Transductions of Graph Classes Admitting Product Structure

Petr Hliněný, Jan Jedelský
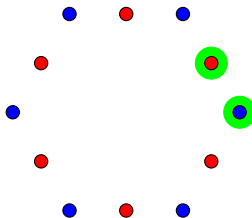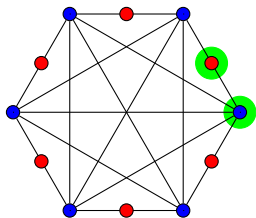
Masaryk University, Brno, Czechia

LOGALG 2025, Vienna, Austria

Outline

- First-order transductions
  - Interpretations, locality, and flips
- Global structure of planar graphs
  - Planar Product Structure Theorem
- Generalizing product structure to dense setting
  - $\mathcal{H}$-clique-width
- The main result
- Corollaries

$\phi(u, v) \equiv$ "(u and v are red and they share common blue neighbor),
or (u and v are adjacent and exactly one of them is red)"

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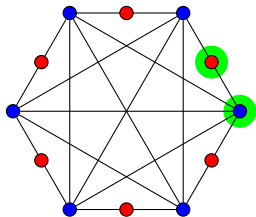

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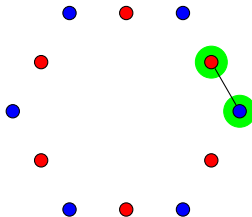G $\qquad\qquad$ $\phi(G)$

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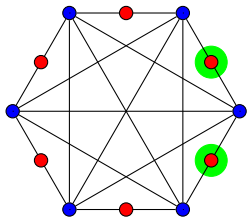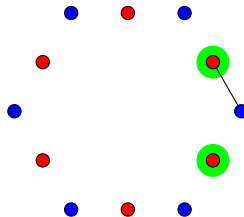
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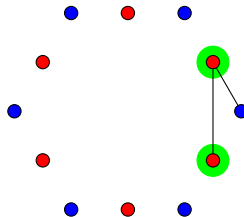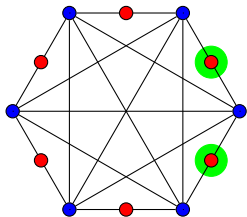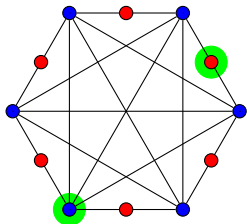G $\qquad$ $\phi(G)$
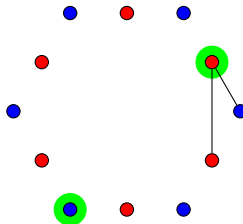
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- FO is local – recall the theorems of Gaifman and Hanf
$\implies$ every FO transduction can be split into local and global part
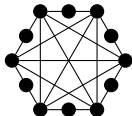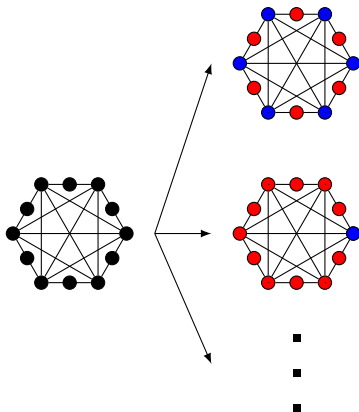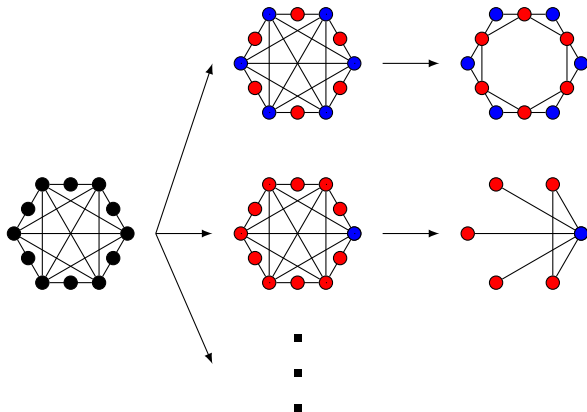
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Definition (Local part)

A transduction τ is *strongly r-local* if it does not create edges between vertices x, y at distance greater than r, and the existence of an edge between x and y depends only on the union of r-neighborhoods of x and y.

Definition (Glocal part)

A graph H is a *k-flip* of a graph G if H can be obtained by coloring G using k colors and possibly flipping adjacency (edges became non-edges and vice versa) between some color classes.

