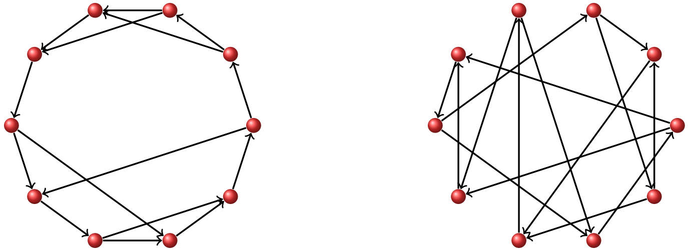
Isomorphism for Tournaments of Small Twin Width

Martin Grohe and Daniel Neuen

MPI Informatics

LogAlg 2025

Graph Isomorphism Problem



Graph Isomorphism Problem (GI)

Input: Two (directed) graphs G_1 and G_2
Problem: Decide whether $G_1 \cong G_2$

Existing Results

Many graph classes admit polynomial-time isomorphism tests:

- planar graphs [Hopcroft, Wong '74]
- graphs of maximum degree d [Luks '82]
- graphs of bounded tree-width [Bodlaender '88]
- graphs excluding a fixed minor [Ponomarenko '91]
- graphs excluding a fixed topological subgraph [Grohe, Marx '12]
- graphs of bounded rank width [Grohe, Schweitzer '15]

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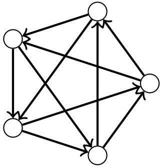
Theorem (Babai 2016)

The Graph Isomorphism Problem can be solved in time $n^{\text{polylog}(n)}$.

Tournament Isomorphism

Tournaments: For every pair of distinct vertices v, w either (v, w) or (w, v) is an edge.



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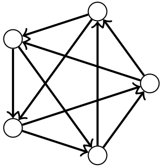


The Tournament Isomorphism Problem seems to be easier:

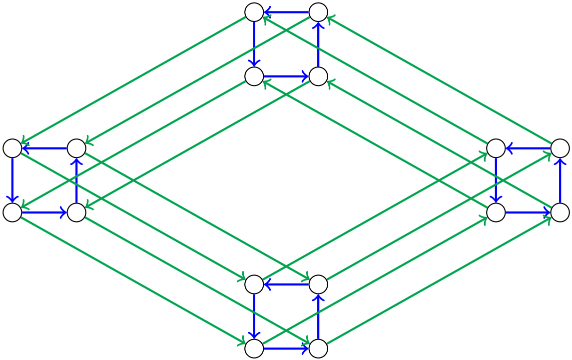
- $\mathrm{Aut}(T)$ is solvable for every tournament T
- randomized reduction from tournament isomorphism to tournament rigidity [Schweitzer '17]

Theorem (Babai, Luks 1983)

The Tournament Isomorphism Problem can be solved in time $n^{O(\log n)}$.

Isomorphism for *k*-Spanning Tournaments

An edge-colored tournament is *d*-spanning if the union of all edge-color classes of maximum degree *d* is strongly connected.



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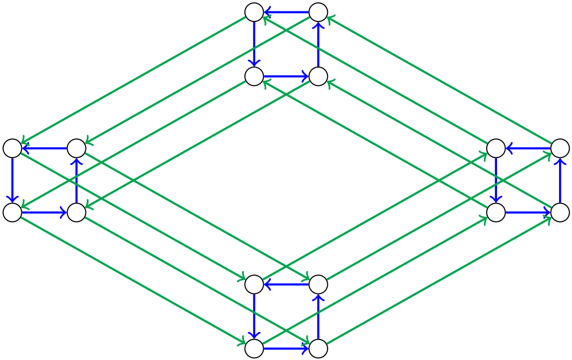
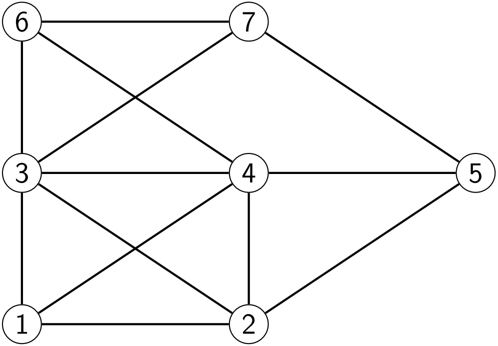


Theorem (Arvind, Ponomarenko, Ryabov 2022)

Isomorphism for d-spanning tournaments can be tested in $d^{O(\log d)} n^{O(1)}$.

