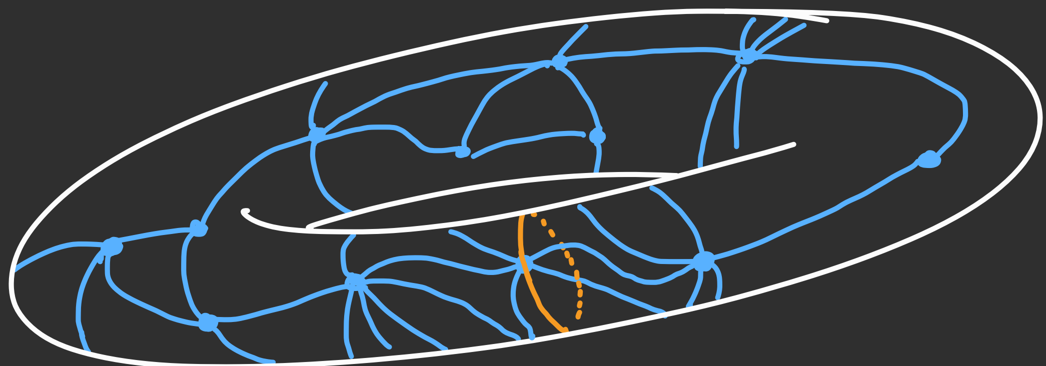
Catching Rats in H-minor-free graphs



joint work with: Giannos Stamoulis, Dimitrios Thilikos, Sebastian Wiederrecht
Université Paris Cité, CNRS, LIRMM, CNRS, KAIST

Width Parameters

Most of the work is done for branchwidth.

Because embedded graphs interact with branchwidth better than treewidth.

(Because of the rats!)

<u>Proposition</u> [Robertson & Seymour '91]

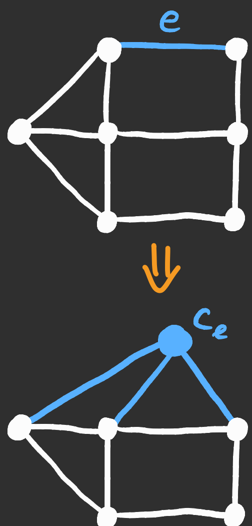
Let G be a non-acyclic graph.

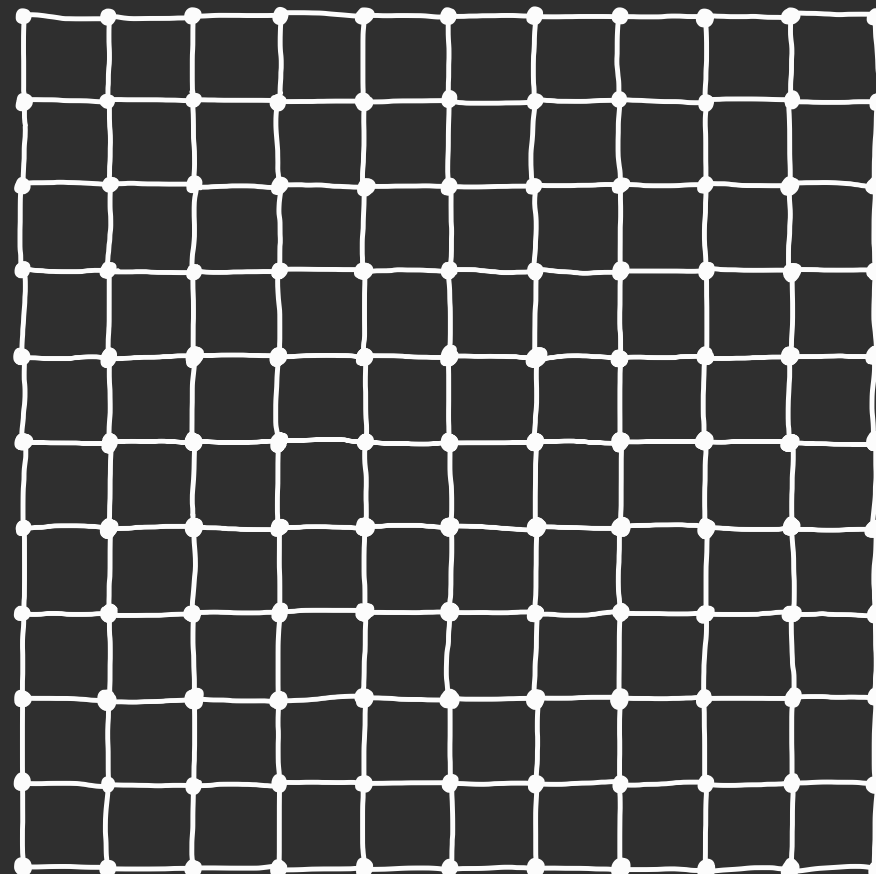
Then $\underbrace{bw}_{\text{branchwidth}}(G) - 1 \leq \underbrace{tw}_{\text{treewidth}}(G) \leq \lfloor 3/2 \; bw(G) \rfloor - 1.$

When does a graph have high treewidth?

Minor

Take a subgraph and contract some edges.





11×11-grid

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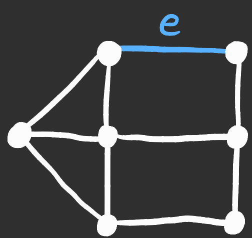
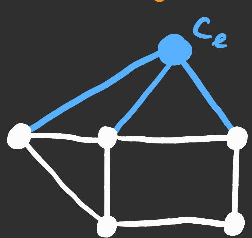
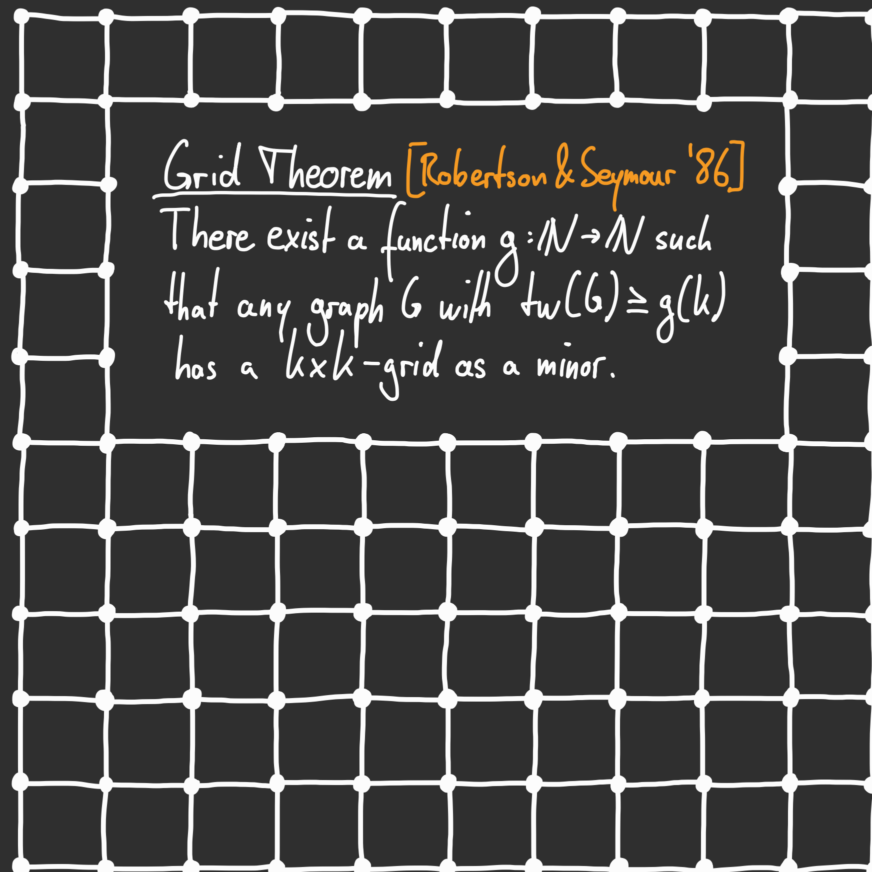


