3D-grids are not transducible from planar graphs

Jakub Gajarský, Michał Pilipczuk, Filip Pokrývka

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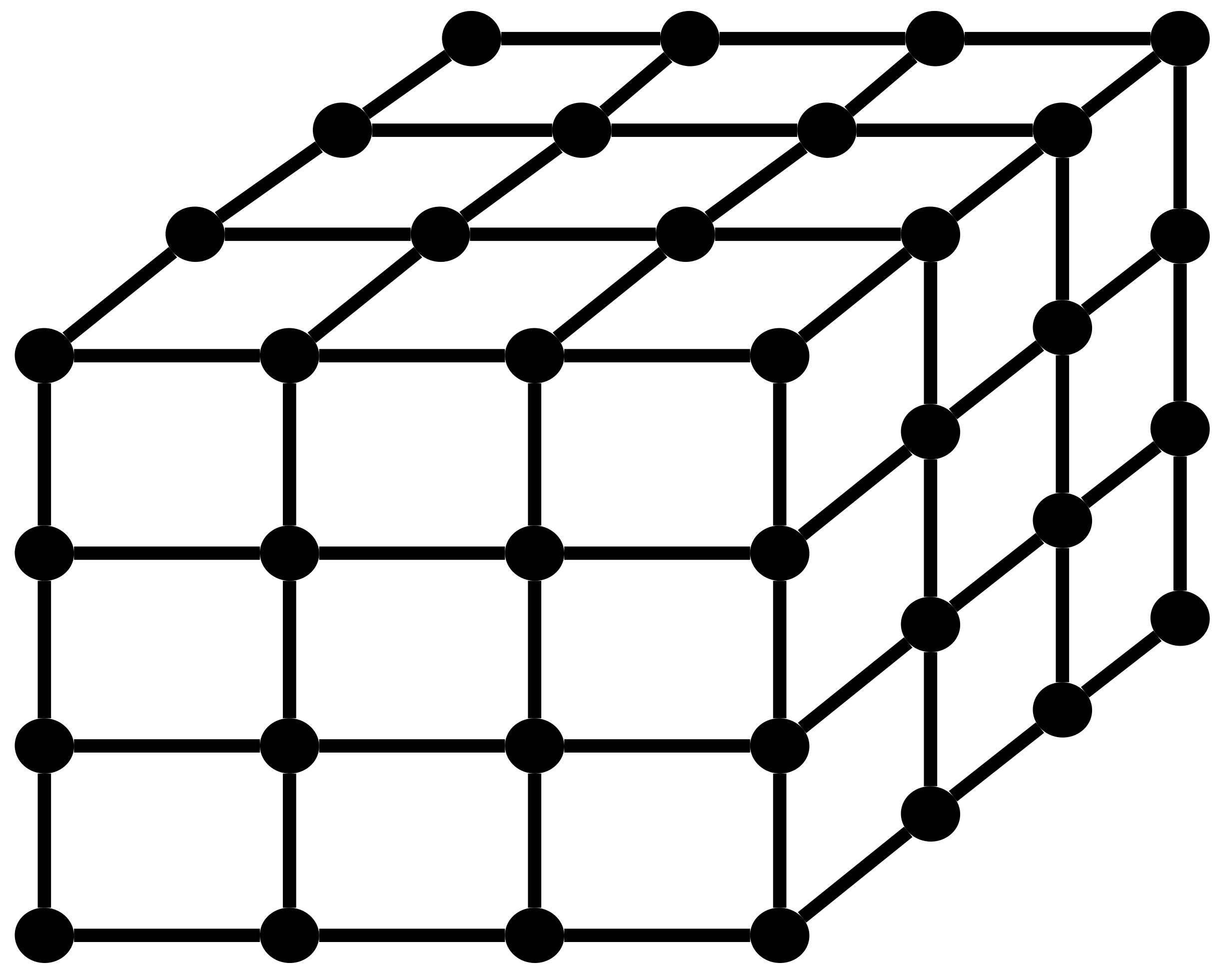

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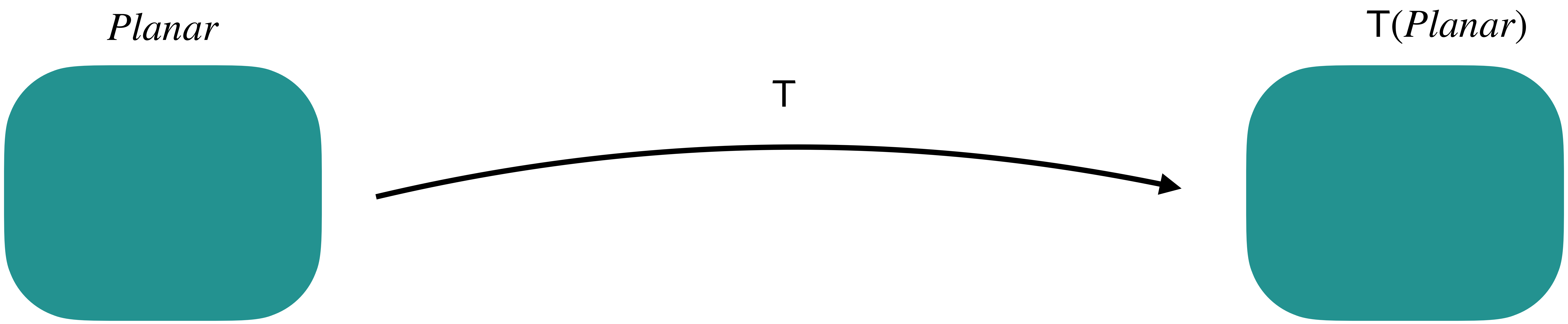

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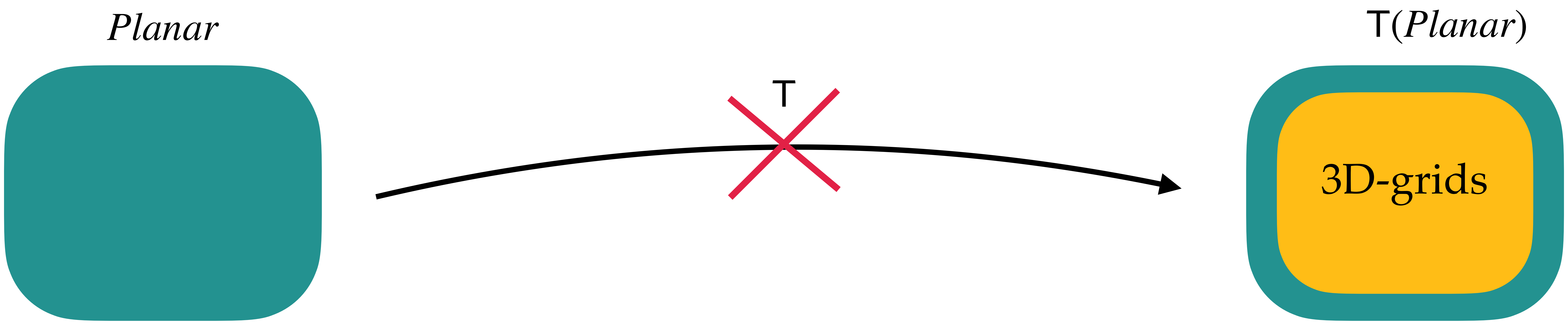

How and why prove such results?

Why are transductions important?

Transductions play a key role in the newly established field that can be called **“structural logical graph theory”**.

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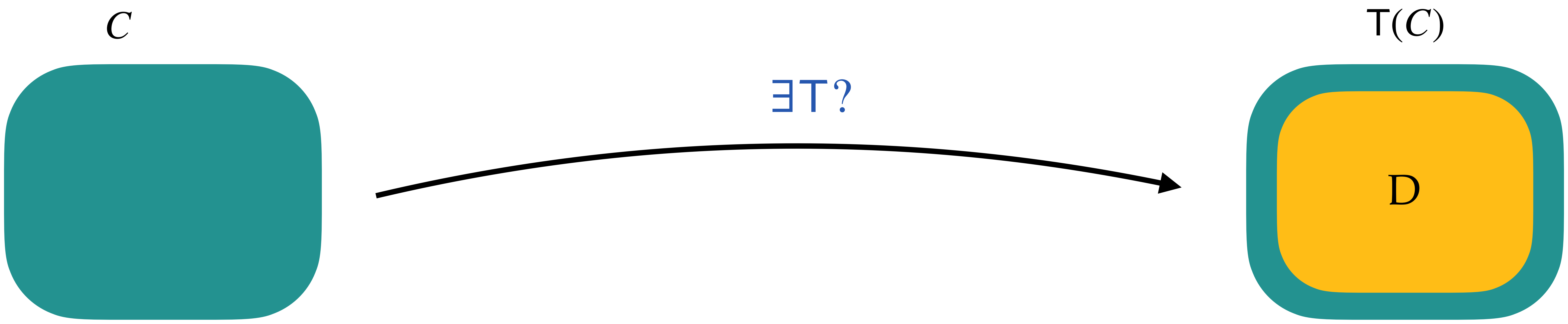
The core idea:

Study graph classes and the relationships between them using transductions.

Structural logical graph theory

Very basic question:

Given two graph classes C and D, does there exist a transduction T such that $D \subseteq \mathsf{T}(C)$?



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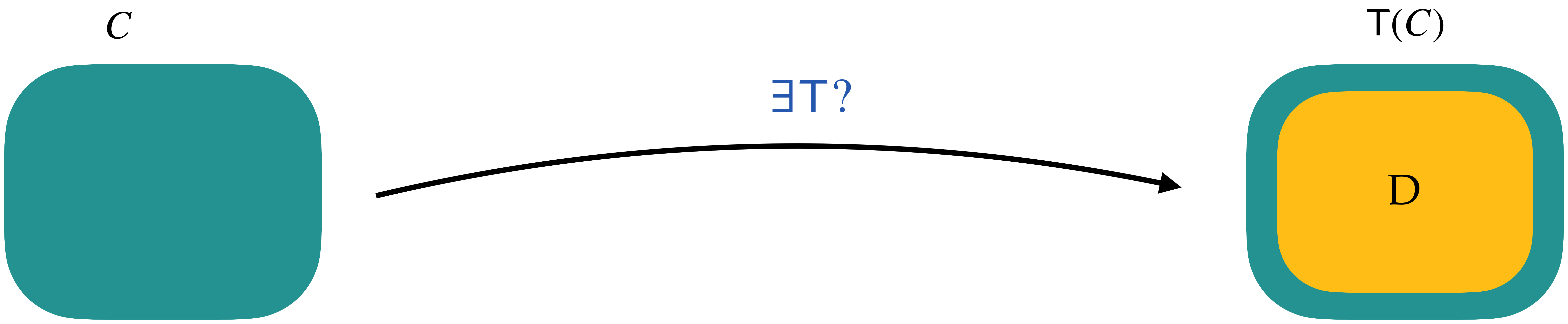

More concisely:

Given two graph classes C and D, is D transducible from C?

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- For any *k*, the class of graphs of pathwidth k is transducible from planar graphs.
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- **The class of 3D-grids is not transducible from planar graphs.**

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- For $k \geq 4$, the class of graphs of treewidth k is not transducible from planar graphs.
- **The class of 3D-grids is not transducible from planar graphs.**
- The class of graphs of treewidth $k+1$ not transducible from treewidth k.

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One has to expose the simplicity of D and the richness of C.

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- D does not have Π.	Planar graphs have unbounded cliquewidth.

Transduction closed properties:
cliquewidth, twin-width, shrub-depth, merge-width, monadic dependence, monadic stability…

The proof

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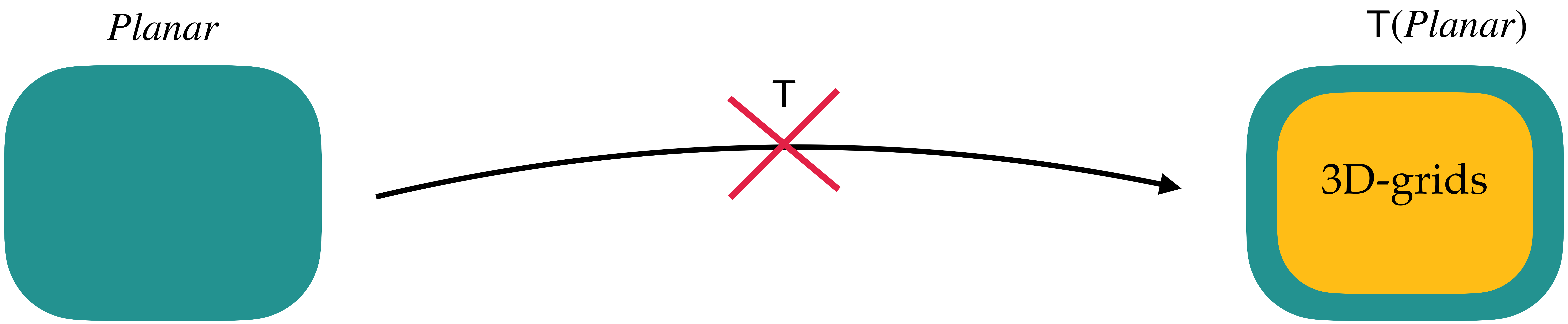

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Transductions of bounded range

We say that a transduction T has bounded range if there exists $b \in \mathbb{N}$ such that the following holds for any G:

If u, v are vertices of G such that $dist_G(u, v) > b$, then u, v are not adjacent in $\mathsf{T}(G)$.