Twin Width [Bonnet et al., 2020]

Contraction Sequence: Repeatedly merge vertices v, w; introduce red edges to all vertices in $N(v) \oplus N(w)$.



The **twin width** of G is the minimum k for which there is a contraction where every graph has red degree at most k.

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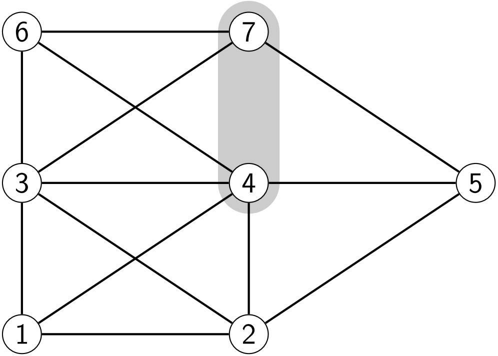

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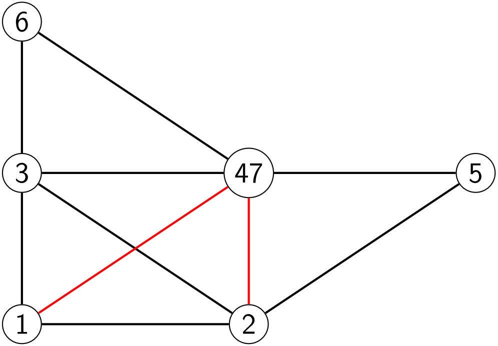

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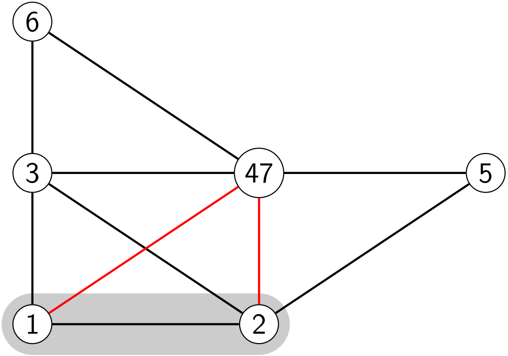

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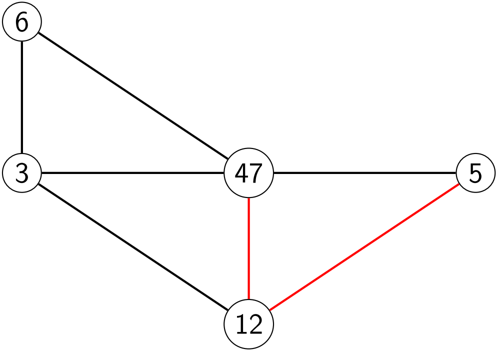

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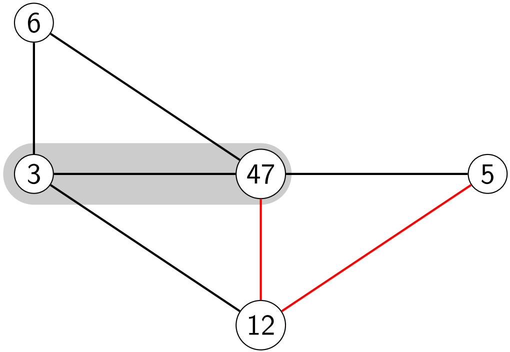

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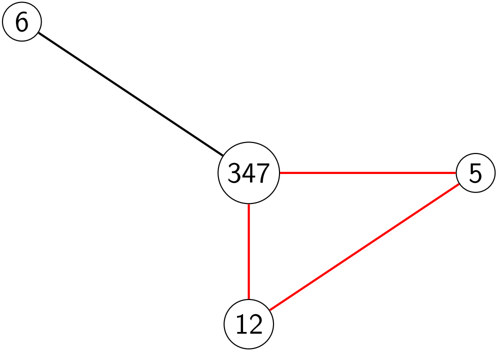