$\Downarrow$





Grid Theorem [Robertson & Seymour '86]

There exist a function $g : \mathbb{N} \to \mathbb{N}$ such that any graph G with $tw(G) \geq g(k)$ has a $k \times k$-grid as a minor.

11×11-grid

$n \times n$-grids have treewidth roughly n

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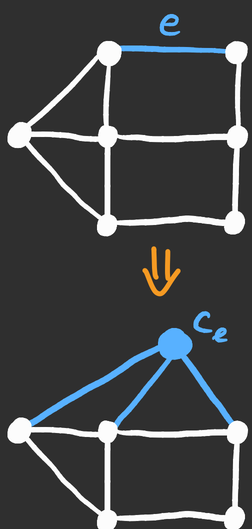

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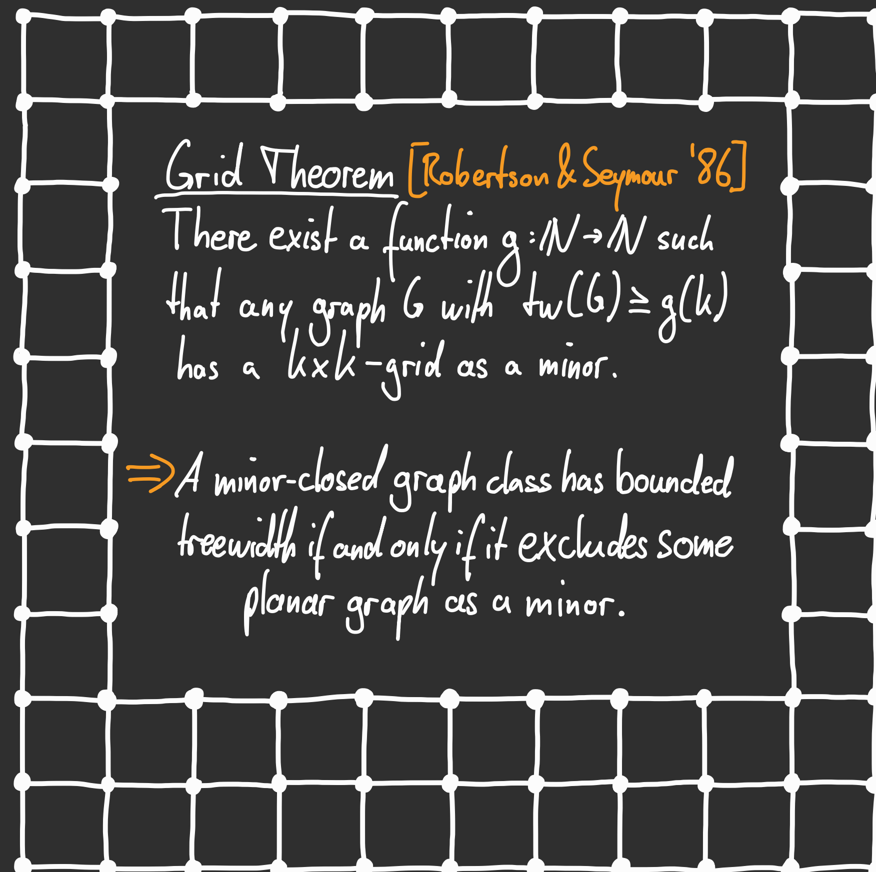

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$\Rightarrow$ A minor-closed graph class has bounded treewidth if and only if it excludes some planar graph as a minor.

The relationship between treewidth and grid-minors

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upper bounds

- $g(k) \in 2^{O(k^5)}$ [Robertson, Seymour & Thomas '94]
- $g(k) \in 2^{O(k^2 \log k)}$ [Leaf & Seymour '15]
- $g(k) \in O(k^{98} \log^{c'} k)$ [Chekuri & Chuzhoy '16]
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What if we exclude additional non-planar minors?

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What if we exclude additional non-planar minors?

- also exclude K_5 and $K_{3,3}$
  $\Rightarrow g(k) \leq 6k - 5$ [Robertson, Seymour & Thomas '94] + [Wagner '37]
- also exclude K_5 (or $K_{3,3}$)
  $\Rightarrow g(k) \leq 6k - 5$ [Robertson, Seymour & Thomas '94] + [Wagner '37] (+ Hall)

Finding a large grid- or clique-minor

ca. 2000

Diestel's proof for the grid theorem:
For any graph G, if $tw(G) \in \Omega(k^{t^2 k})$
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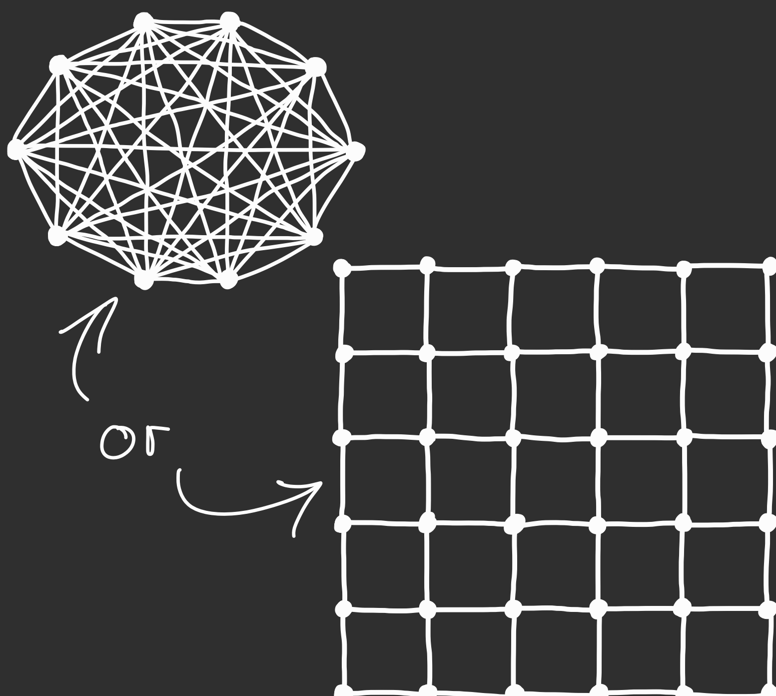

