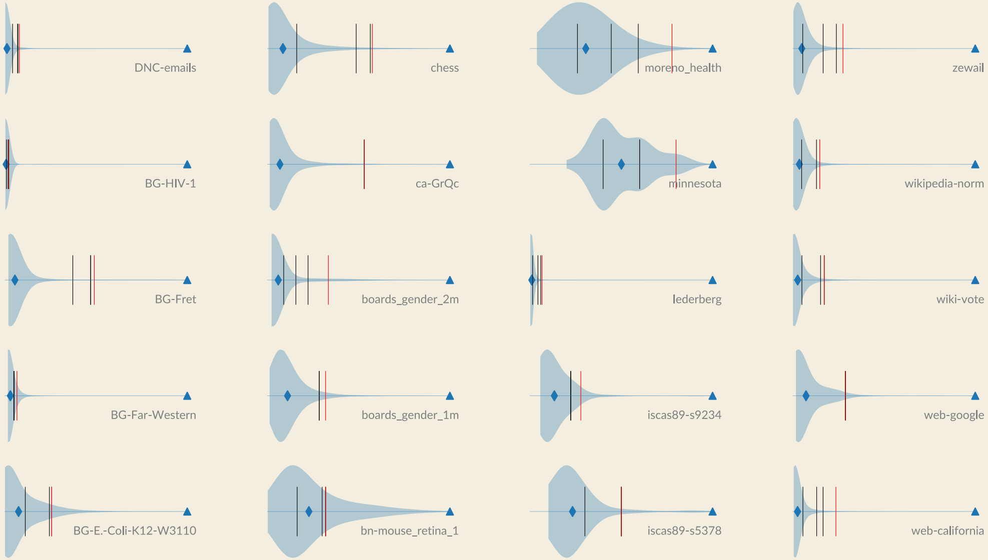


Real-world admissibility



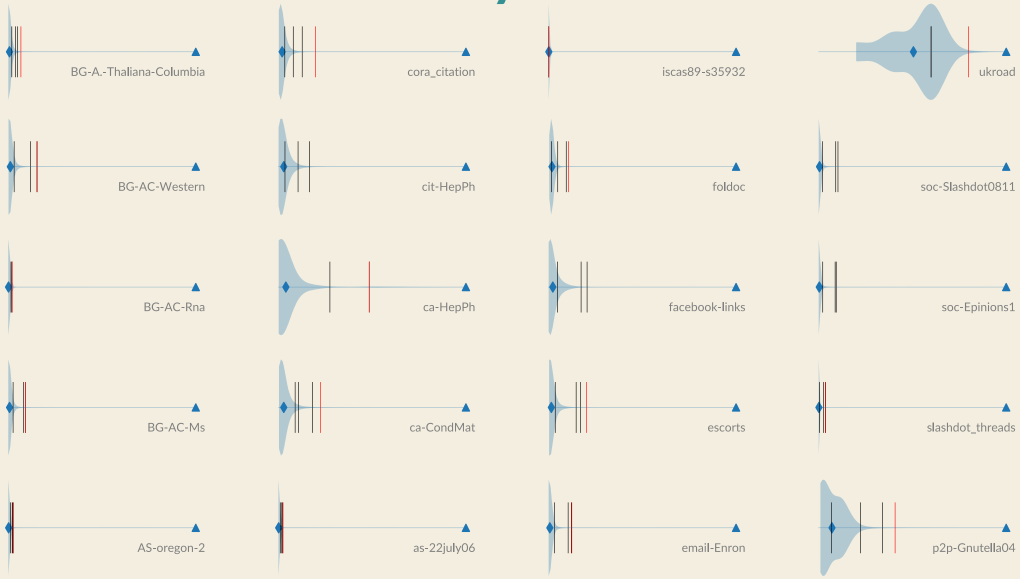
Selection of networks with less than 1000 nodes

Real-world admissibility



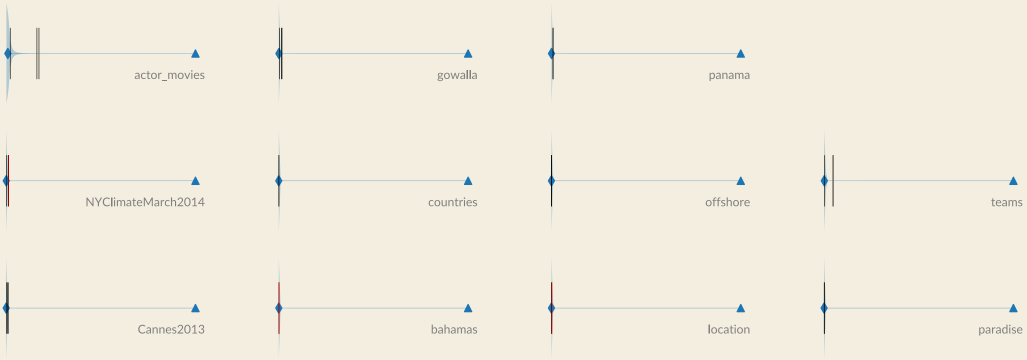
Selection of networks with less than 10 000 nodes