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Theorem (Nešetřil, Ossona de Mendez, Siebertz)

If a graph class $\mathcal{C}$ is FO-transducible from a class $\mathcal{D}$ (without copying), then there are number k and r, and a strongly r-local transduction τ such that $\mathcal{C}$ is contained in a k-flip of $\tau(\mathcal{D})$.

Planar Product Structure Theorem
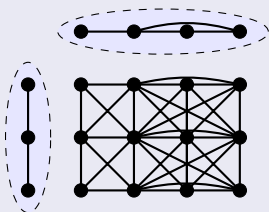
Definition (Strong product $G \boxtimes H$ of graphs G and H)

- $V(G \boxtimes H) = V(G) \times V(H)$
- $[g_1, h_1][g_2, h_2] \in E(G \boxtimes H)$ if one of the following conditions holds:
  - $g_1 = g_2$ and $h_1 h_2 \in E(H)$,
  - $g_1 g_2 \in E(G)$ and $h_1 = h_2$, or
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Definition (Product structure)

A graph class $\mathcal{C}$ *admits product structure* if there is a constant k such that every graph $G \in \mathcal{C}$ is a subgraph of the strong product $P \boxtimes M$ of a path P and a graph M of tree-width at most k.

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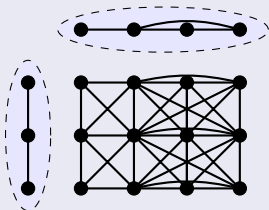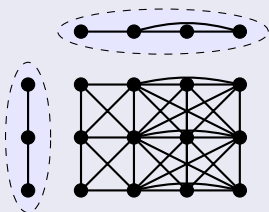

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Theorem (Dujmovic, Joret, Micek, Morin, Ueckerdt, Wood)

Planar graphs as well as graphs embeddable on a fixed surface admit product structure.

Dense analogue of product structure

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Definition (Product structure for dense graphs)

A graph class $\mathcal{C}$ *admits* **hereditary** *product structure* if there is a constant k such that every graph $G \in \mathcal{C}$ is an **induced** subgraph of the strong product $P \boxtimes M$ of a path P and a graph M of clique-width at most k.

Dense analogue of product structure

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A graph class C *admits product structure* if there is a constant k such that every graph $G \in C$ is a subgraph of the strong product $P \boxtimes M$ of a path P and a graph M of tree-width at most k.

Definition (Product structure for dense graphs)

A graph class C *admits* **hereditary** *product structure* if there is a constant k such that every graph $G \in C$ is an **induced** subgraph of the strong product $P \boxtimes M$ of a path P and a graph M of clique-width at most k.

Theorem

Let C be a graph class admitting product structure, and let D be a graph class FO-transducible from C. Then, D is a flip of some graph class D' which admits hereditary product structure.

Product structure – another view
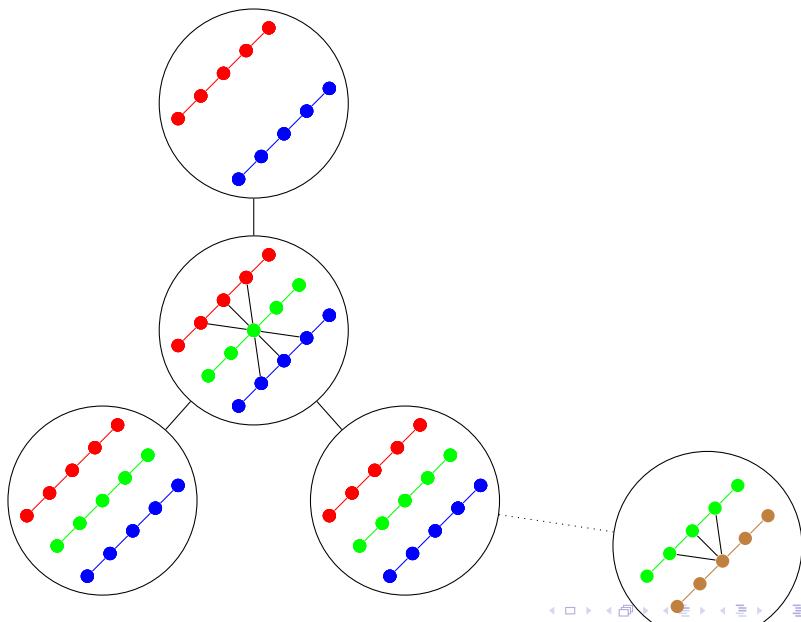
Hereditary product structure = $\mathcal{H}$-clique-width