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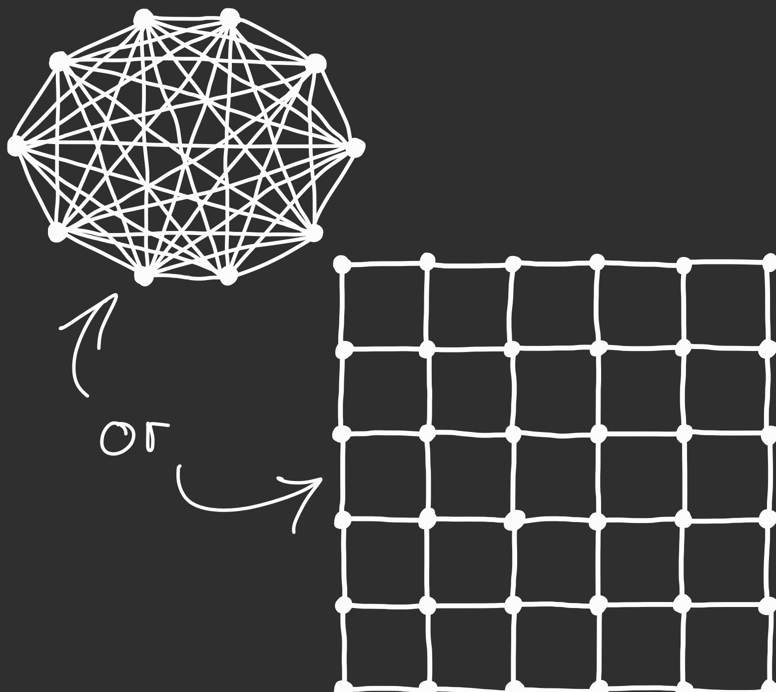

Actually, this relation is linear in k!

- $tw(G) \in \Omega_t(k)$ [Demaine & Hajiaghayi '08]

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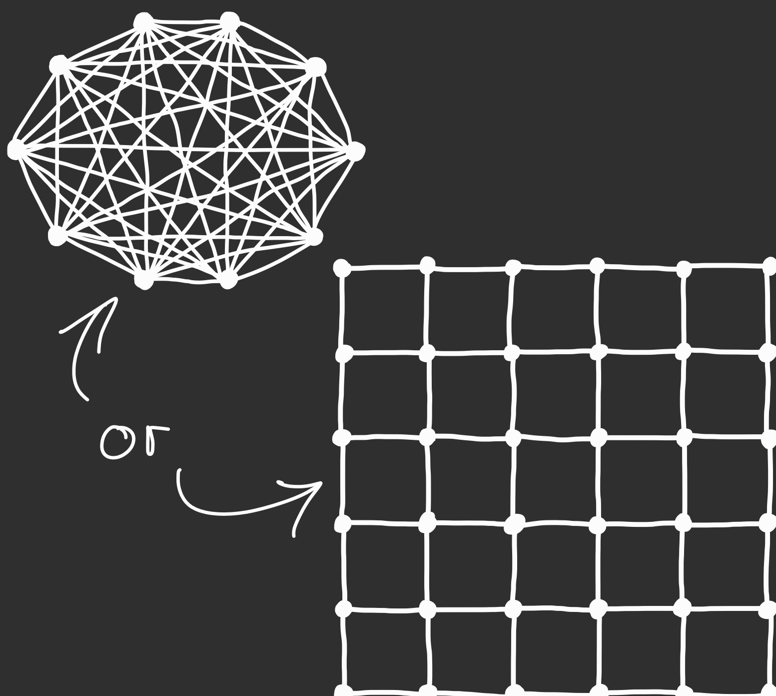

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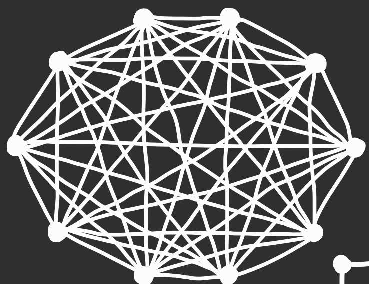

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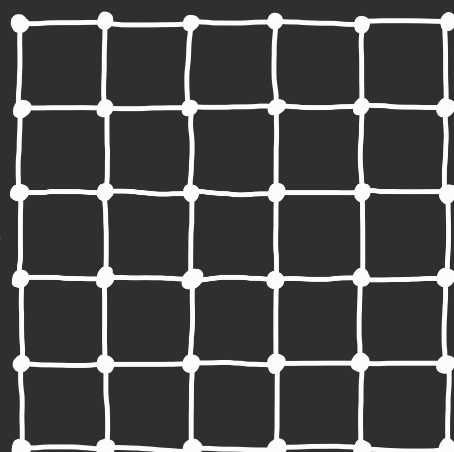

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- $tw(G) \in 2^{\Omega(t \log t)} k$ [Kawarabayashi & Kobayashi '20]
- $tw(G) \in \Omega(t^2 k + t^{2304})$ [G., Stamoulis, Thilikos, & Wiederrecht '25+]