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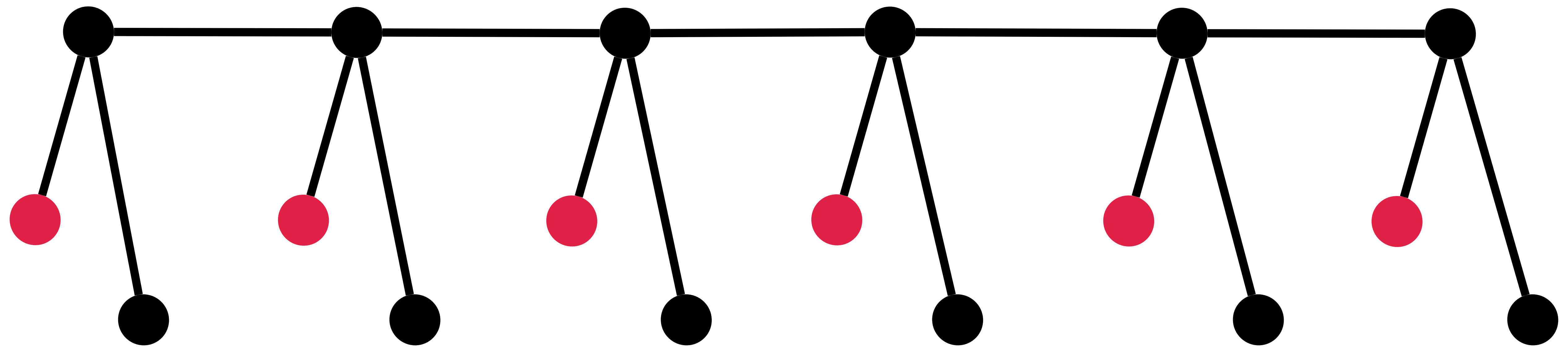

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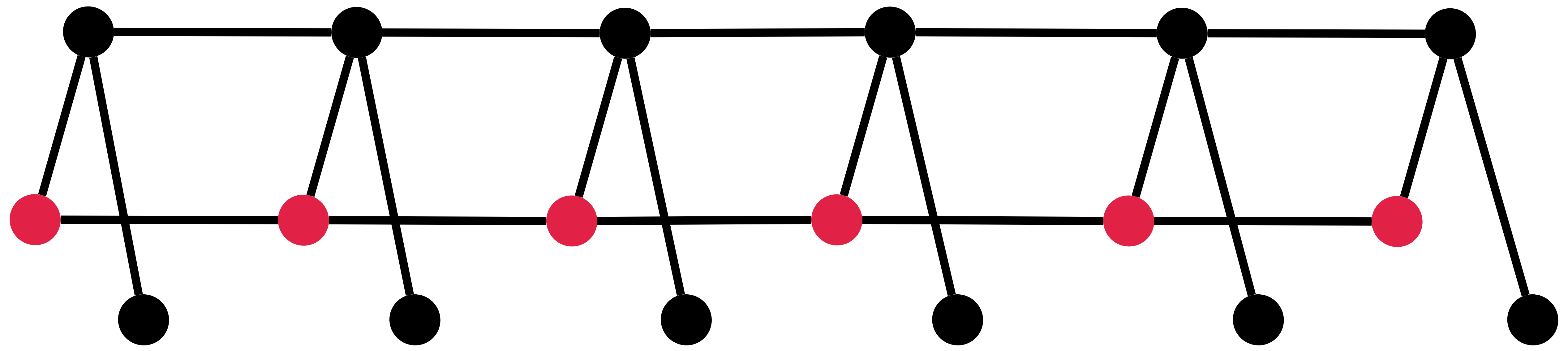

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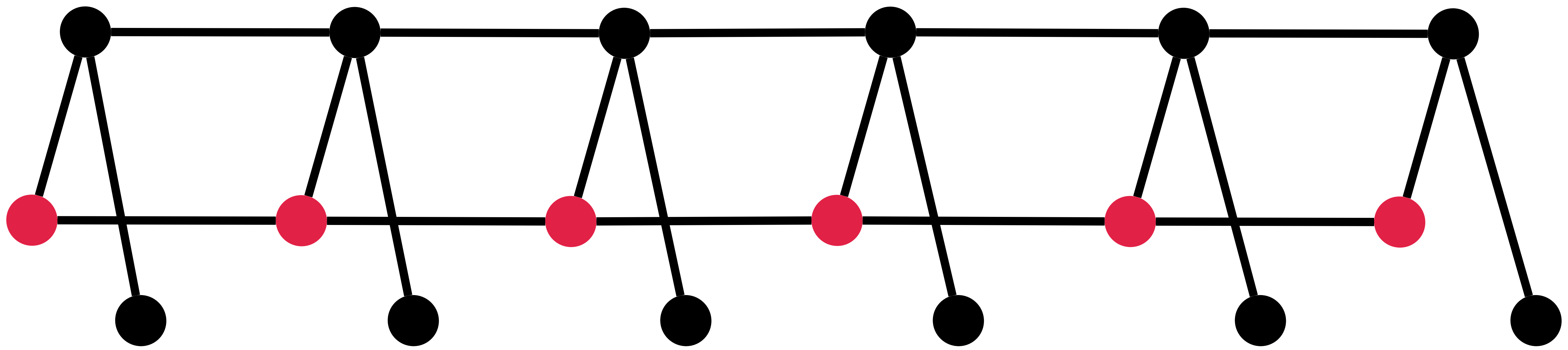

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The range of T is 3.

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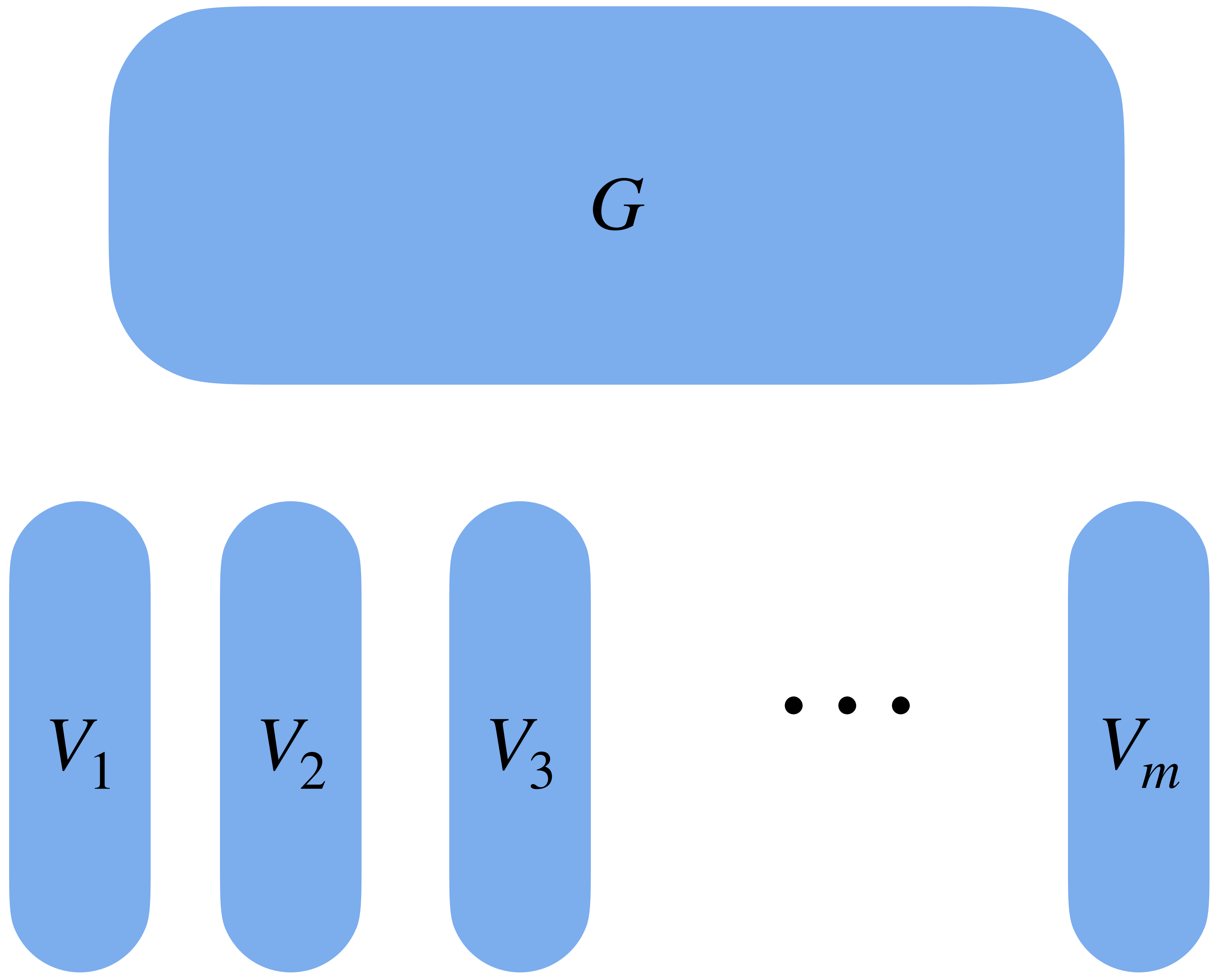

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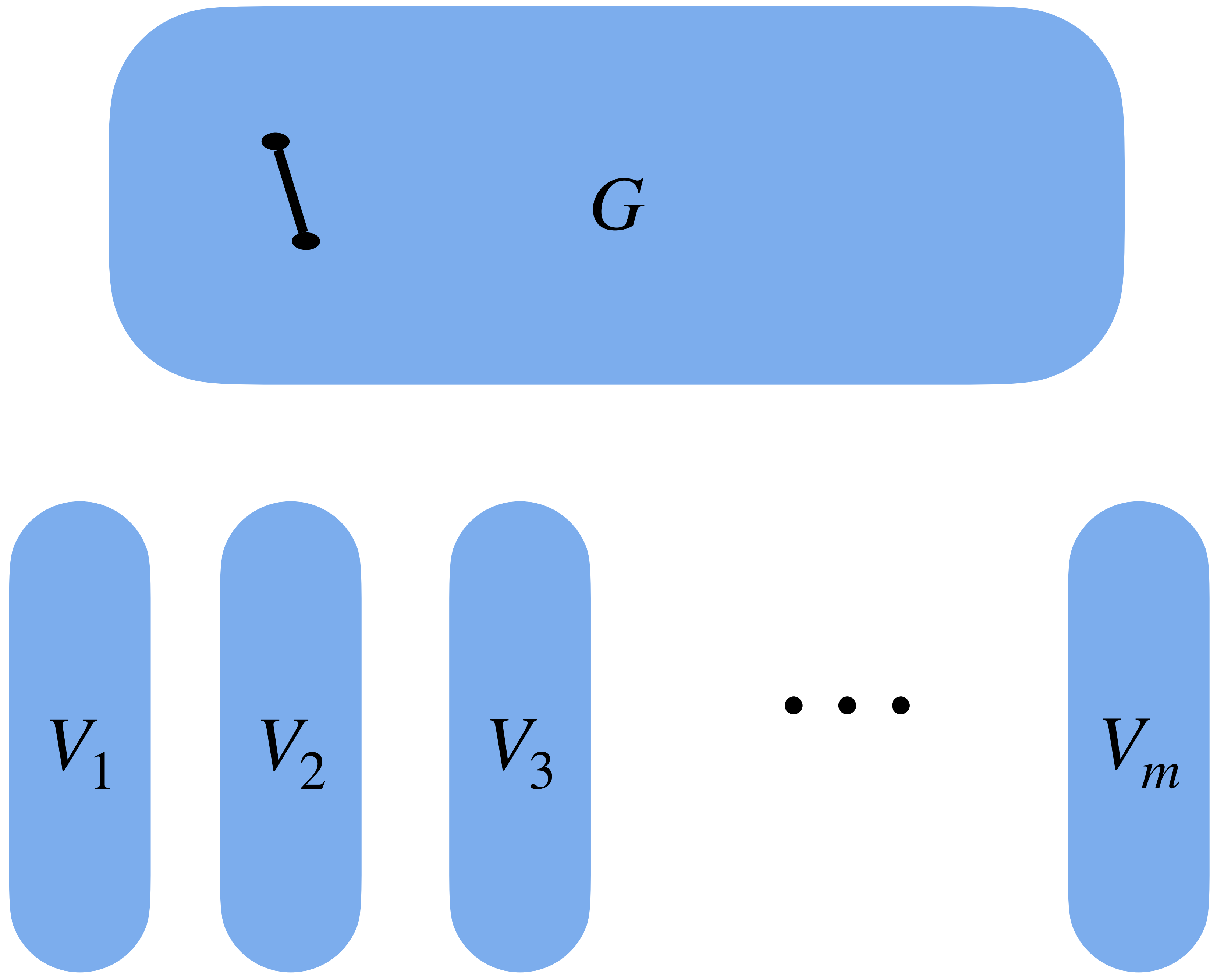