Definition

A graph G has *clique-width* at most k if there is a k-expression valued G.

k-expression: k colors and the following operations:

- Given $c \in [k]$, create a graph having single vertex with *color c*
- Take disjoint union
- Given a pair of colors $c_1 \neq c_2$, add edges between every pair of vertices u, v satisfying that:
    - color of u is c_1, and
    - color of v is c_2

- Recolor c_1 to c_2

Hereditary product structure $= \mathcal{H}$-clique-width

Definition

A graph G has *$\mathcal{H}$-clique-width* at most k if there is a loop graph $H \in \mathcal{H}$ and a (H,k)-expression valued G. If no such expression exists, then we say that $\mathcal{H}$-Clique-Width is ∞.

(H,k)-expression: k colors and the following operations:

- Given $c \in [k]$ and $p \in V(H)$, create a graph having single vertex with *color c* and *parameter vertex $p \in V(H)$*
- Take disjoint union
- Given a pair of colors $c_1 \neq c_2$, add edges between every pair of vertices u, v satisfying that:
  - color of u is c_1, and
  - color of v is c_2, and
  - the parameter vertices of u and v are adjacent in H
- Recolor c_1 to c_2 **without** changing parameter vertices

Example: 2D grid

- $\mathcal{P}^\circ :=$ reflexive paths (all vertices have loop)
- 2D grid has $\mathcal{P}^\circ$-clique-width at most 5
- Create path colored "modulo 2":

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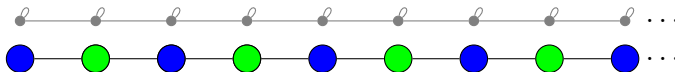- Create path colored "modulo 2":

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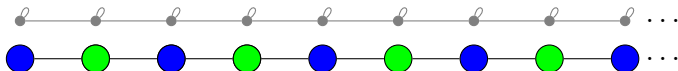- Create path colored "modulo 2":

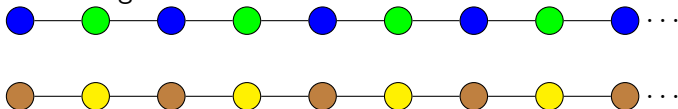## Example: 2D grid

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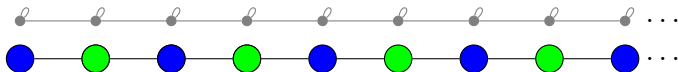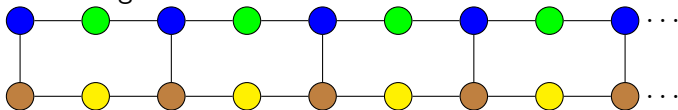- Create path colored "modulo 2":

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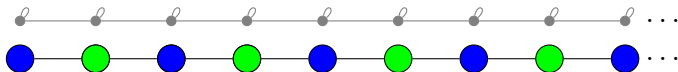## Example: 2D grid

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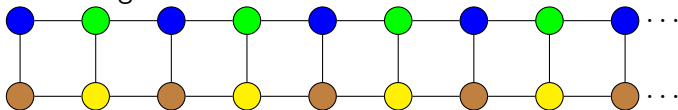- Create path colored "modulo 2":

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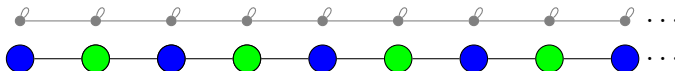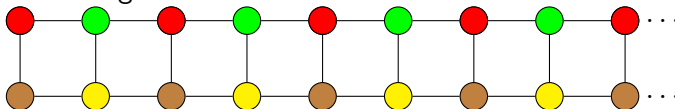- Create path colored "modulo 2":

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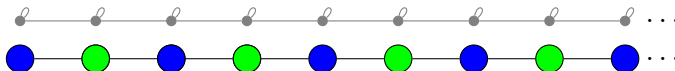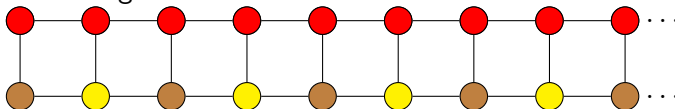- Create path colored "modulo 2":