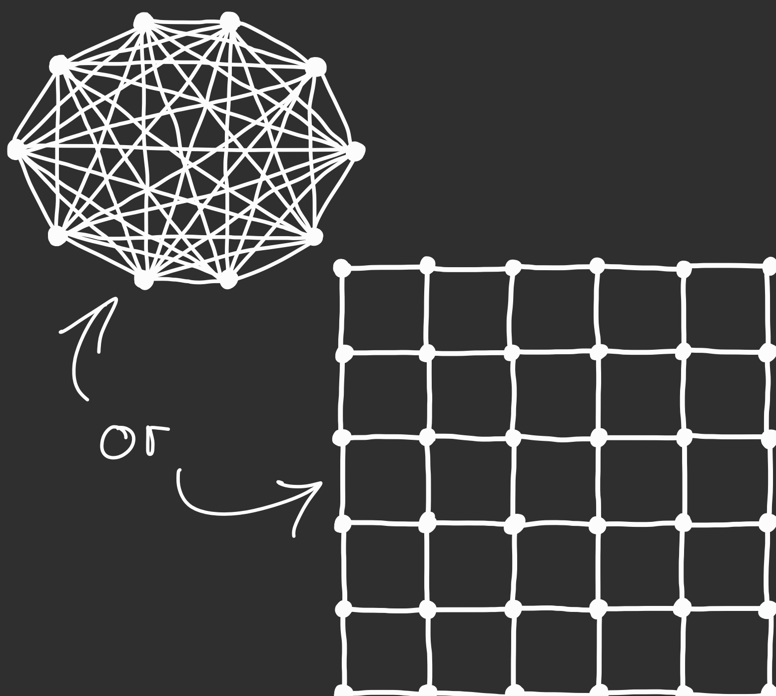
general answer to

What if we exclude additional non-planar minors?

Finding a large grid- or clique-minor

Theorem [G., Stamoulis, Thilikos, & Wiederrecht '25+]

For any graph G, if $tw(G) \in \Omega(t^2 k + t^{2304})$ then G contains a K_t- or a $(k \times k)$-grid-minor.



In fact in $2^{poly(t)} poly(|V(G)|)$-time we can find:

- a K_t-minor
- a $(k \times k)$-grid-minor, or
- a branch-decomposition of G with approximately the correct width for G.

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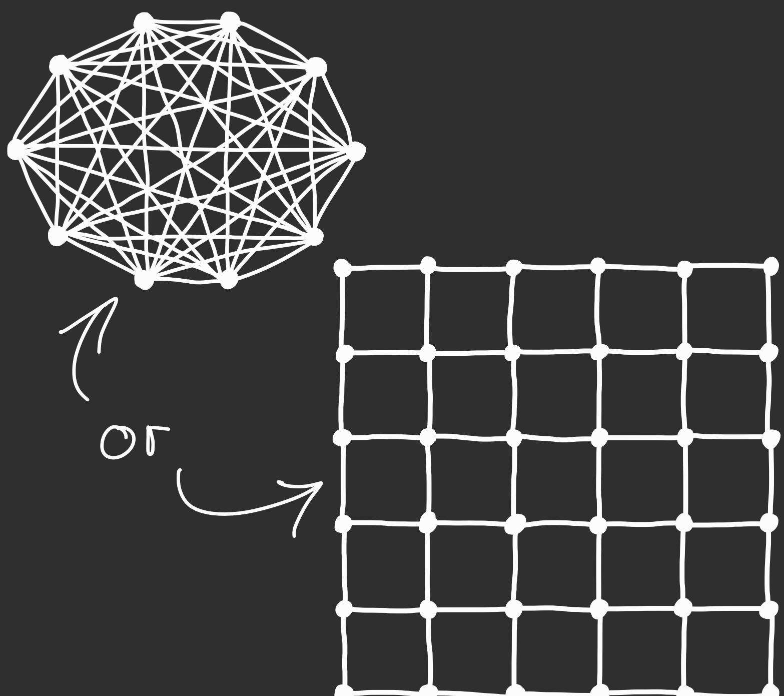

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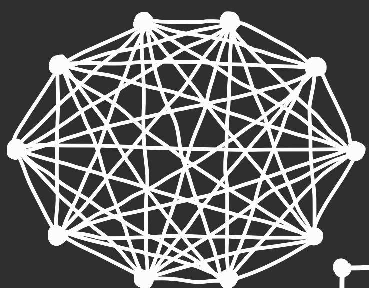

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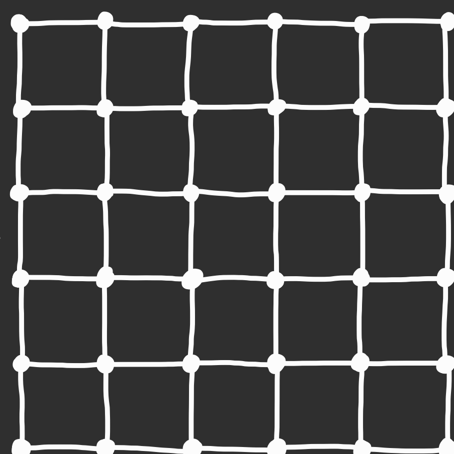

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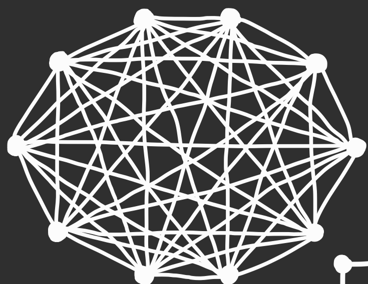

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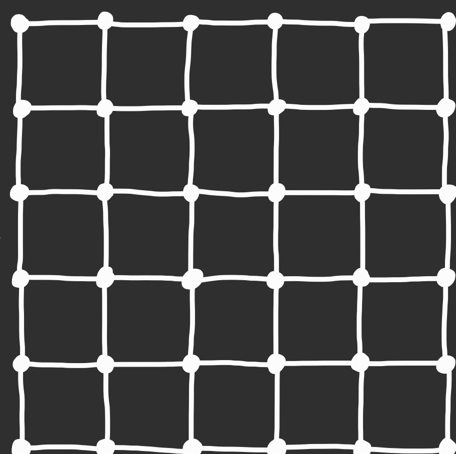

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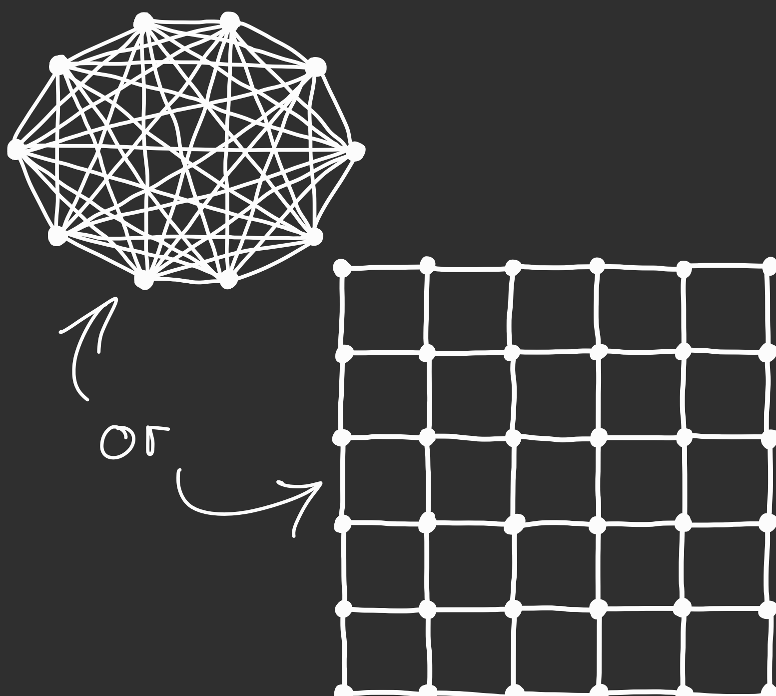