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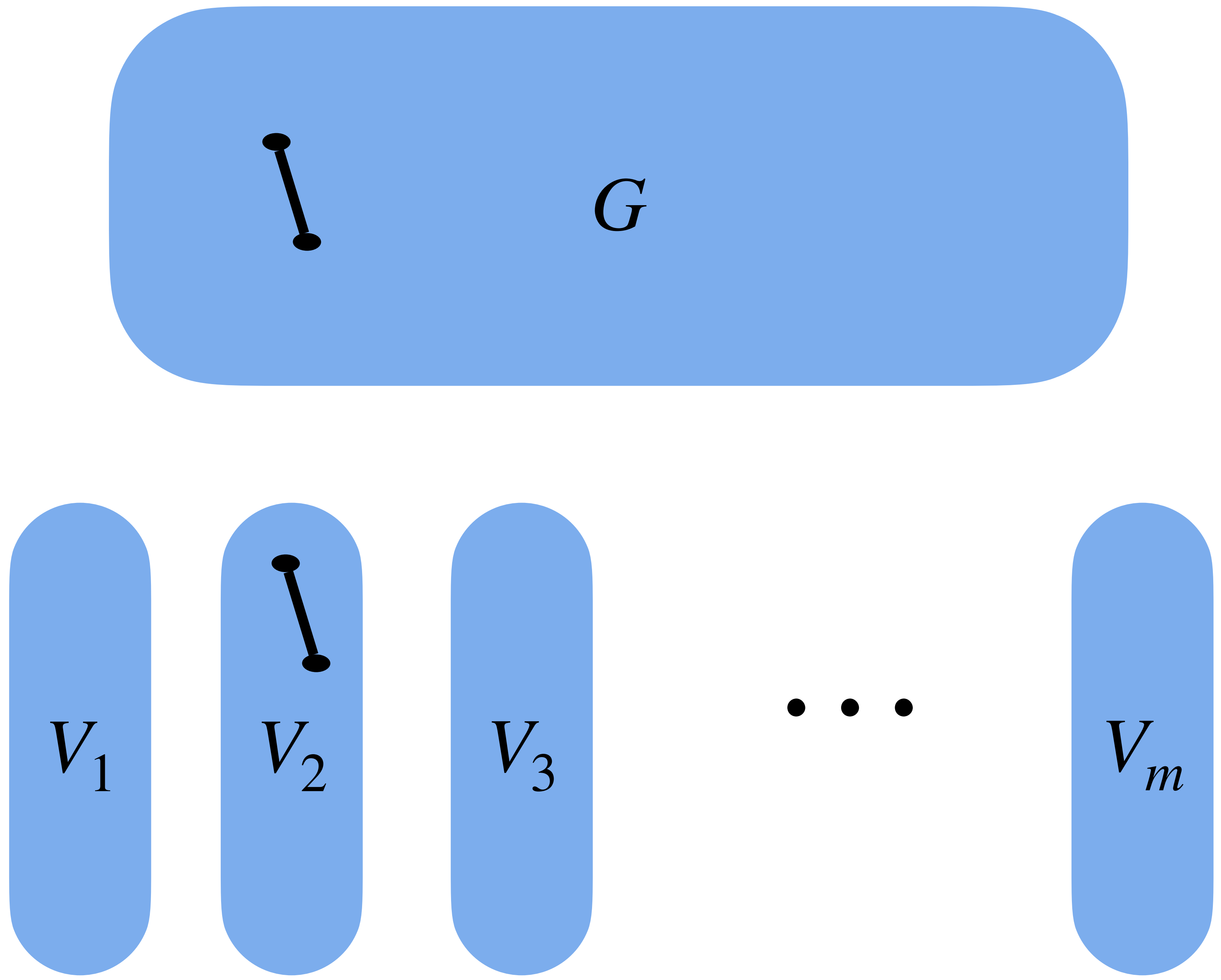

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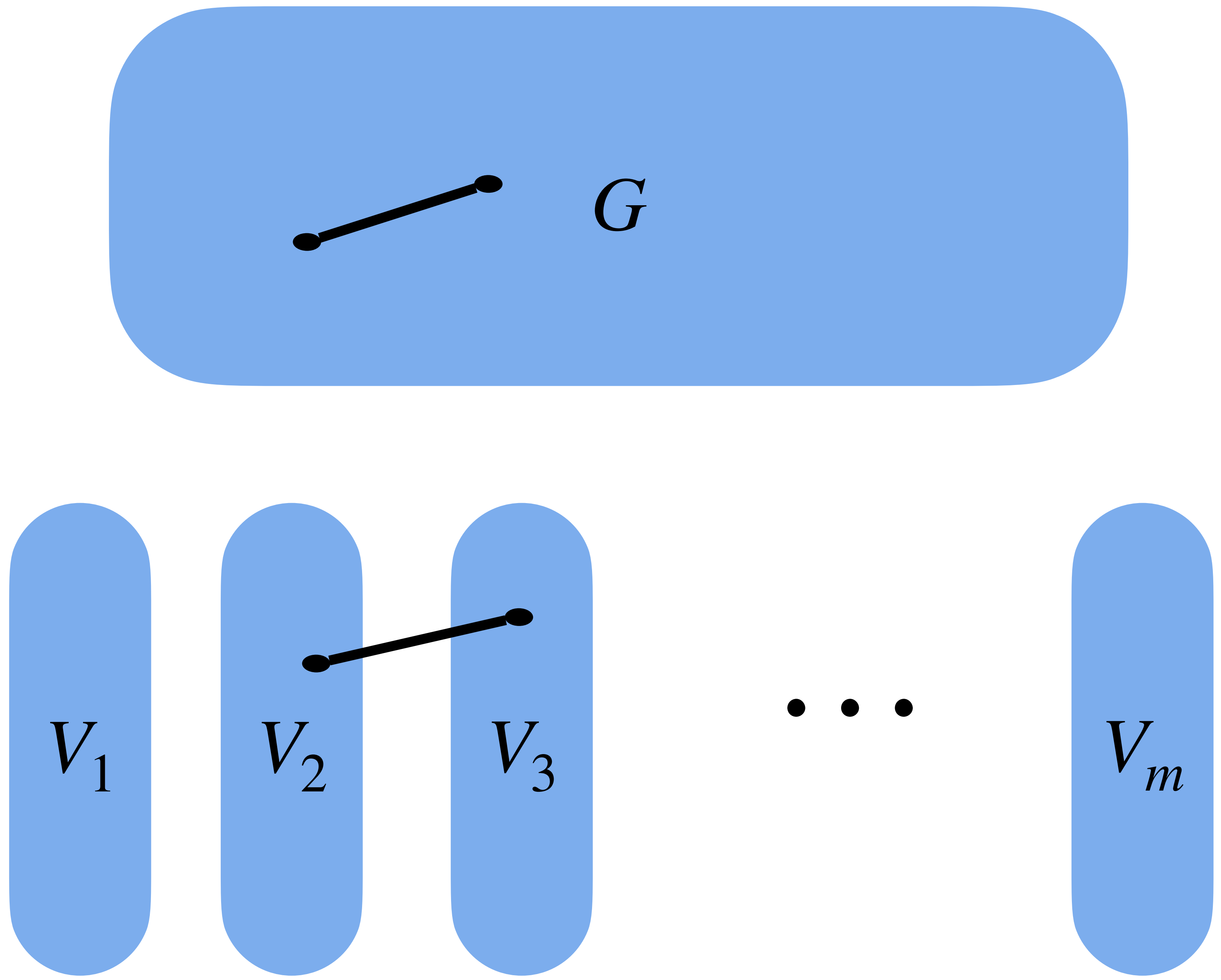

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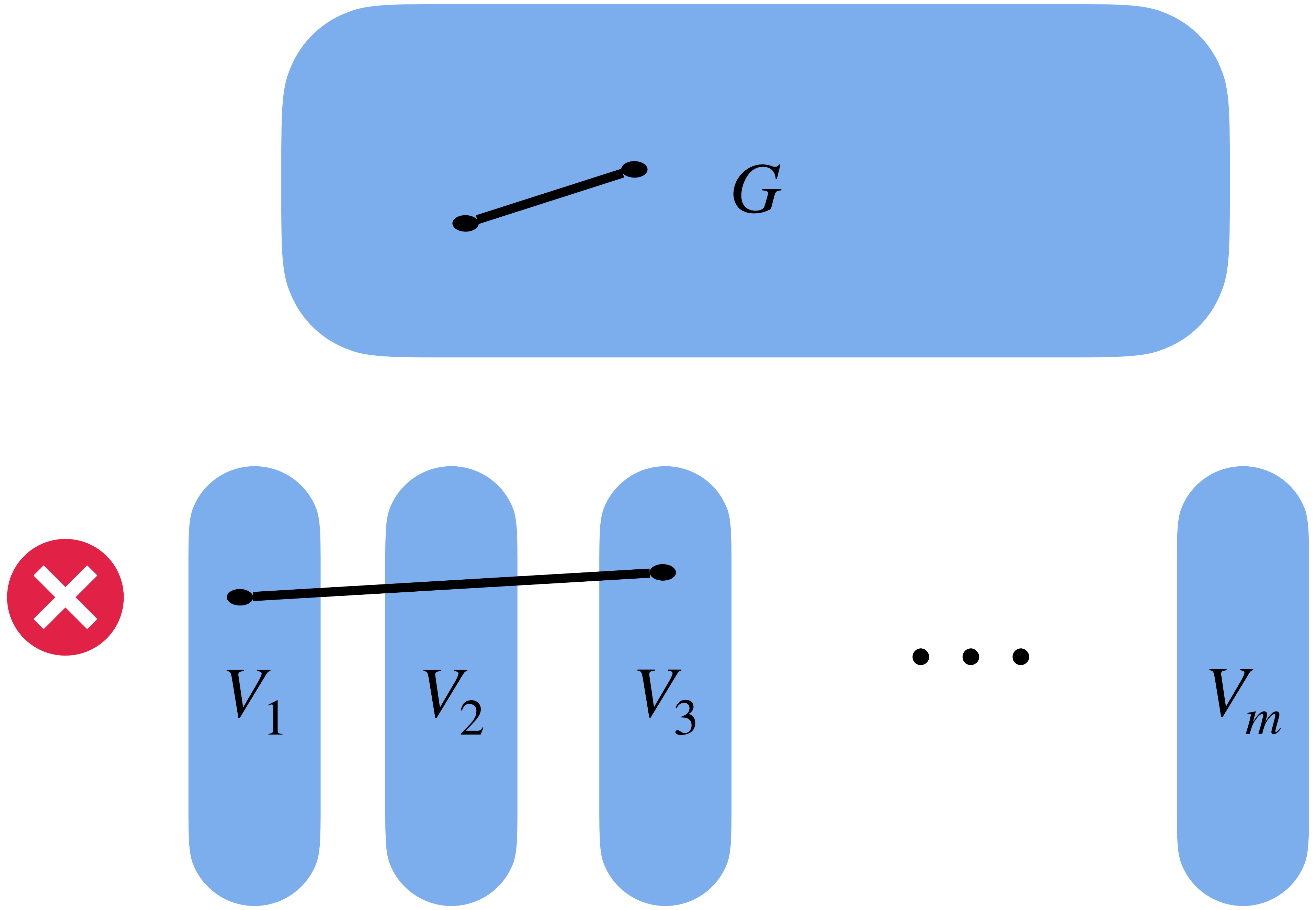

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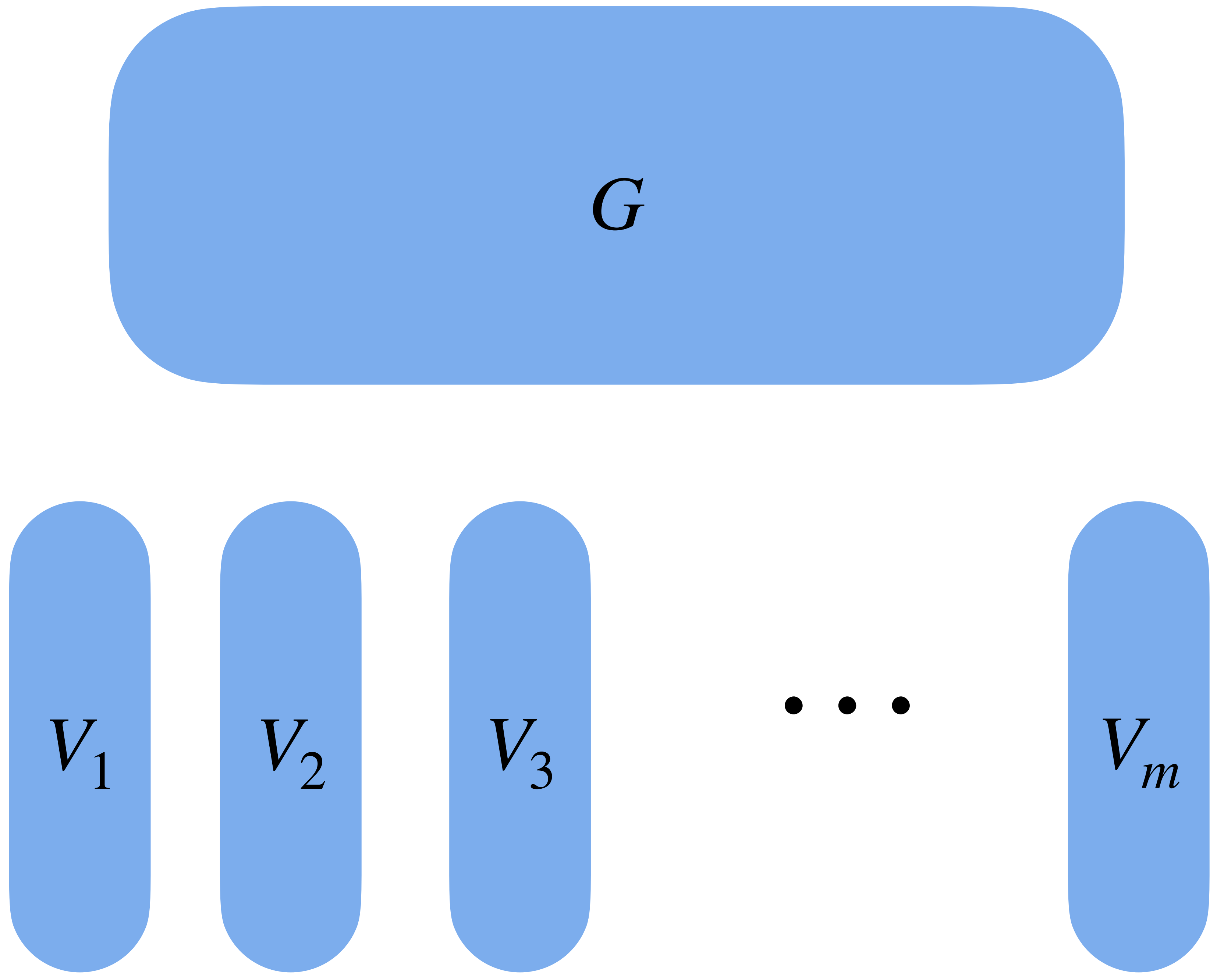

The sets $V_1, \ldots, V_m$ are called **slices**.