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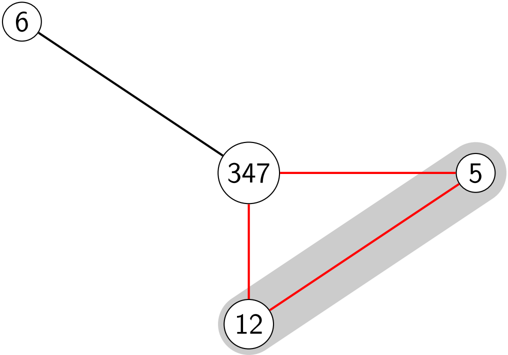

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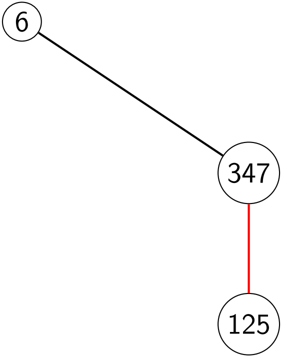

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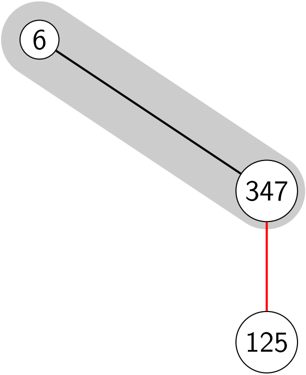

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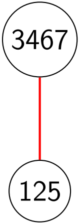

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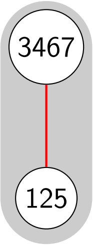

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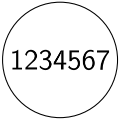

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Isomorphism for Graphs of Bounded Twin Width

Isomorphism for graphs of bounded twin width is GI complete.

Theorem

Graph Isomorphism $\leq_{PTIME}$ *Graph Isomorphism for graphs of twin width* ≤ 4

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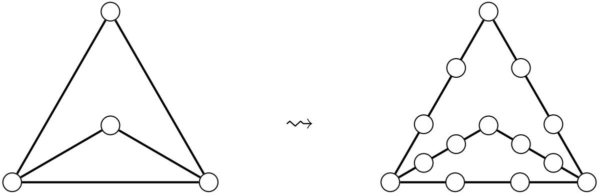
Proof: Replace every edge with path of length $\Omega(\log n)$.



The resulting graph has twin width ≤ 4 [Bergé et al., 2022].

Isomorphism for Tournaments of Bounded Twin Width

Theorem

Isomorphism for tournaments of twin width k can be tested in time $k^{O(\log k)} n^{O(1)}$.

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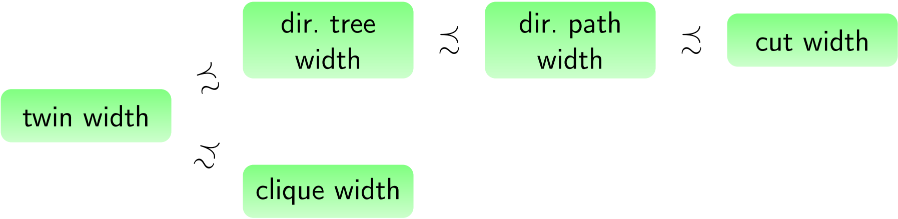
Connections between graph parameters on tournaments:



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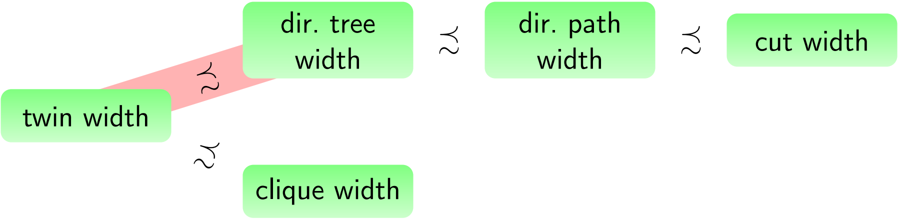

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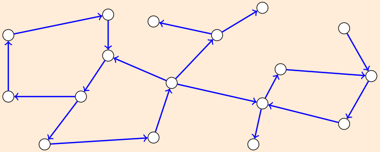
Overview on Algorithm