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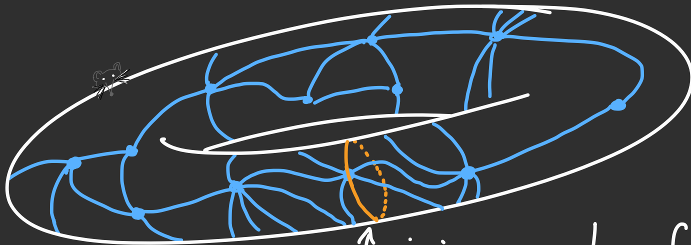
branchwidth can be verified to either be at least k or $\in O(t^2k + t^{2302})$

Embedded graphs

representativity (of an embedded graph):



minimum number of vertices intersected by a non-contractible curve

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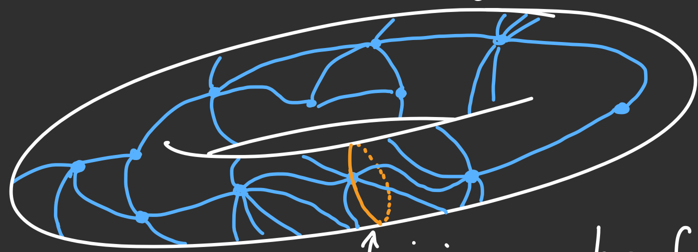

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Theorem [Demaine, Fomin, Hajiaghayi & Thilikos '05]
Let G be an embedded graph (on a surface with positive genus) with representativity $4k$.
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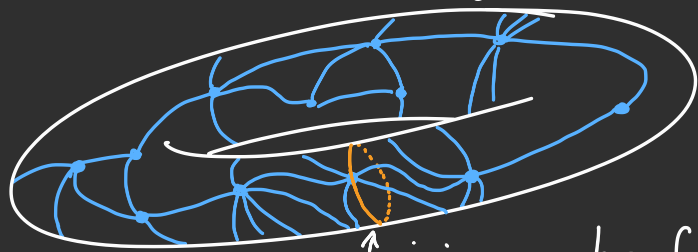

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Lemma [folklore]
Let G be a graph and $X \subseteq V(G)$ s.t. $bw(G-X) \geq 2$.
Then $bw(G-X) \geq bw(G) - |X|$.

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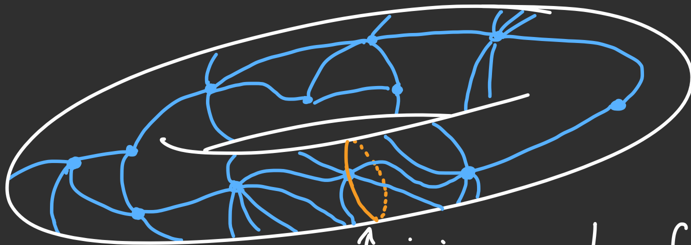

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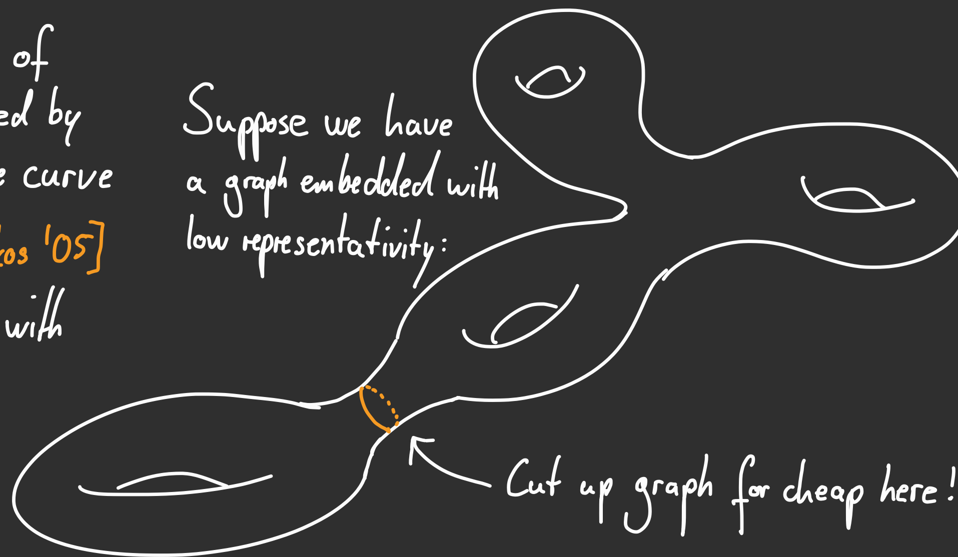
Suppose we have a graph embedded with low representativity:



Cut up graph for cheap here!

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The bounded genus case

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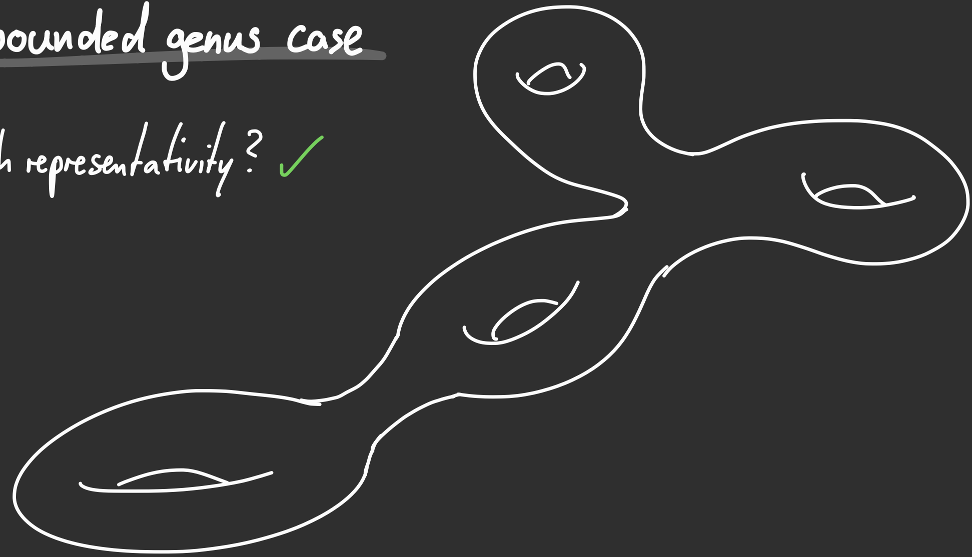
High representativity? ✓