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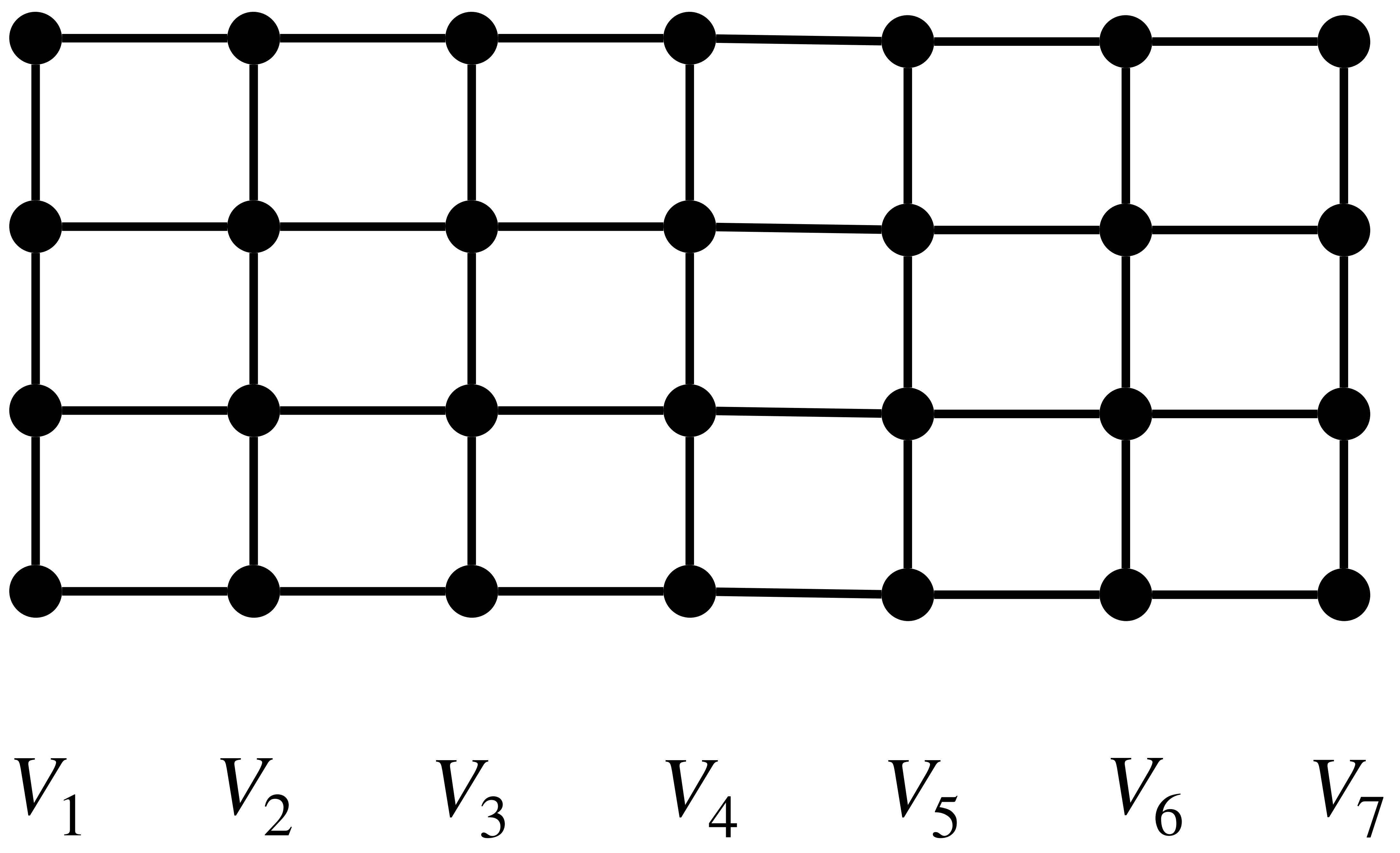

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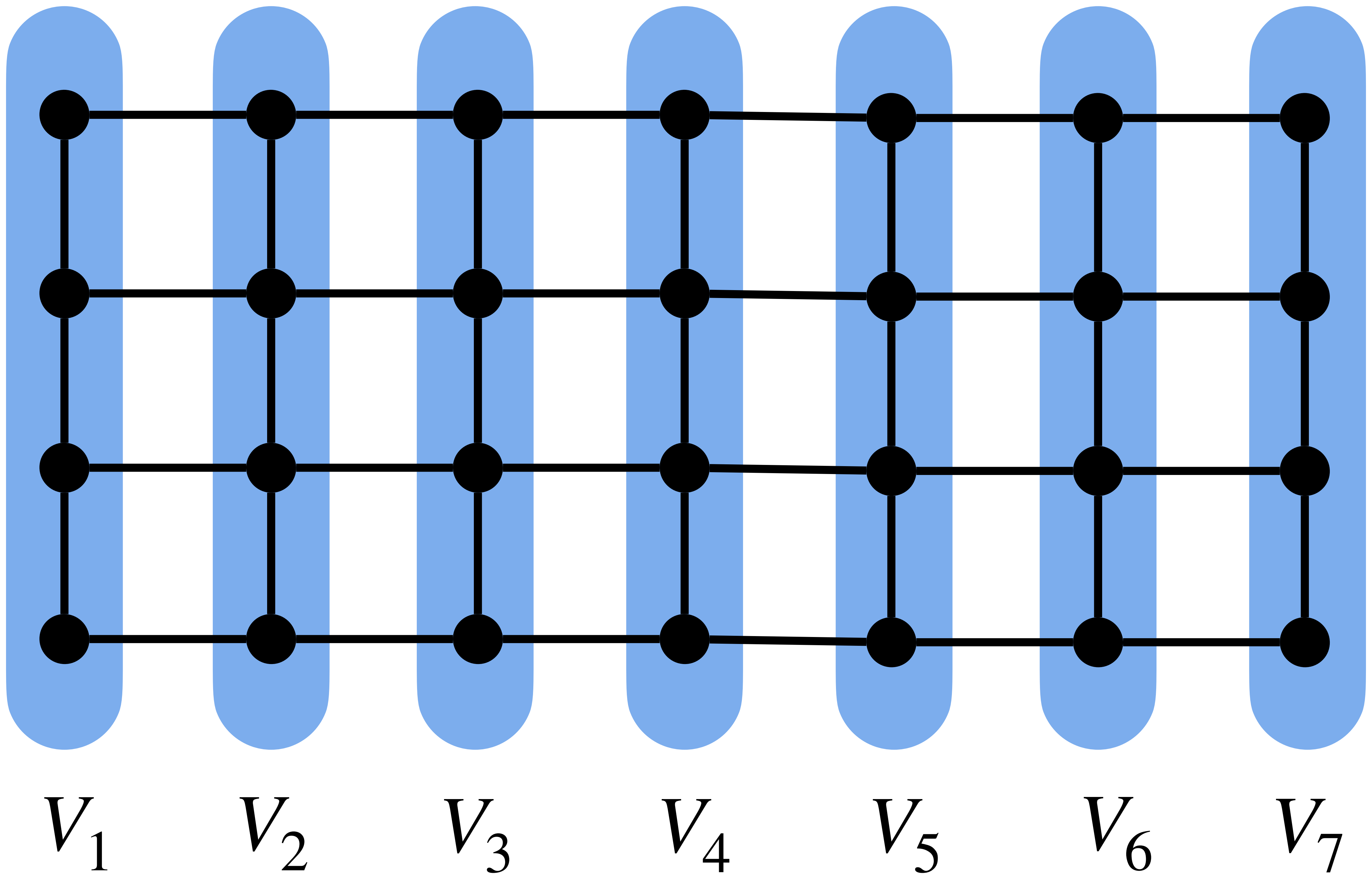

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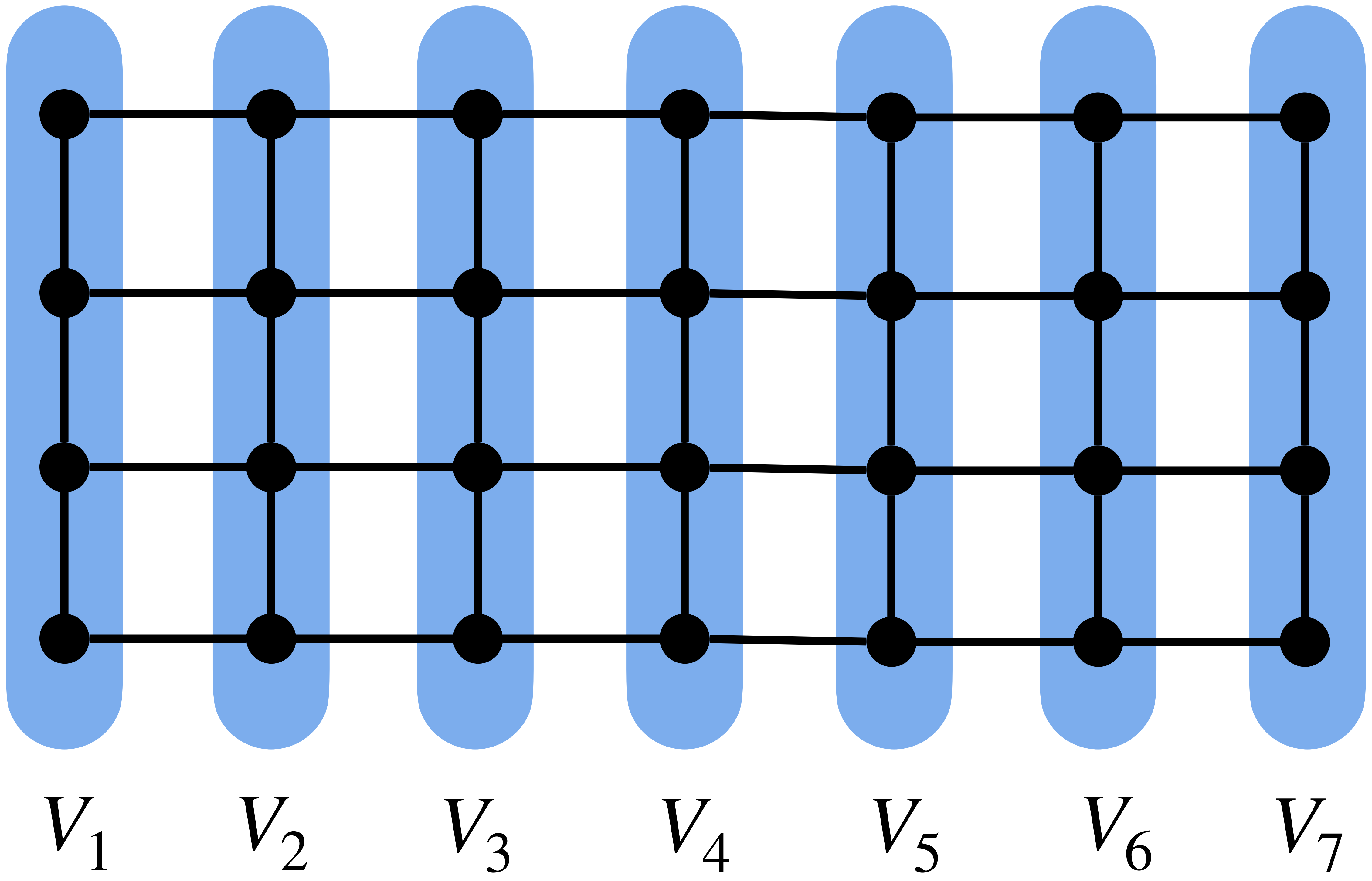