Phase 1: **Structure Extraction**

- compute coloring of edges
- find sequence of colors that induce a "near bounded degree" structure



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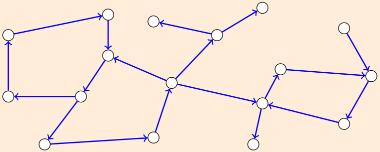
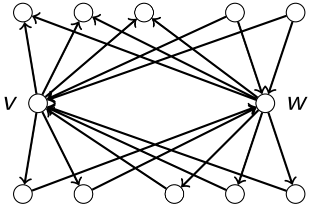


Phase 2: **Group-Theoretic Methods**

- compute isomorphisms based on "near bounded degree" structure
- use group-theoretic methods dating back to [Luks 1982]
- extend methods from [Arvind et al., 2022]

Mixed Neighbors

We define $M(v, w) \coloneqq (N_+(v) \cap N_-(w)) \cup (N_-(v) \cap N_+(w))$



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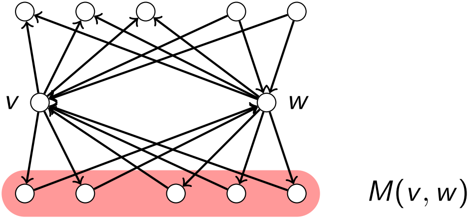

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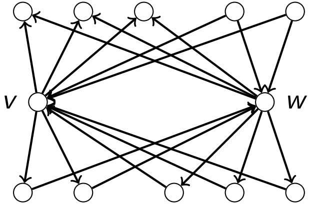
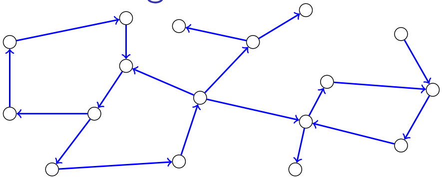


Lemma

There are $v, w \in V(T)$ such that $|M(v, w)| \leq \text{tww}(T)$.

Finding a Structure of Near Bounded Degree

We mark all edges (v, w) such that $M(v, w) \leq \text{tww}(T)$.



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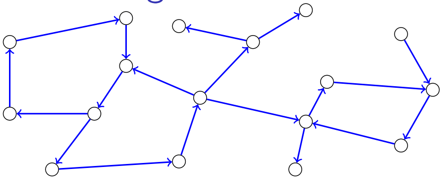
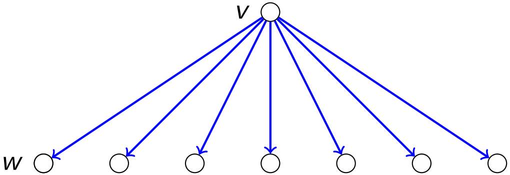


Lemma

The marked subgraph has maximum out-degree $2 \cdot \text{tww}(T) + 1$.



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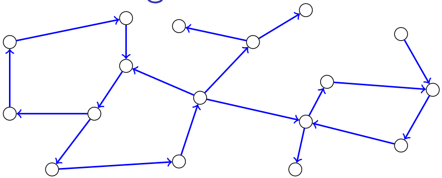

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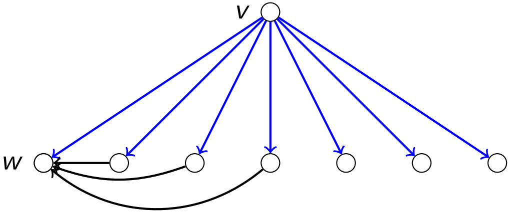

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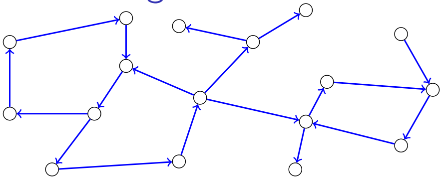