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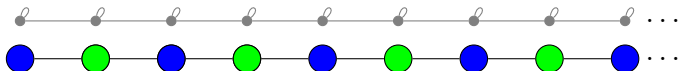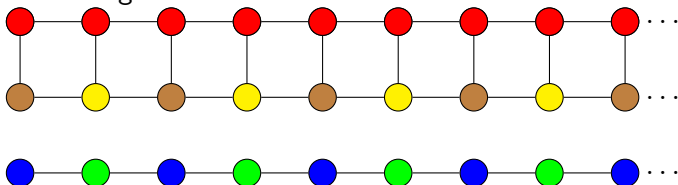Create a grid:

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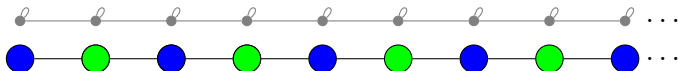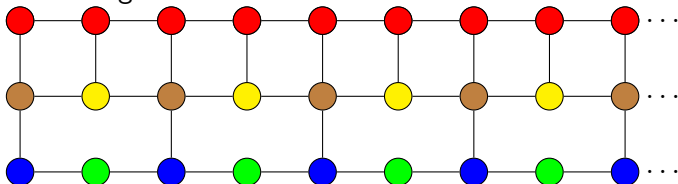- Create path colored "modulo 2":

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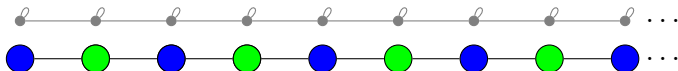

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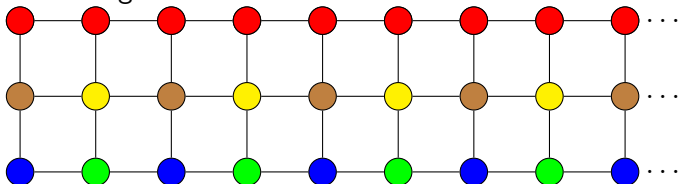
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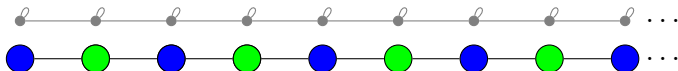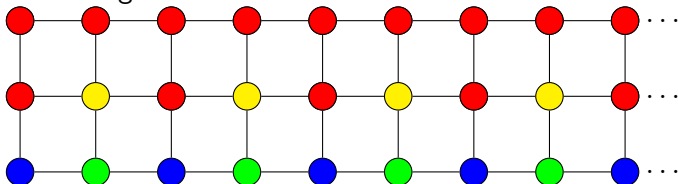

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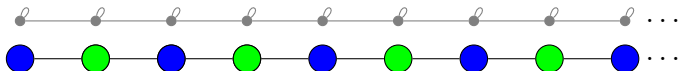- Create path colored "modulo 2":

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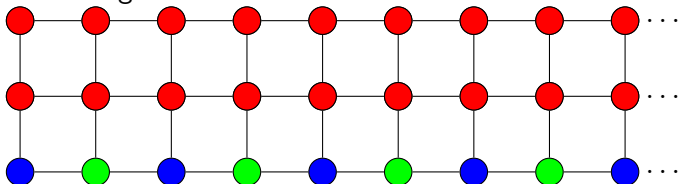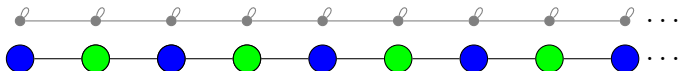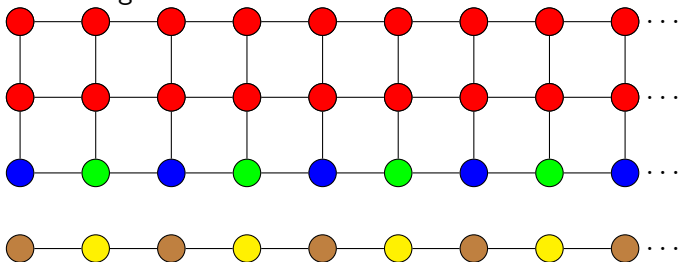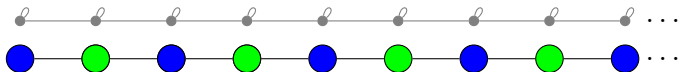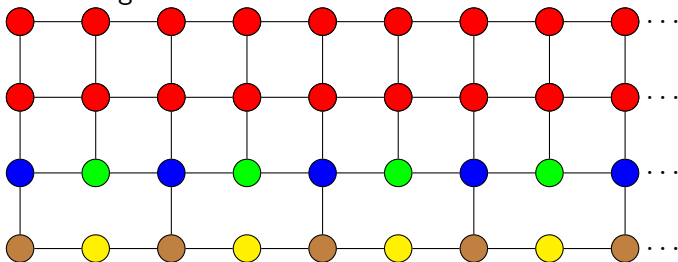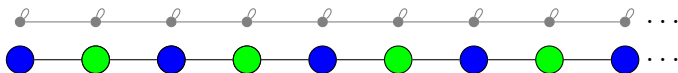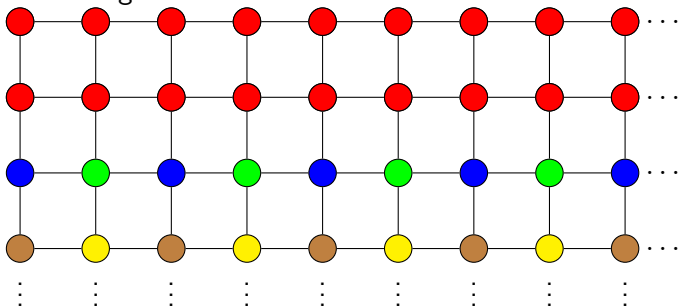
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Main result (again and more generally)