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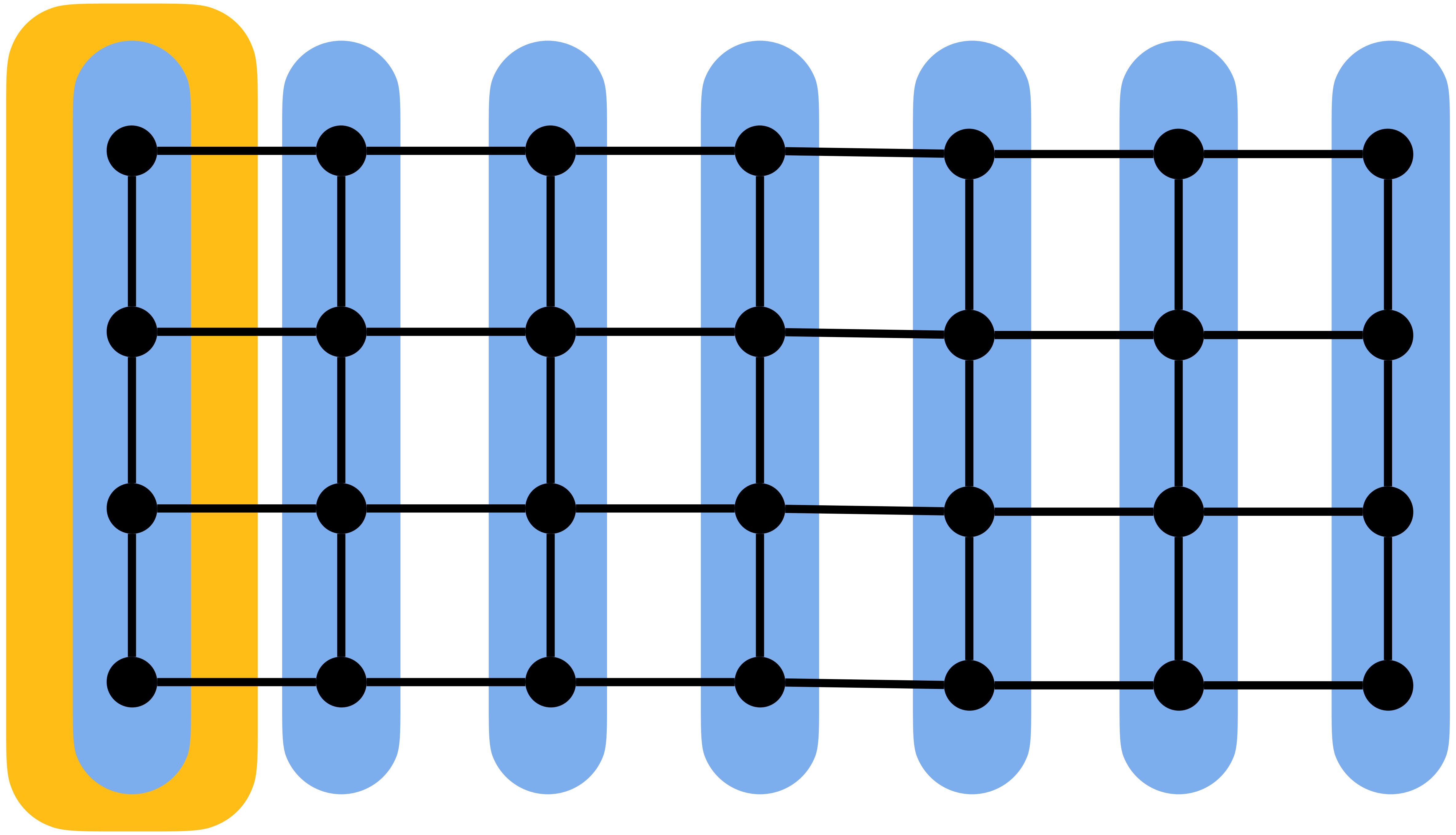

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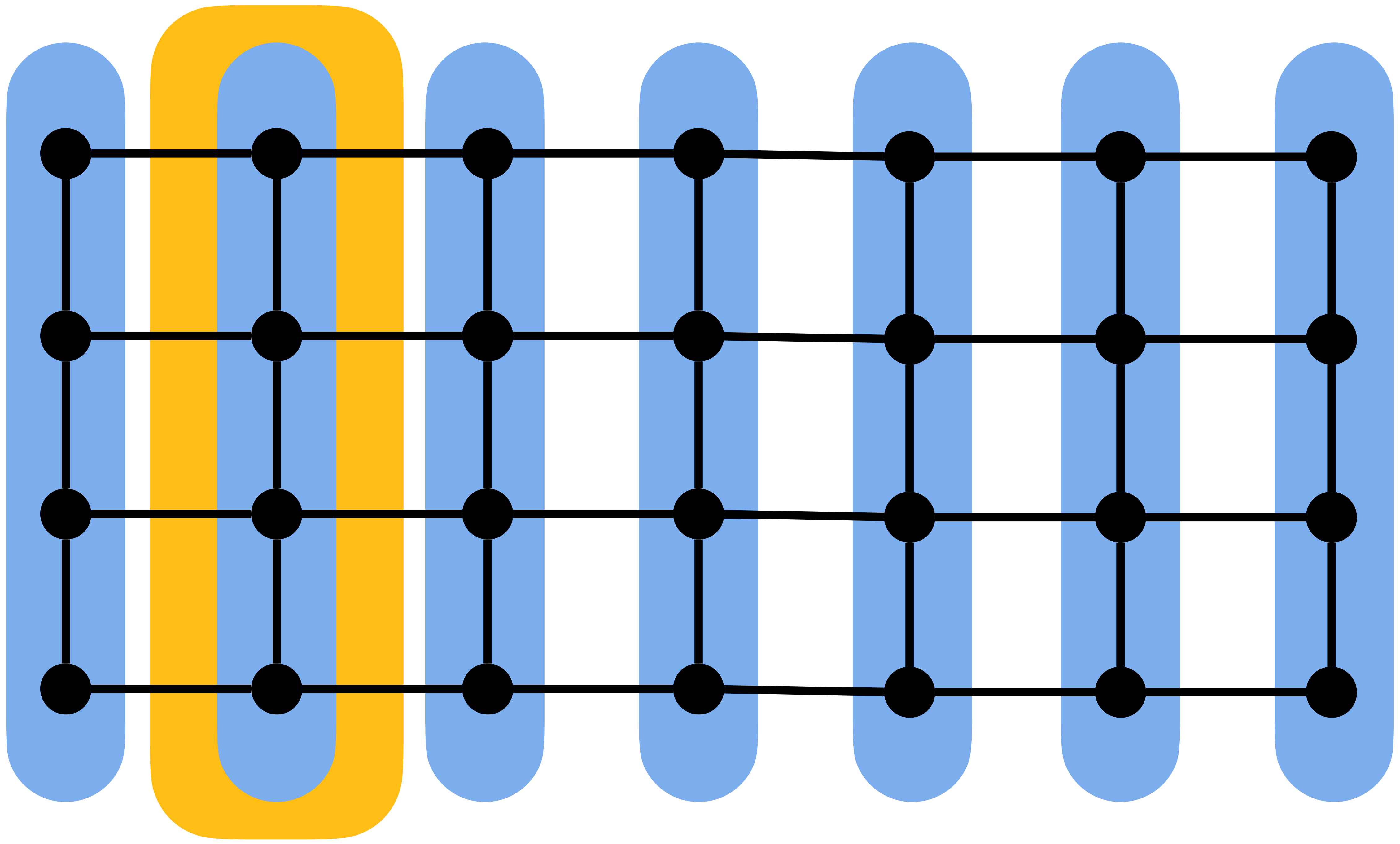

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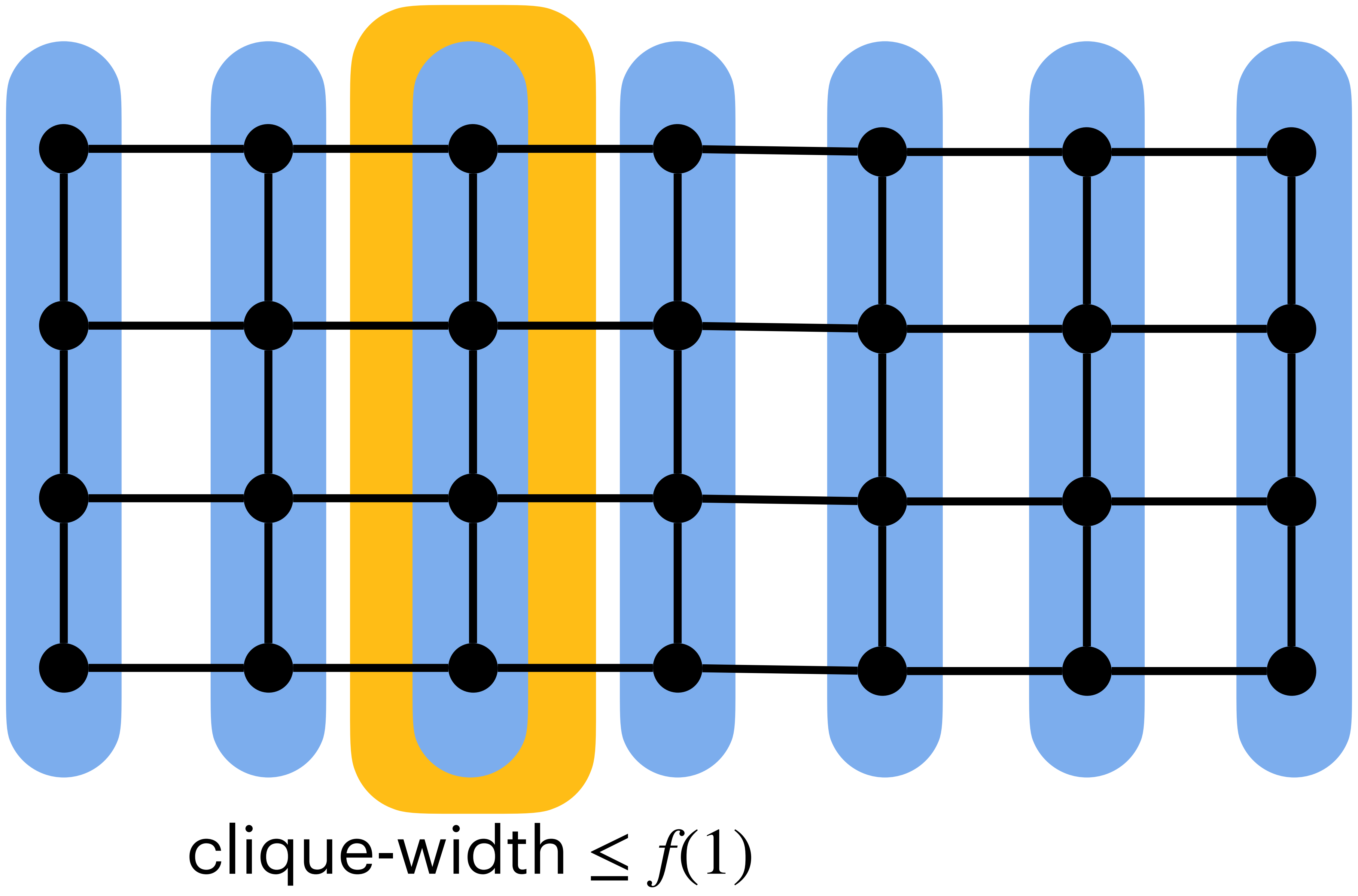

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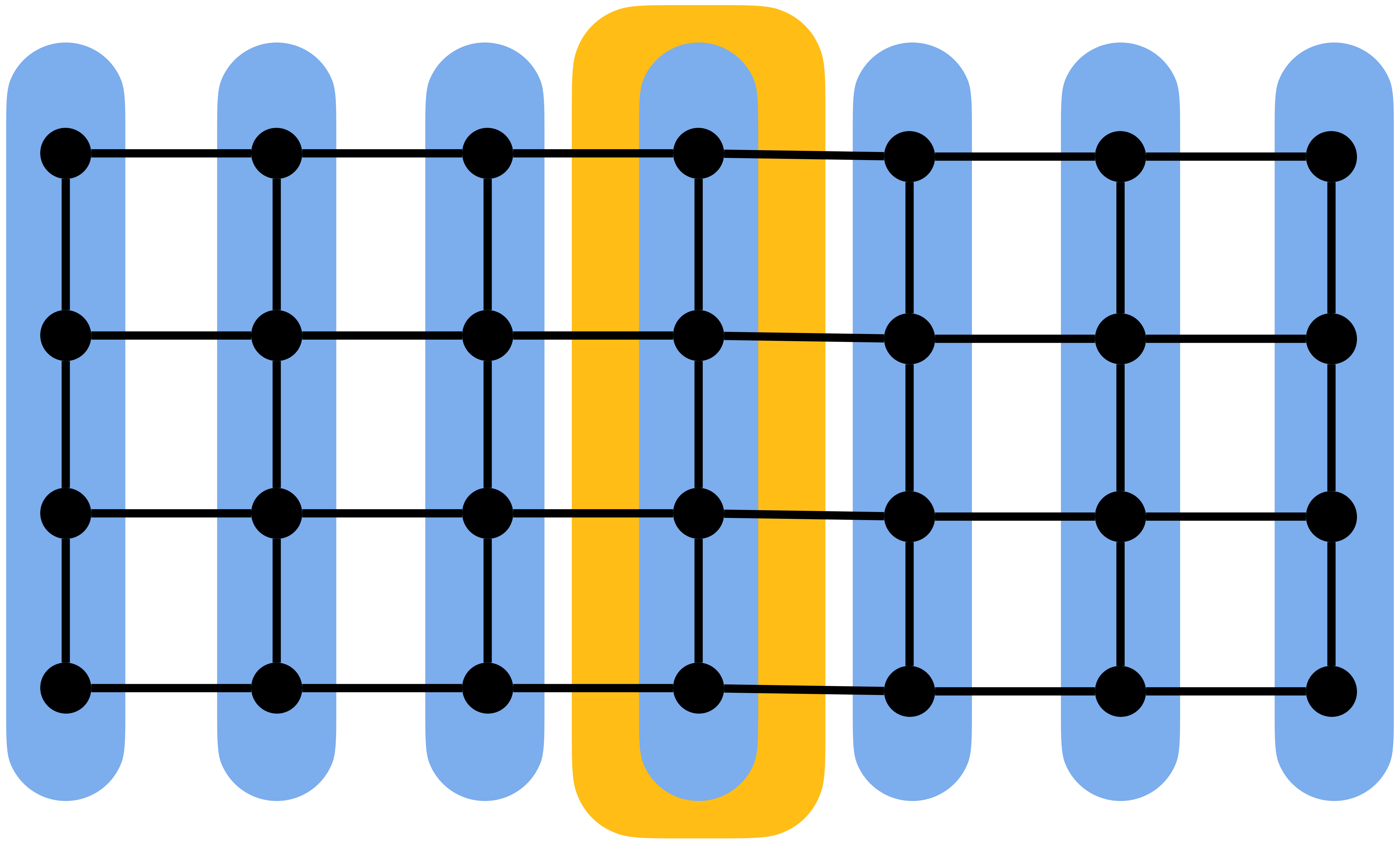