Real-world admissibility



Selection of networks with less than 100 000 nodes

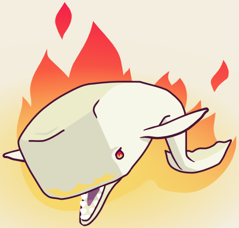
Real-world admissibility



Selection of networks with more than 100 000 nodes

The true value of admissibility

- Help spread the 'bounded expansion' toolkit to other communities
- Helps us gauge heuristics for more useful measures like scol/wcol
- Helps me finally settle/catch my burning whale:



The true value of admissibility

- Help spread the 'bounded expansion' toolkit to other communities
- Helps us gauge heuristics for more useful measures like scol/wcol
- Helps me finally set[illegible]tch my burning whale:

Do real-world networks have **'bounded expansion'?**

YES! Even better:

Most real-world networks have **small ∞-admissibility**

THANKS!

Questions?

Bibliography I

[AGLLR] Christine Awofeso et al. "A sufficient condition for characterizing the one-sided testable properties of families of graphs in the Random Neighbour Oracle Model". In: submitted.

[AGLLR25] Christine Awofeso et al. "Testing C_k-Freeness in Bounded Admissibility Graphs". In: *52nd International Colloquium on Automata, Languages, and Programming, ICALP 2025, July 8-11, 2025, Aarhus, Denmark*. Ed. by Keren Censor-Hillel et al. Vol. 334. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2025, 15:1–15:20. DOI: 10.4230/LIPICS.ICALP.2025.15. URL: https://doi.org/10.4230/LIPIcs.ICALP.2025.15.

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[CS19] Artur Czumaj and Christian Sohler. "A Characterization of Graph Properties Testable for General Planar Graphs with one-Sided Error (It's all About Forbidden Subgraphs)". In: *2019 IEEE 60th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE. 2019, pp. 1525–1548.

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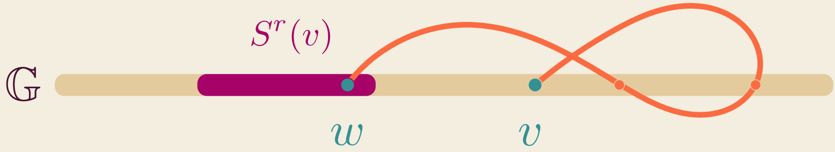
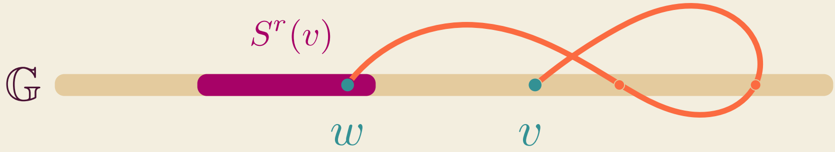
Strong colouring number



The set of *strongly* r*-reachable* vertices $S^r(v)$ from a vertex v contains all vertices w which can be reached via a path that

1) has length at most r, and
2) whose interior vertices are to the right of v.

Strong colouring number



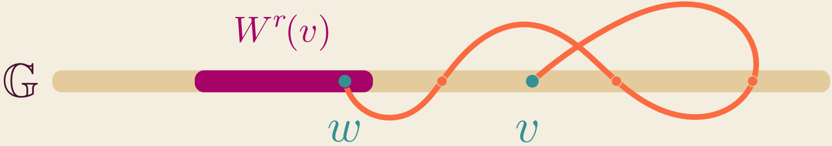
The *strong r-colouring number* $\mathrm{scol}_r(\mathbb{G})$ of an ordered graph $\mathbb{G}$ is

$$\mathrm{scol}_r(\mathbb{G}) = \max_{v \in \mathbb{G}} |S^r(v)|$$

The *strong r-colouring number* of a graph G is the minimum value over all its orderings:

$$\mathrm{scol}_r(G) = \min_{\mathbb{G} \in \pi(G)} \mathrm{scol}_r(\mathbb{G})$$

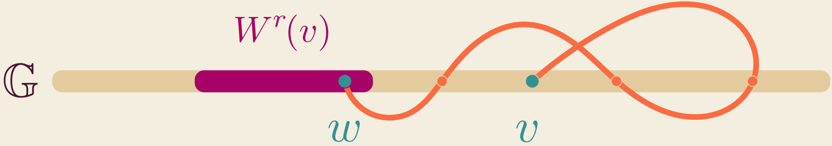
Weak colouring number



The set of *weakly* r*-reachable* vertices $W^r(v)$ from a vertex v contains all vertices w which can be reached via a path that

1) has length at most r, and
2) whose vertices are to the right of w.