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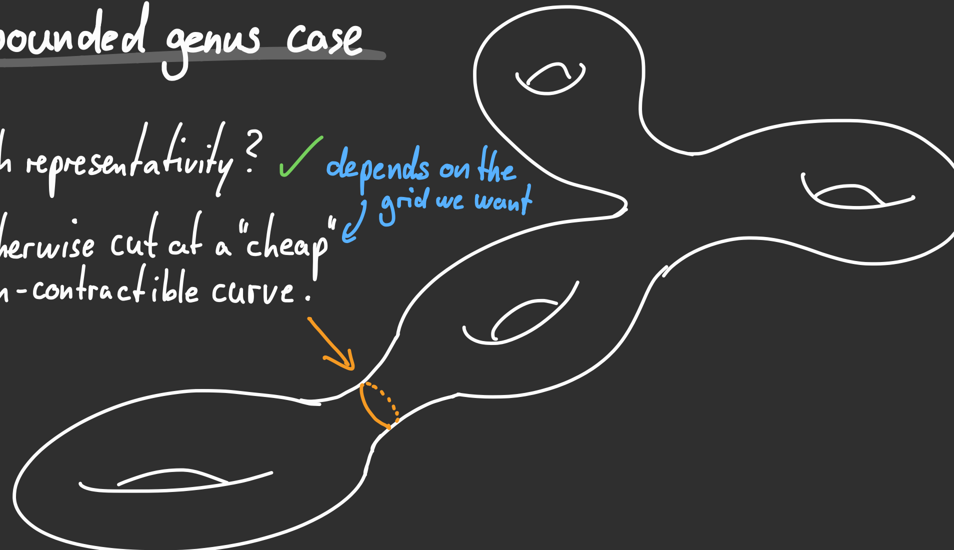
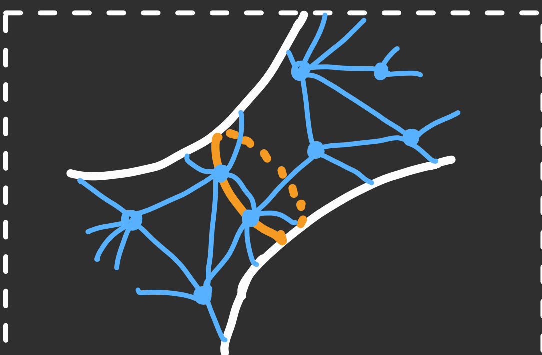
High representativity? ✓ depends on the grid we want

Otherwise cut at a "cheap" non-contractible curve.





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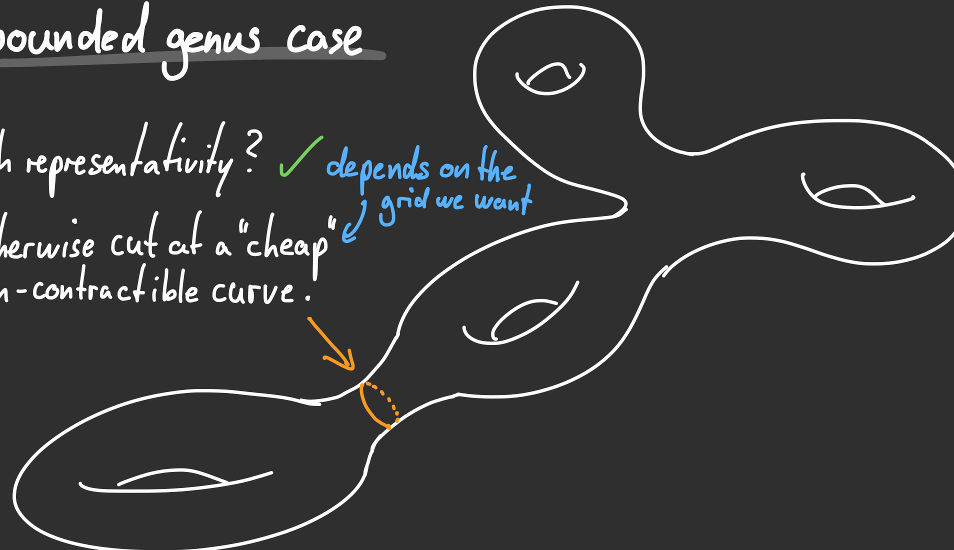

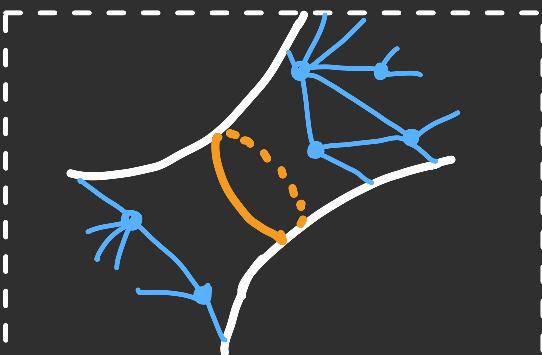

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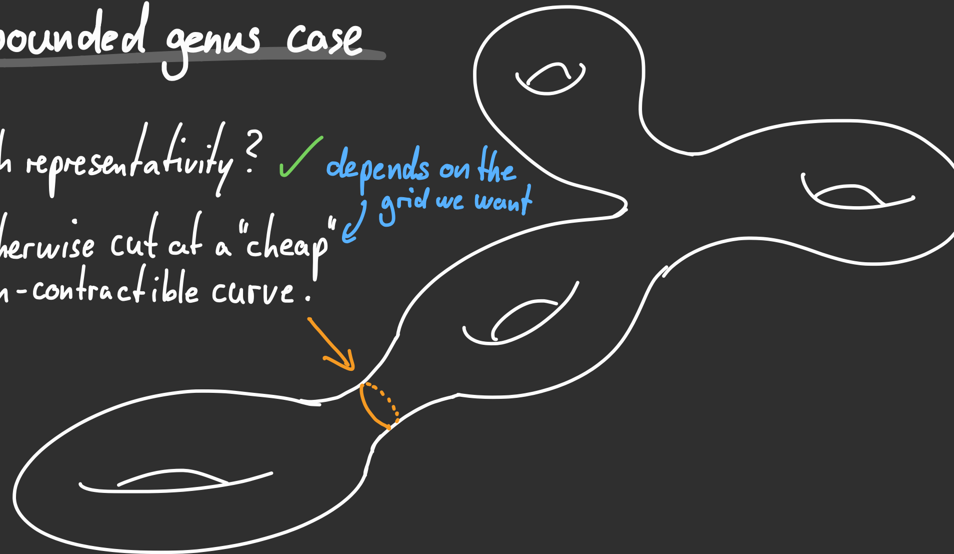