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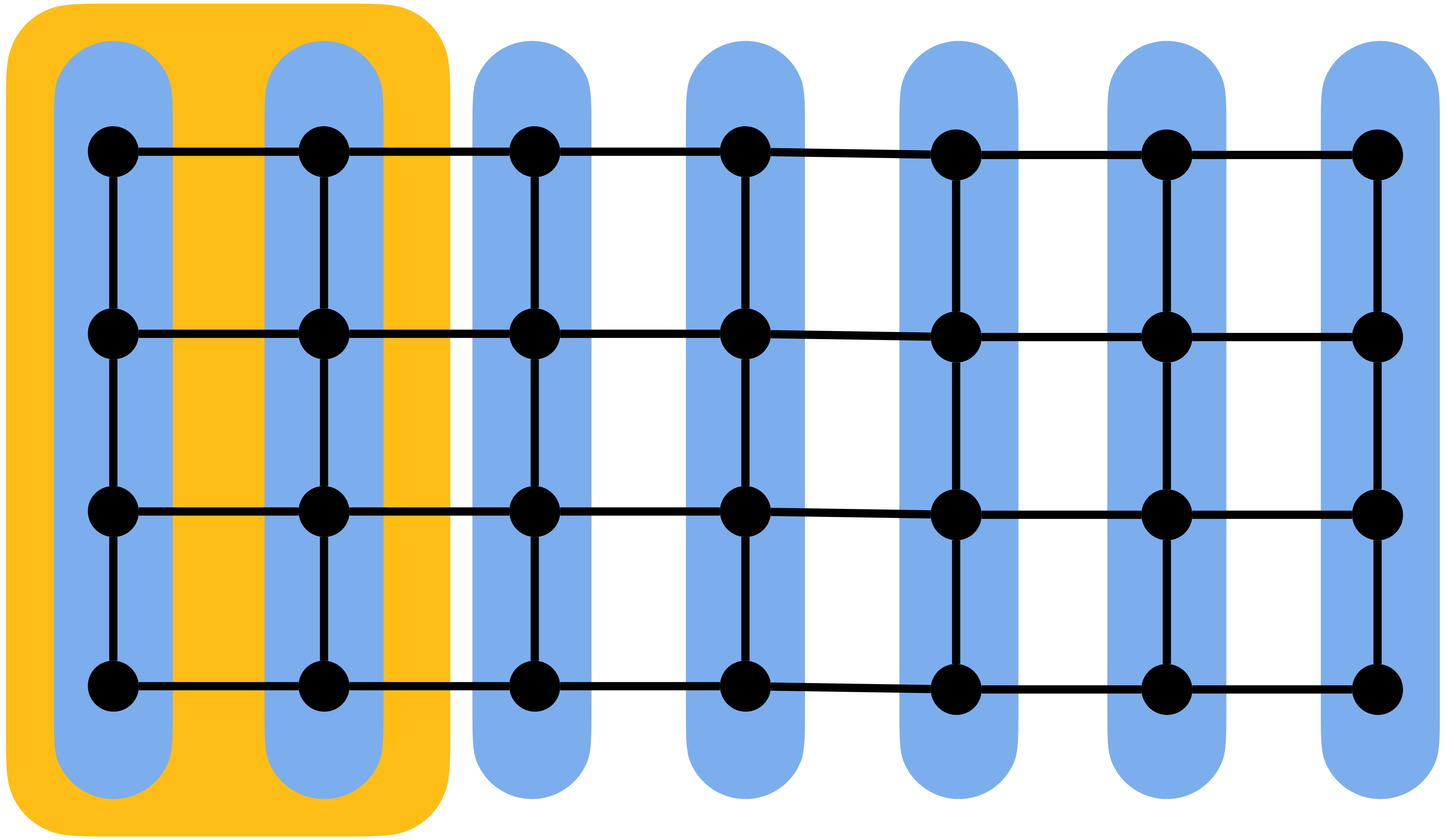

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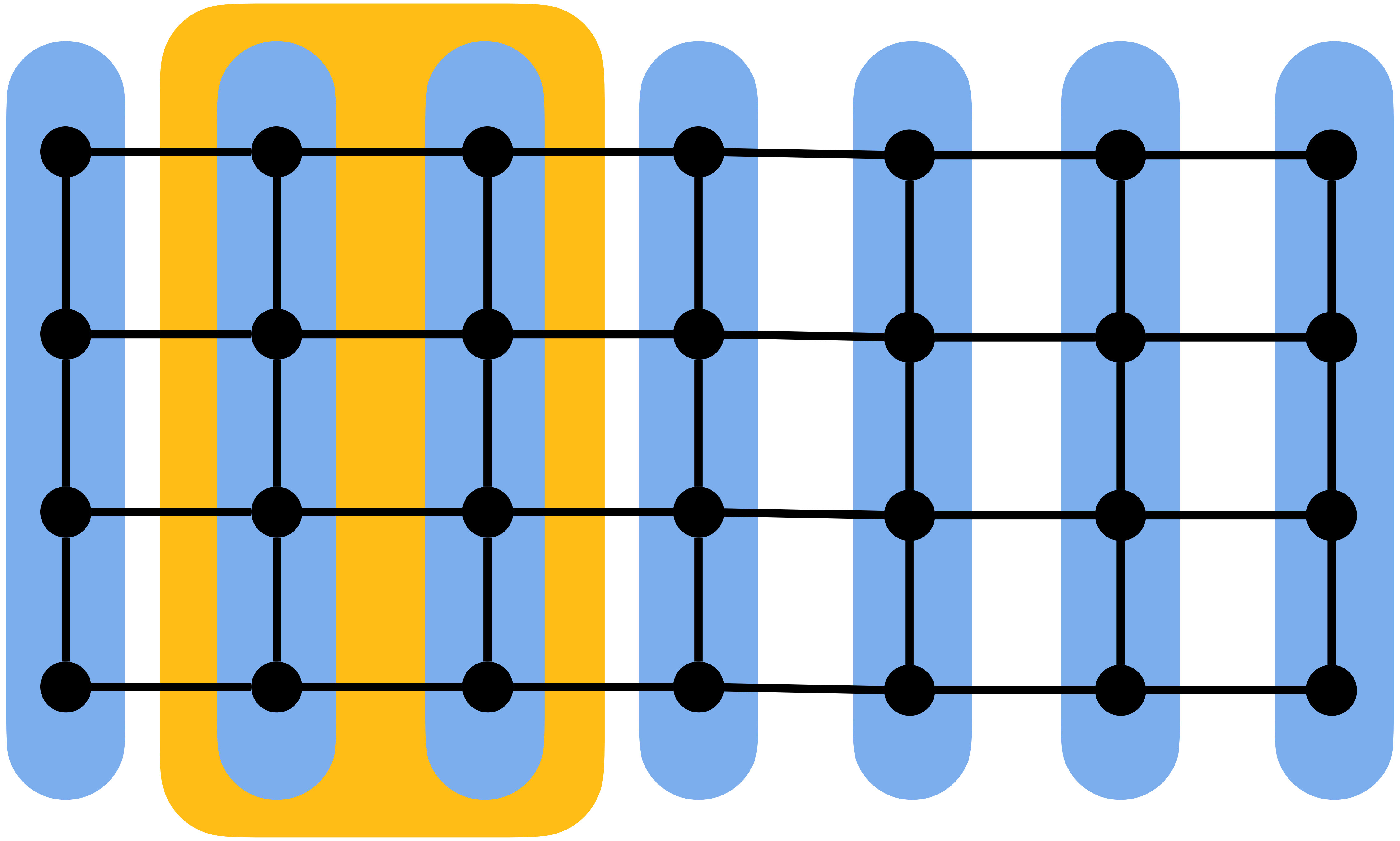

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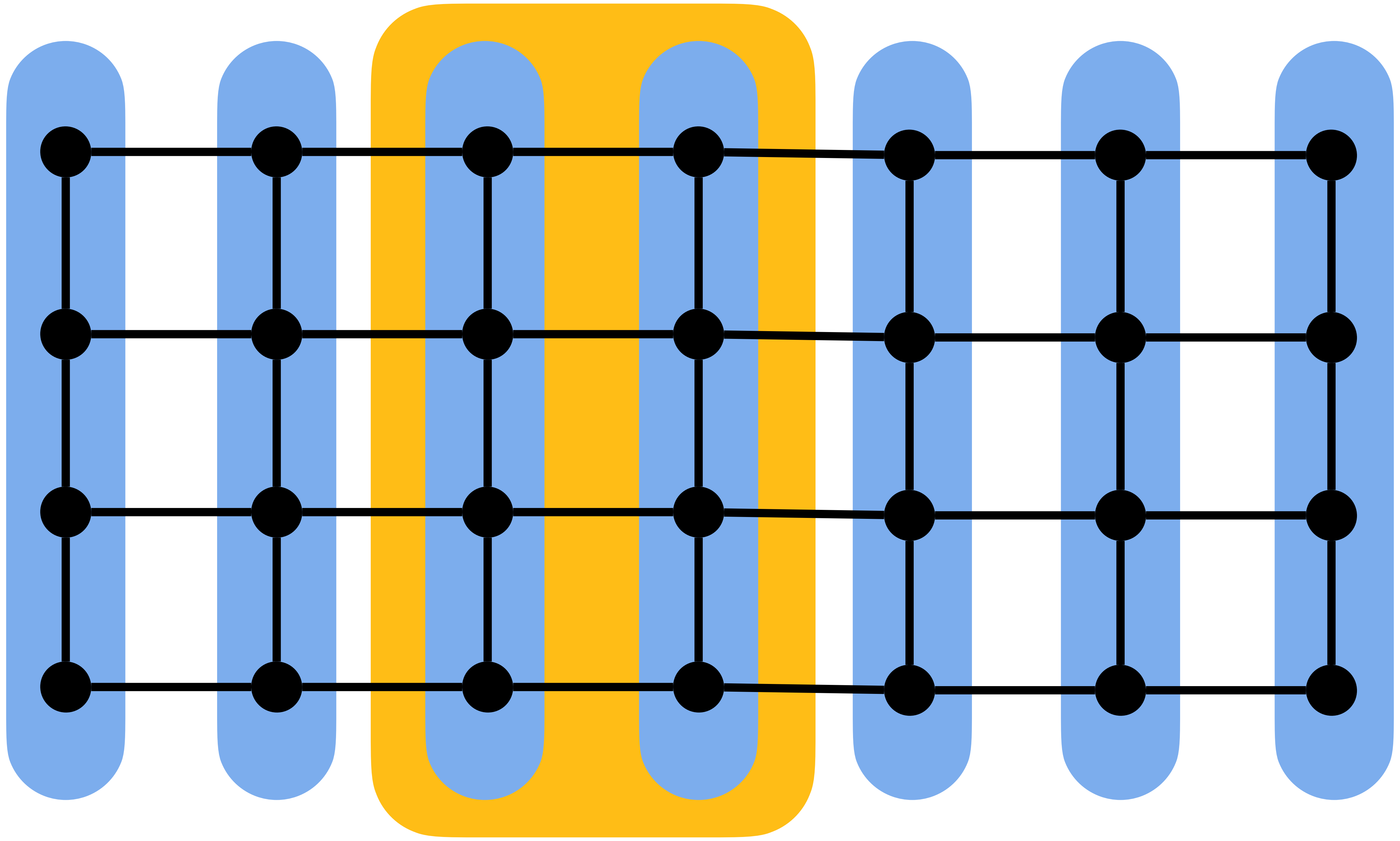