Let $\mathcal{Q}$ be a class of bounded degree simple graphs. We denote by $\mathcal{Q}_r^\circ$ the reflexive closure of the r-th power of $\mathcal{Q}$.

Definition

A graph class $\mathcal{C}$ admits $\mathcal{Q}$-product structure if there is a constant k such that every graph $G \in \mathcal{C}$ is a subgraph of the strong product $Q \boxtimes M$ of a graph $Q \in \mathcal{Q}$ and a graph M of tree-width at most k.

Theorem

Let $\mathcal{C}$ be a graph class admitting $\mathcal{Q}$-product structure (e.g. planar graphs). Let $\mathcal{D}$ be a graph class FO-transducible from $\mathcal{C}$. Then, there are constants k, ℓ, and r such that $\mathcal{D}$ is contained in a k-flip (global path) of a class with $\mathcal{Q}_r^\circ$-clique-width at most ℓ (local part).

Definition (Product structure for dense graphs)

A graph class $\mathcal{C}$ *admits hereditary product structure* if there is a constant k such that every graph $G \in \mathcal{C}$ is an induced subgraph of the strong product $P \boxtimes M$ of a path P and a graph M of clique-width at most k.

Theorem

Let $\mathcal{D}$ be a class of bounded stable clique-width (stable = does not FO-transduce all half-graphs). Let $\mathcal{C}$ be a class admitting hereditary product structure such that, the graph M from the above definition can be chosen from $\mathcal{D}$. Then, there is a class $\mathcal{G}$ admitting (the classical) product structure such that $\mathcal{C}$ is FO-transducible from $\mathcal{G}$.

Corollaries – 3D grids

Theorem

The class of all 3D grids is not FO-transducible from planar graphs.

Proof.

- Any balanced bipartiton A, B ($|A| \leq 2|B| \leq 4|A|$) of $a \times a \times a$ grid $G_{a \times a \times a}$ induces a matching of size $\Omega(a^2)$
- $G_{a \times a \times a}$ has diameter $\Theta(a)$
- If a is large enough, then any k-flip of $G_{a \times a \times a}$ contains a large induced subgraph H of diameter $d \in \mathcal{O}_k(a)$ such that any balanced bipartiton of H induces a matching or anti-matching of size $m \in \Omega_k(a^2)$
- Suppose that there is (P, ℓ)-expression ϕ valued H
- Some node of ϕ corresponds to balanced bipartition A, B but the maximum size of both matching and anti-matching at every node is at most $\mathcal{O}(\ell \cdot d) = \mathcal{O}_\ell(a)$

□

Conclusions

- Transductions of graph classes admitting product structure (subgraphs of Path $\boxtimes$ bnd. Tree-Width) are k-flips of a class admitting hereditary product structure (induced subgraphs of Path $\boxtimes$ bnd. Clique-Width)

- $\mathcal{C}$ admits hereditary product structure $\iff$ $\mathcal{C}$ has bounded Path$^\circ$-clique-width

- Using (H, k)-expressions, it is easy to prove some non-transducibility results – eq. 3D grids are not transducible from planar graphs