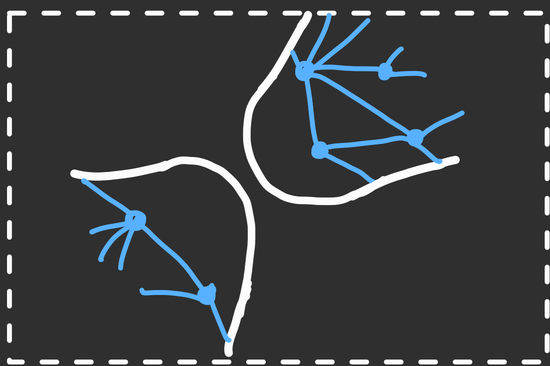

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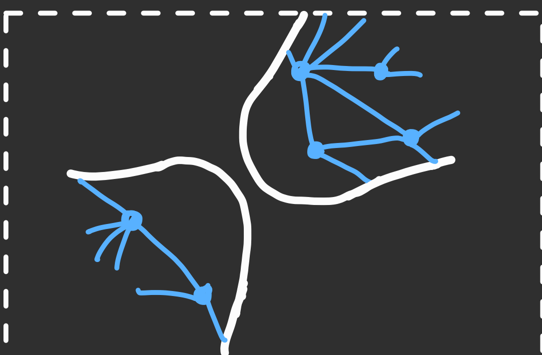

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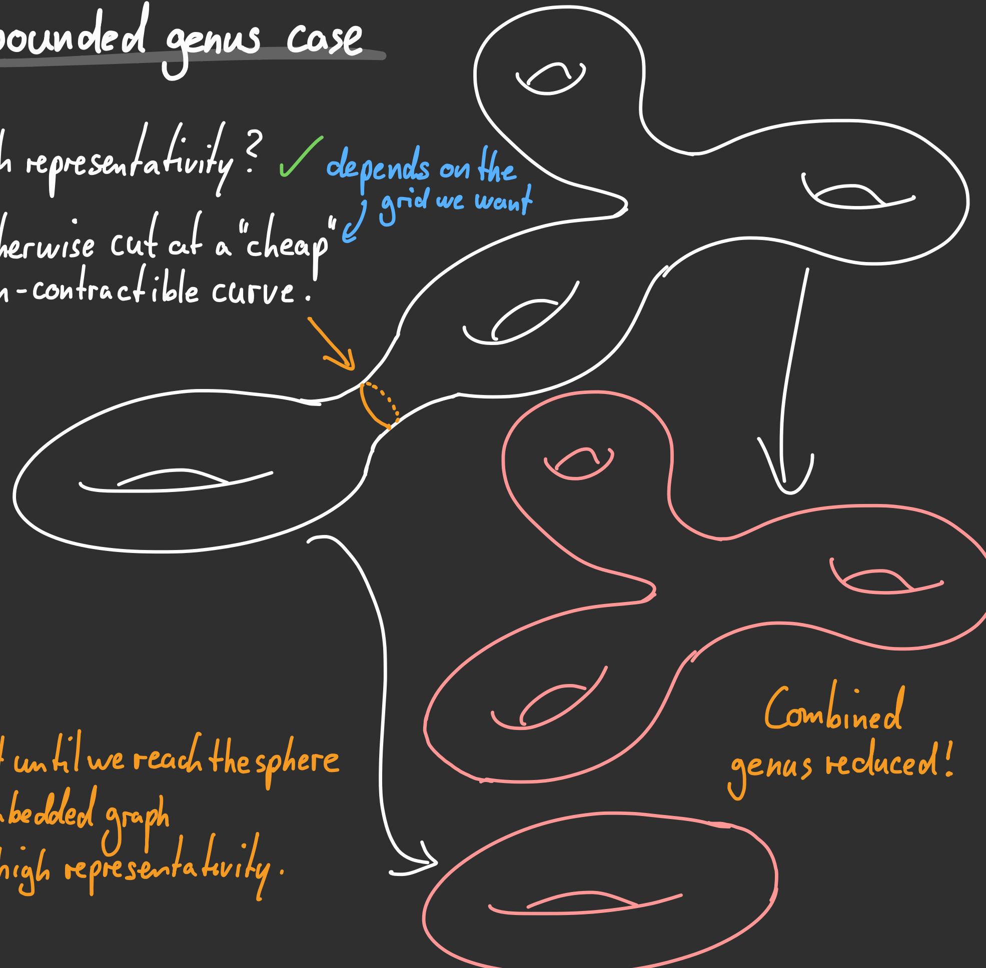

Repeat until we reach the sphere or embedded graph with high representativity.

The Graph Minor Structure Theorem

Theorem [Original – Robertson & Seymour '03]

There exists a function f such that for every graph H, ← $t := |V(H)|$
every H-minor-free G can be constructed via $\leq f(t)$-sums
from graphs that are built via:

i) taking a graph embedded "up to 3-separations" in a surface Σ into which H does not embed,

ii) adding $f(t)$ vortices of width at most $f(t)$, and

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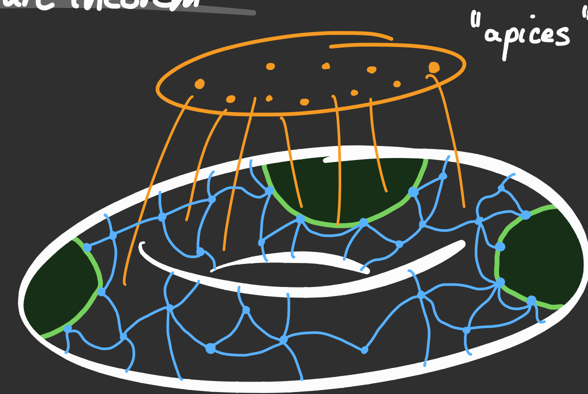

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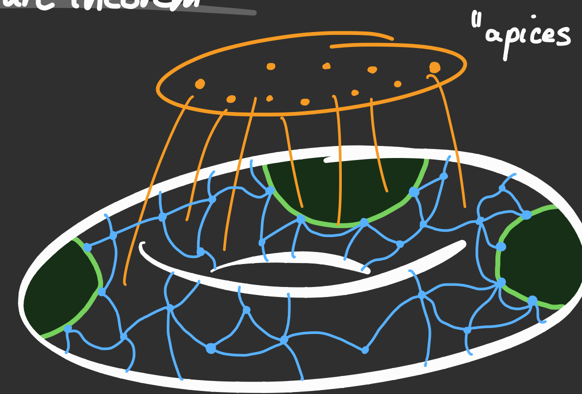
gives a special tree-decomposition of G