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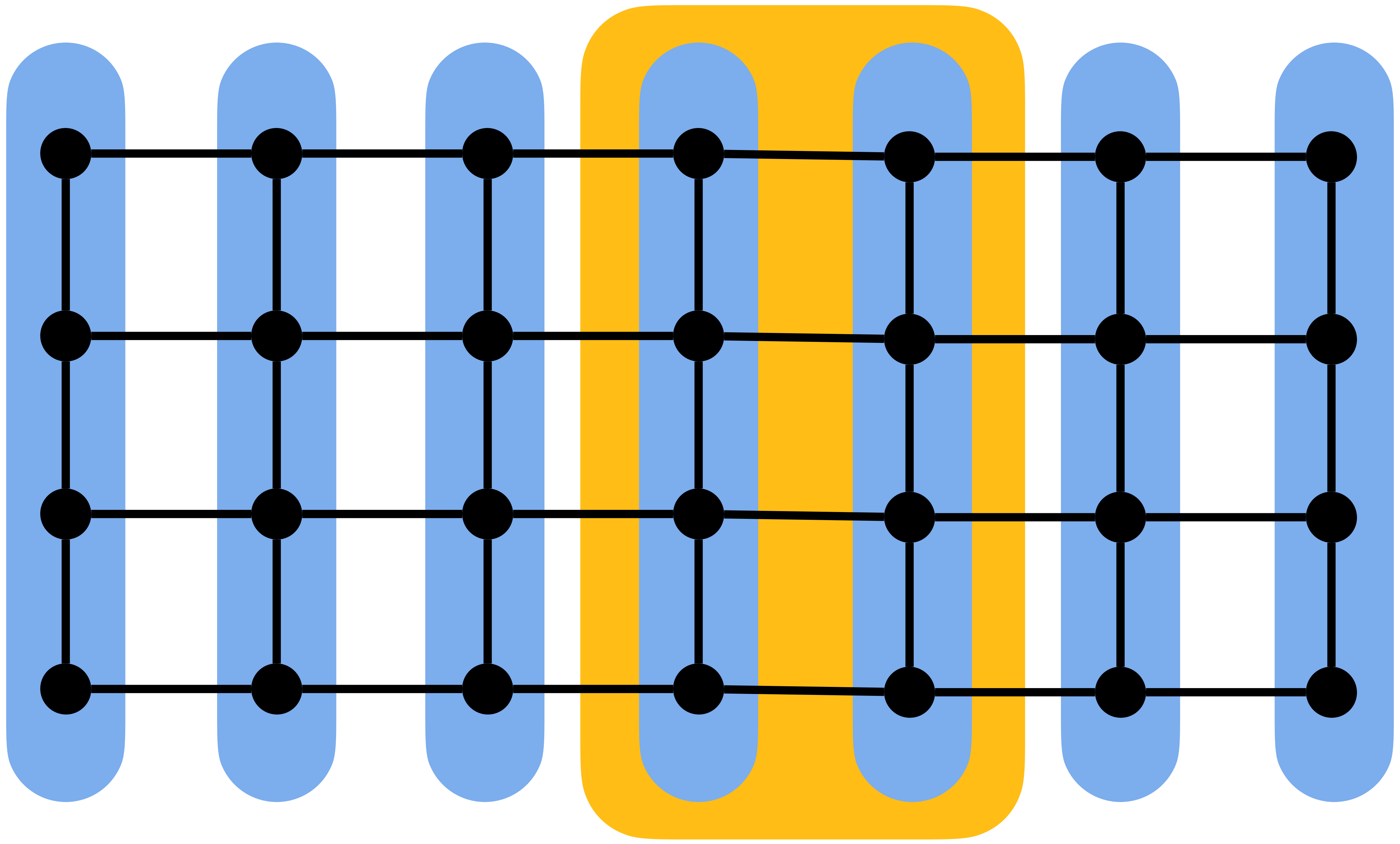

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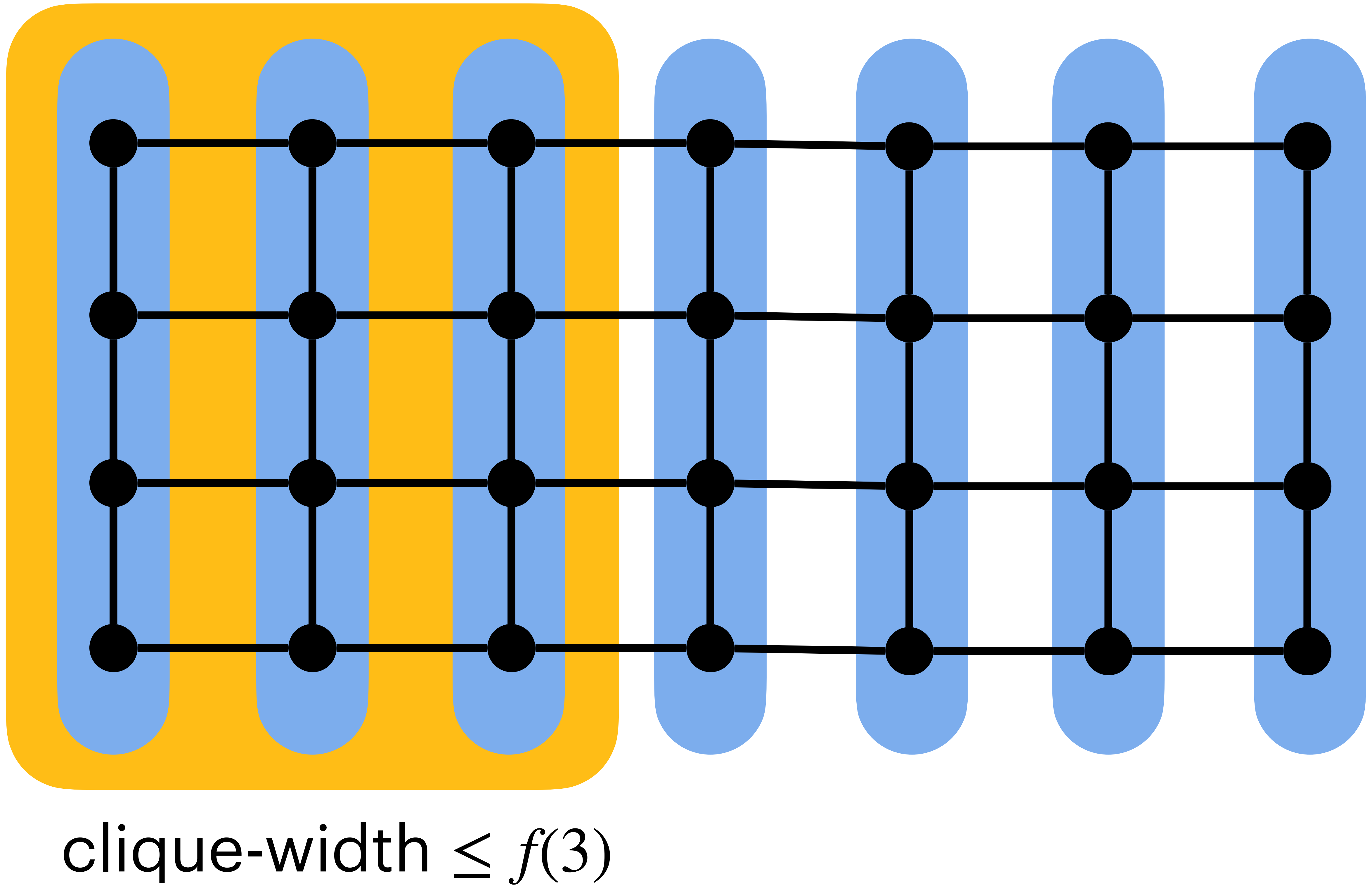

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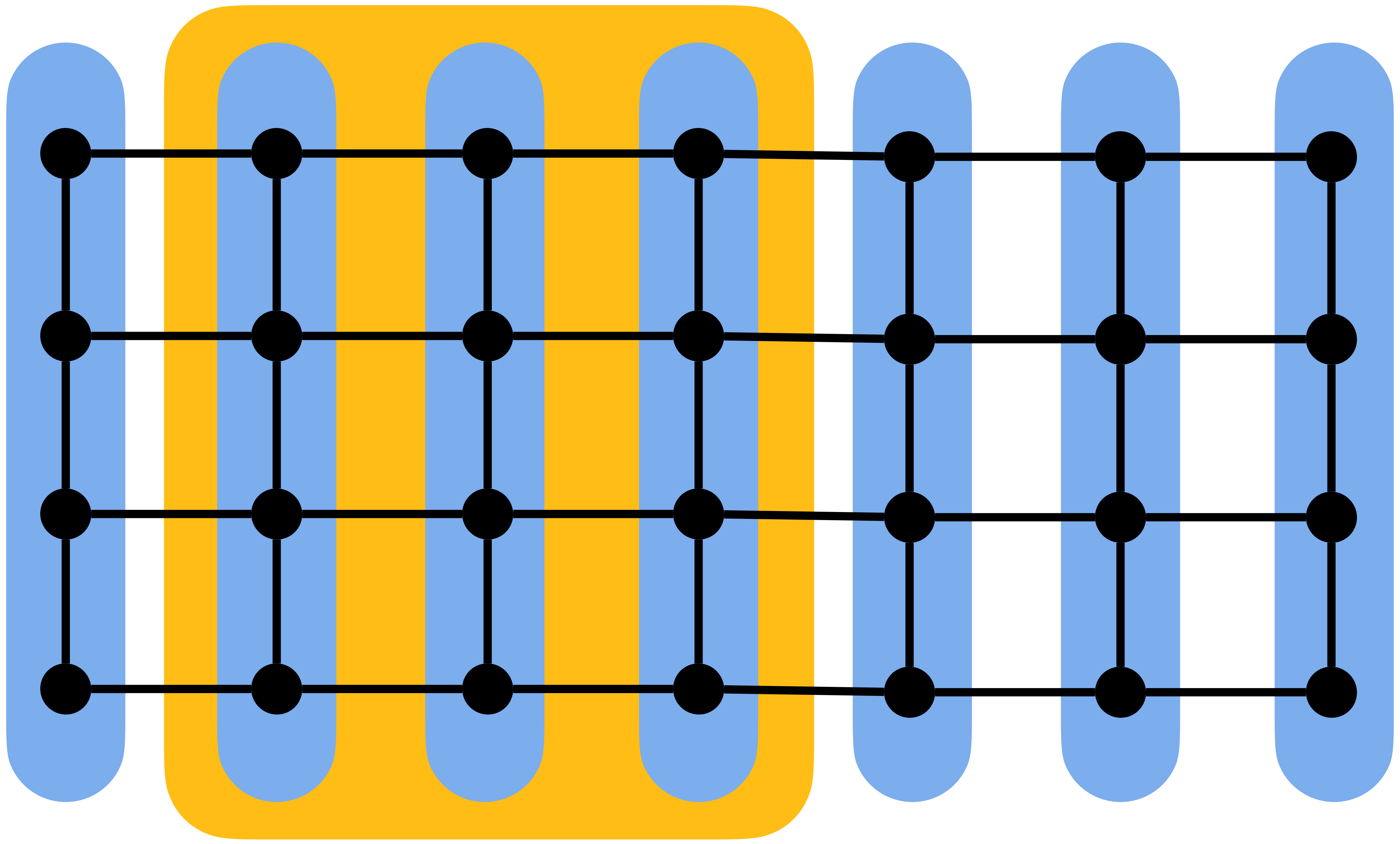

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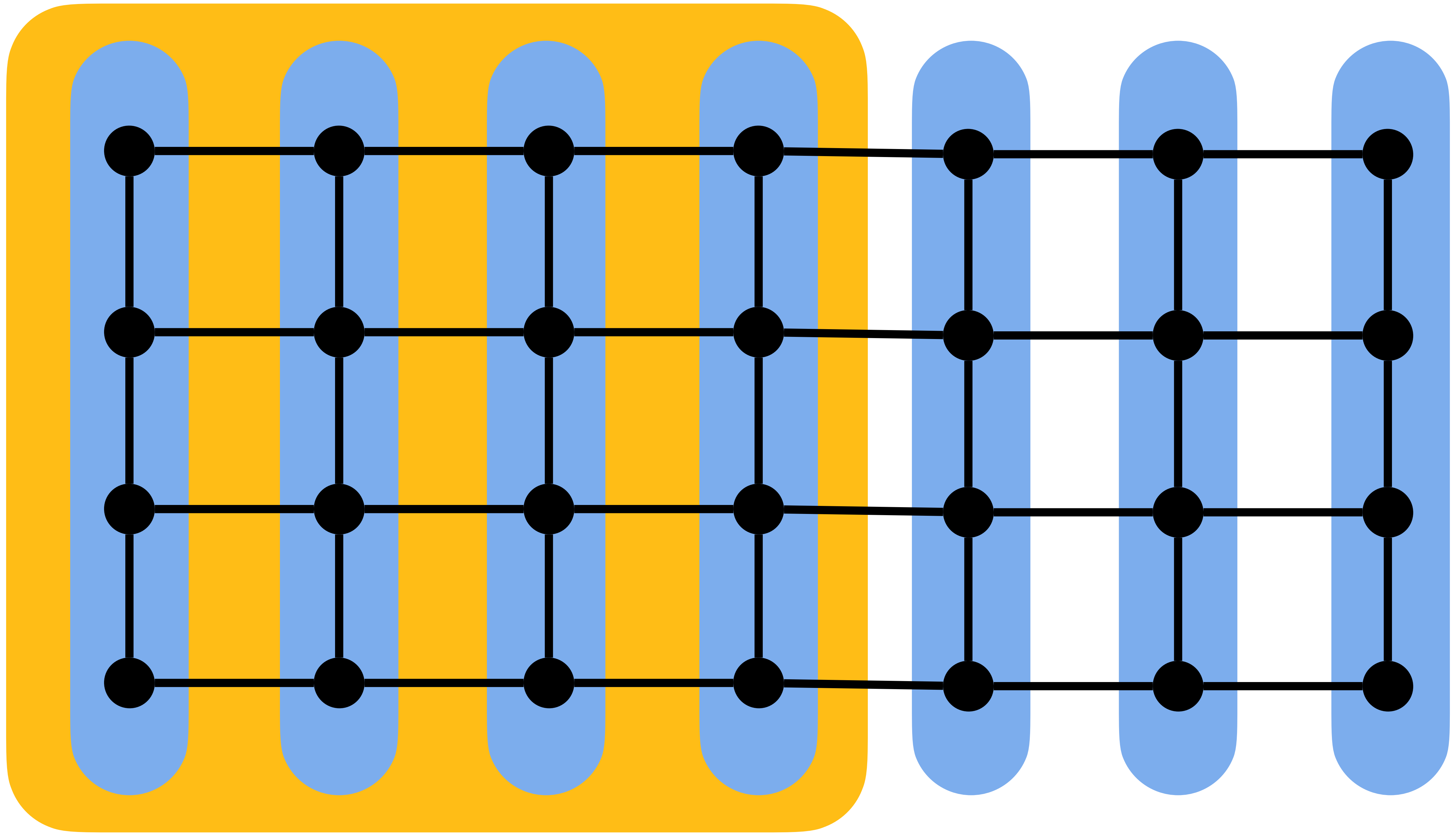

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Slice decompositions

Definition

We say that a **graph class** C **admits slice decompositions** if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every $G \in C$ has a slice decomposition with respect to f.