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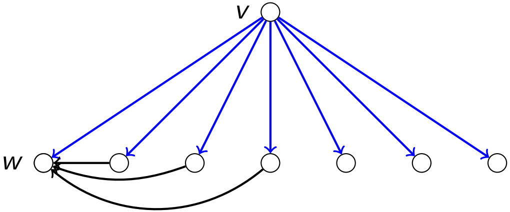
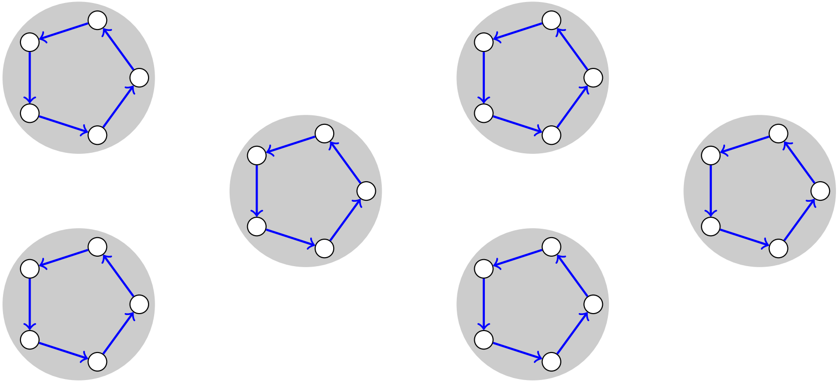


If the marked subgraph is connected, we can use the algorithm from [Arvind et al., 2022] to decide isomorphism.

Finding a Structure of Near Bounded Degree

If the marked subgraph is not connected, we mark further edges between components.



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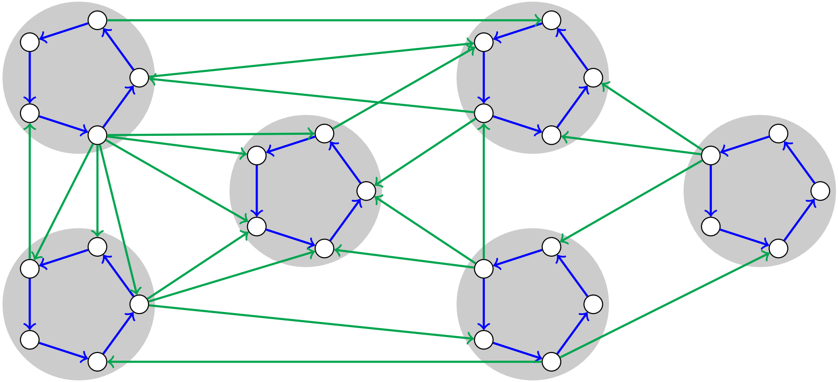

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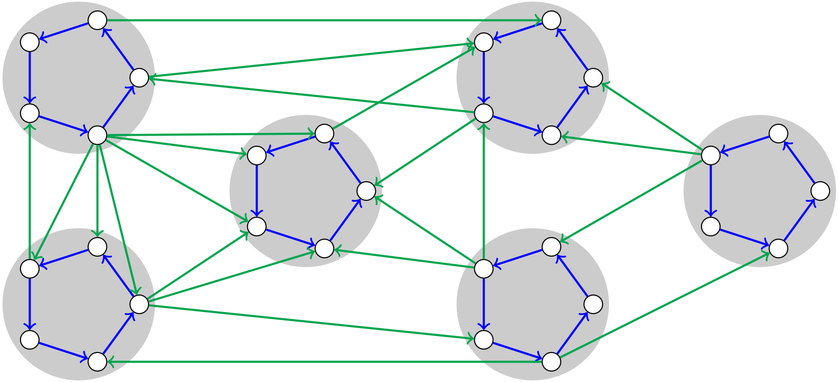

Each vertex has green edges to at most $2\,\mathrm{tww}(T) + 1$ many blue components.

Isomorphism for Tournaments of Bounded Twin Width

Theorem

Isomorphism for tournaments of twin width k can be tested in time $k^{O(\log k)}n^{O(1)}$.

Theorem

The k-dimensional Weisfeiler-Leman algorithm fails to decide isomorphism of tournaments of twin width 35 for $k = o(n)$.

Open Questions

Theorem

Isomorphism for tournaments of twin width k can be tested in time $k^{O(\log k)} n^{O(1)}$.

Open Questions:

- Isomorphism for tournaments with bounded VC dimension of the set system $\{N_+(v) \mid V(T)\}$?
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I Am Hiring

One Postdoc & one PhD position at TU Dresden