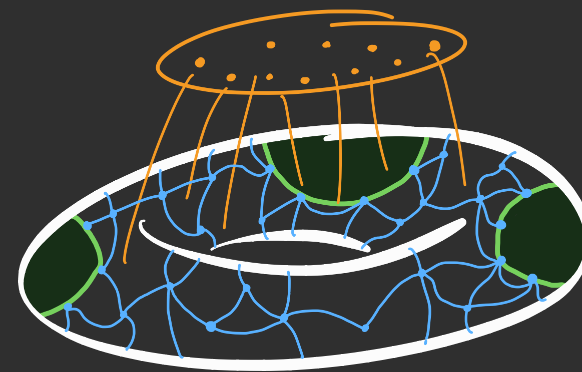


$f \in \mathcal{O}(t^{2300})$ [G., Seweryn & Wiederrecht 25'+] and * this part of the graph is a minor of G. Proved using [Grohe '16]

Apices:

<u>Lemma</u> [folklore]
Let G be a graph and $X \subseteq V(G)$ s.t. $bw(G-X) \geq 2$.
Then $bw(G-X) \geq bw(G) - |X|$.

General case



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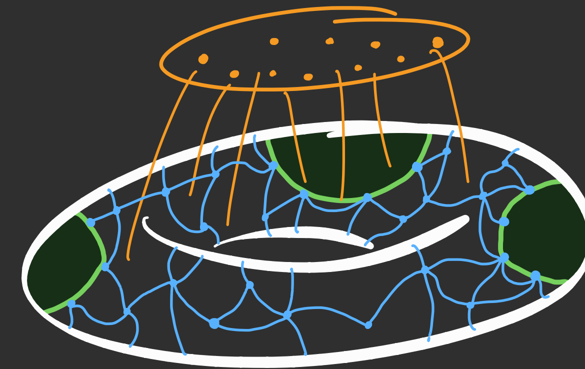
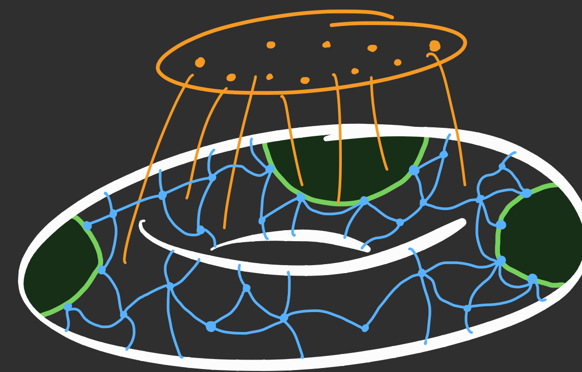


Vortices:

Theorem [Thilikos & Wiederrecht '25]
Let G be a graph embedded on the sphere with b vortices of width $\leq w$. Let G' be the result of deleting the "inside" of all vortices.
Then $bw(G) \leq bw(G') + 2wb$.

General case



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Proof uses sphere-cut decompositions & tangles.

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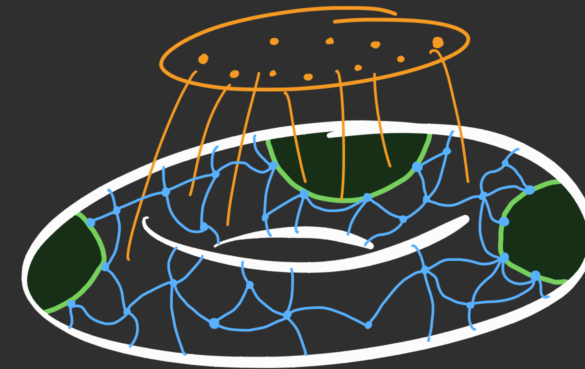

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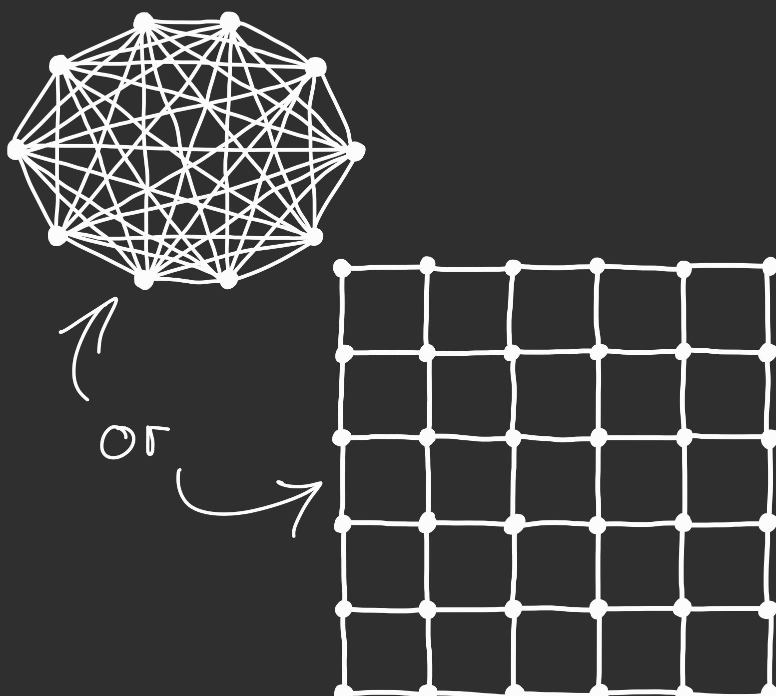
$\Rightarrow$ We can "more or less" reduce to the bounded genus case and proceed as explained earlier.

Finding a large grid- or clique-minor

Theorem [G., Stamoulis, Thilikos, & Wiederrecht '25+]

For any graph G, if $tw(G) \in \Omega(t^2 k + t^{2304})$ then G contains a K_t- or a $(k \times k)$-grid-minor.



In fact in $2^{poly(t)} poly(|V(G)|)$-time we can find:

or randomized $(t+n)^{O(1)}$-time (with worse but still poly-bounds)

- a K_t-minor
- a $(k \times k)$-grid-minor, or
  (In fact an induced wall!)
- a branch-decomposition of G with approximately the correct width for G.
  $\in O(t^7 k + t^{2309})$
  branchwidth can be verified to either be at least k or $\in O(t^2 k + t^{2302})$