Weak colouring number



The *weak r-colouring number* $\mathrm{wcol}_r(\mathbb{G})$ of an ordered graph $\mathbb{G}$ is

$$\mathrm{wcol}_r(\mathbb{G}) = \max_{v \in \mathbb{G}} |W^r(v)|$$

The *weak r-colouring number* of a graph G is the minimum value over all its orderings:

$$\mathrm{wcol}_r(G) = \min_{\mathbb{G} \in \pi(G)} \mathrm{wcol}_r(\mathbb{G})$$

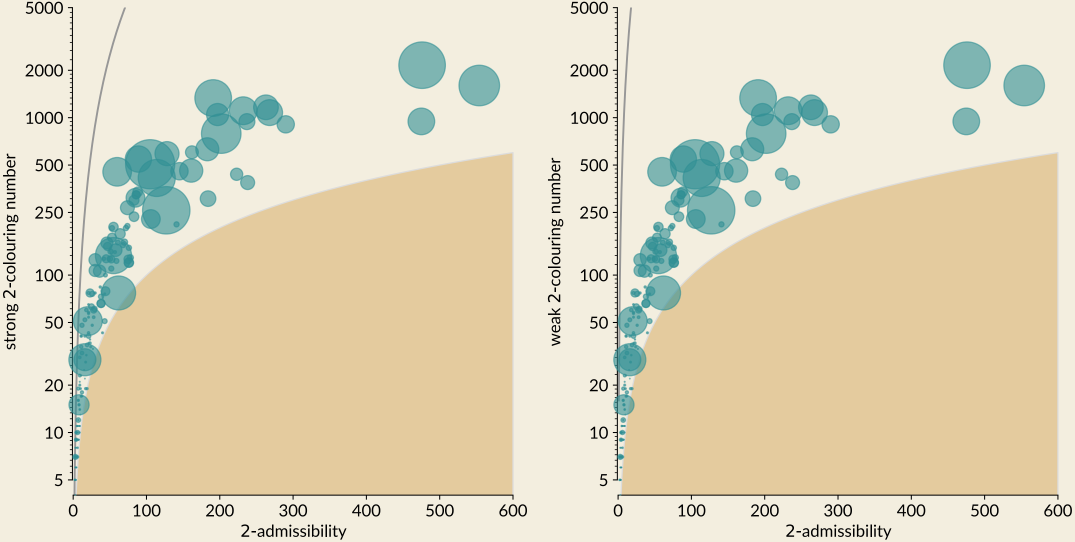
Computing good wcol/scol orders

In past experiments we found that sorting the vertices by descending degree or computing a simple degeneracy ordering often gives us a good wcol/scol ordering for small r.

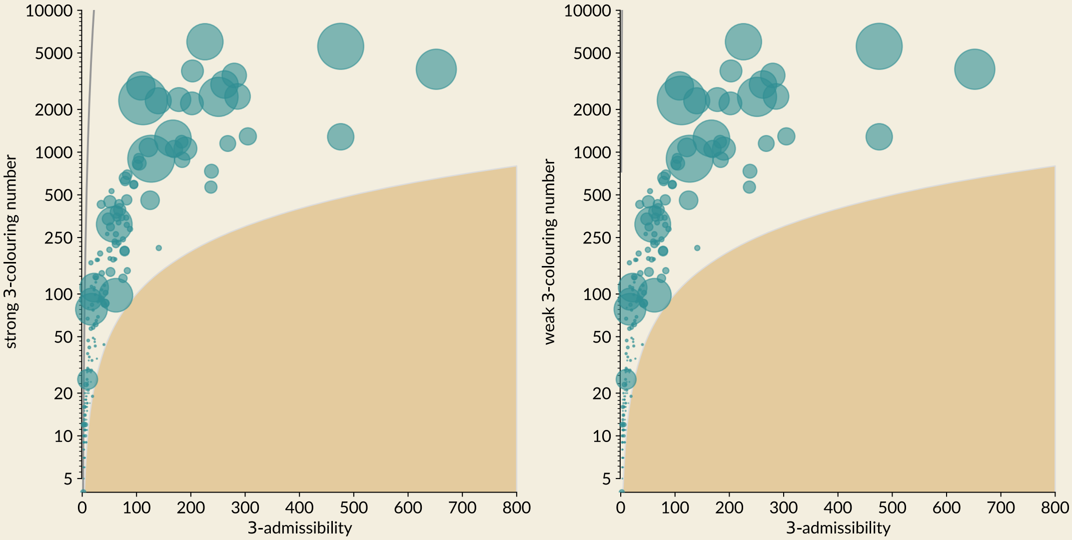
Empirical evaluation of approximation algorithms for generalized graph coloring and uniform quasi-wideness [NPRRS18]
W. Nadara, M. Pilipczuk, R. Rabinovich, FR, S. Siebertz (SEA'18)

But *how good* are these heuristics?

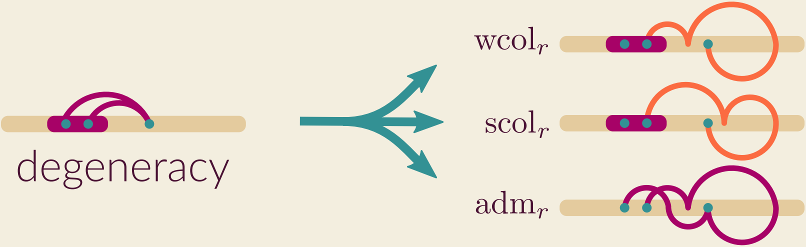
Computing good wcol/scol orders



Computing good wcol/scol orders



Lifting degeneracy



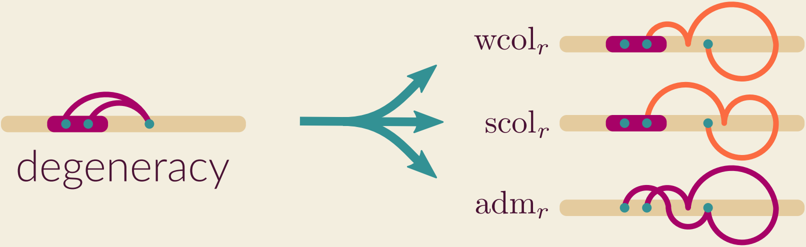
If $\mathbb{G}$ is a d-degenerate ordering, then for every vertex $v \in \mathbb{G}$ it holds that

$$|N^-(v)| = \mathrm{pp}^1(v) = |S^1(v)| = |W^1(v)|.$$

Therefore

$$d = \mathrm{adm}_1(\mathbb{G}) = \mathrm{scol}_1(\mathbb{G}) = \mathrm{wcol}_1(\mathbb{G}).$$

Lifting degeneracy



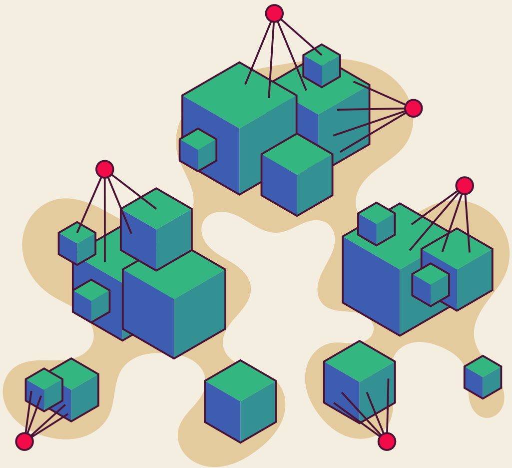
For every ordered graph $\mathbb{G}$, every integer $r \geq 1$ and vertex $v \in \mathbb{G}$ it holds that

$$\mathrm{pp}^r(v) \leq |S^r(v)| \leq |W^r(v)|.$$

Therefore

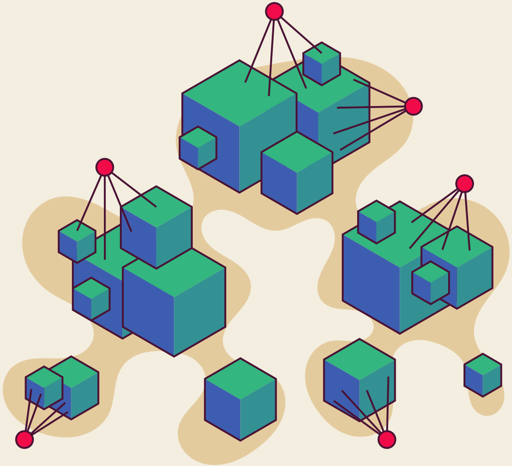
$$\mathrm{adm}_r(\mathbb{G}) \leq \mathrm{scol}_r(\mathbb{G}) \leq \mathrm{wcol}_r(\mathbb{G}).$$

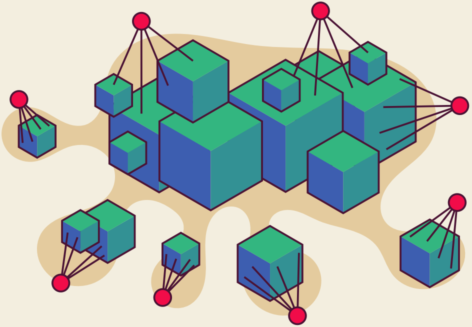
∞-admissibility decomposition



[Dvrk12]

∞-admissibility decomposition





‘Degenerate’ decomposition:
only one large bag

[Dvrk12]