Theorem

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Proof plan:

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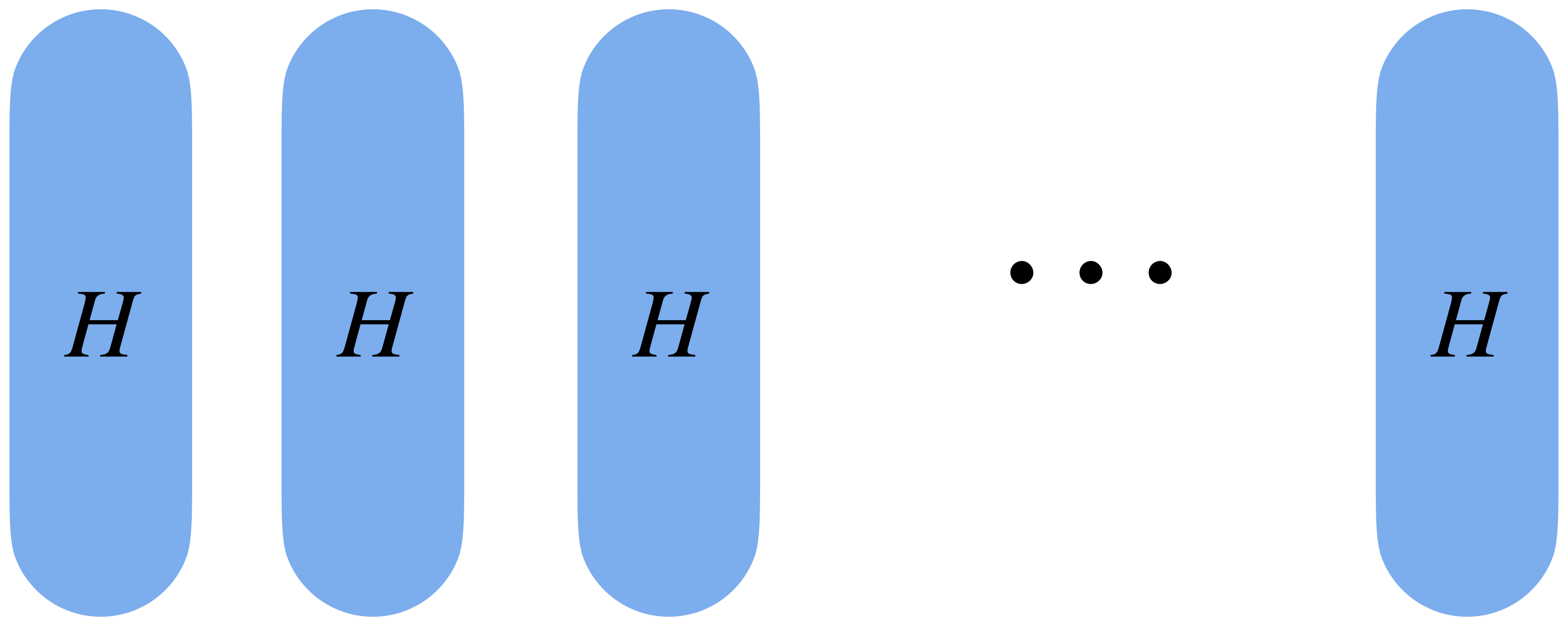

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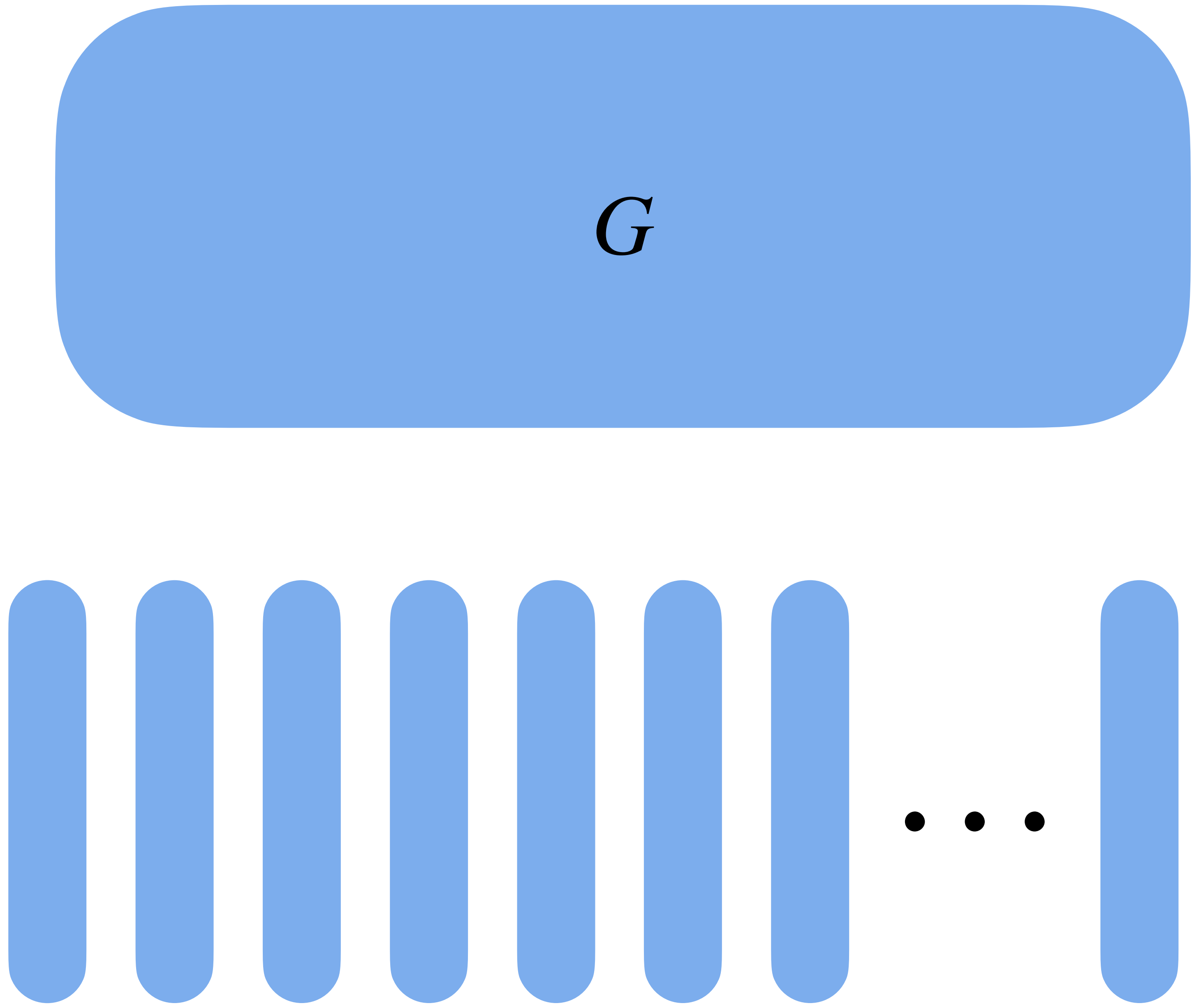

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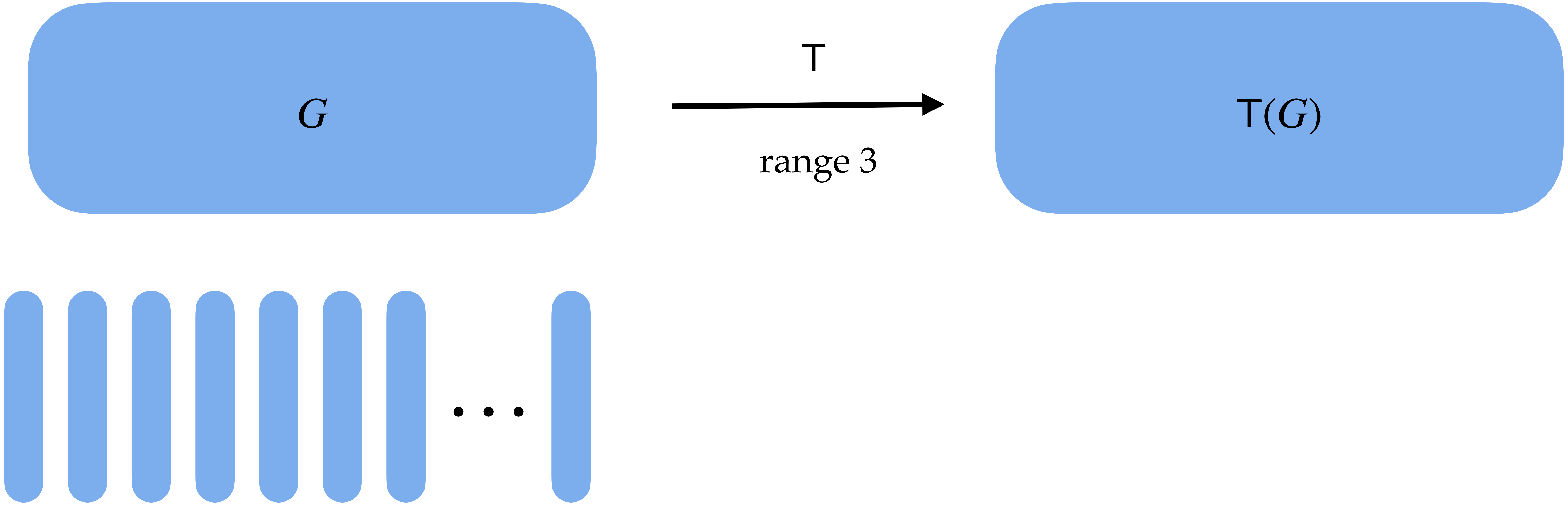

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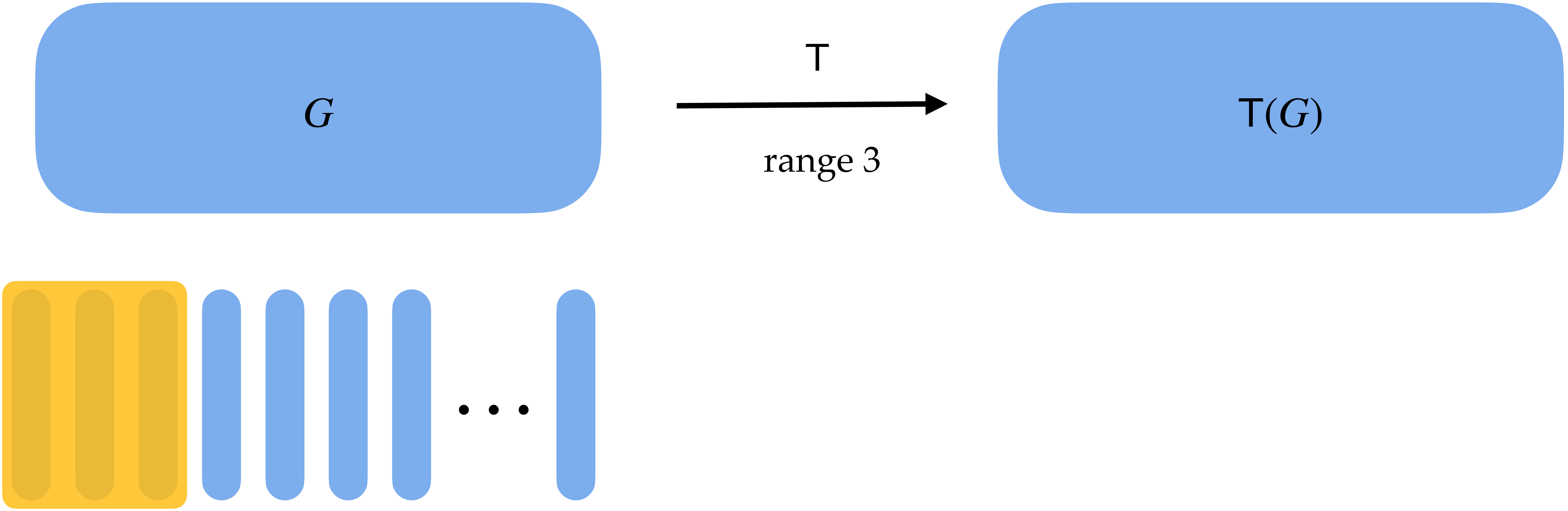

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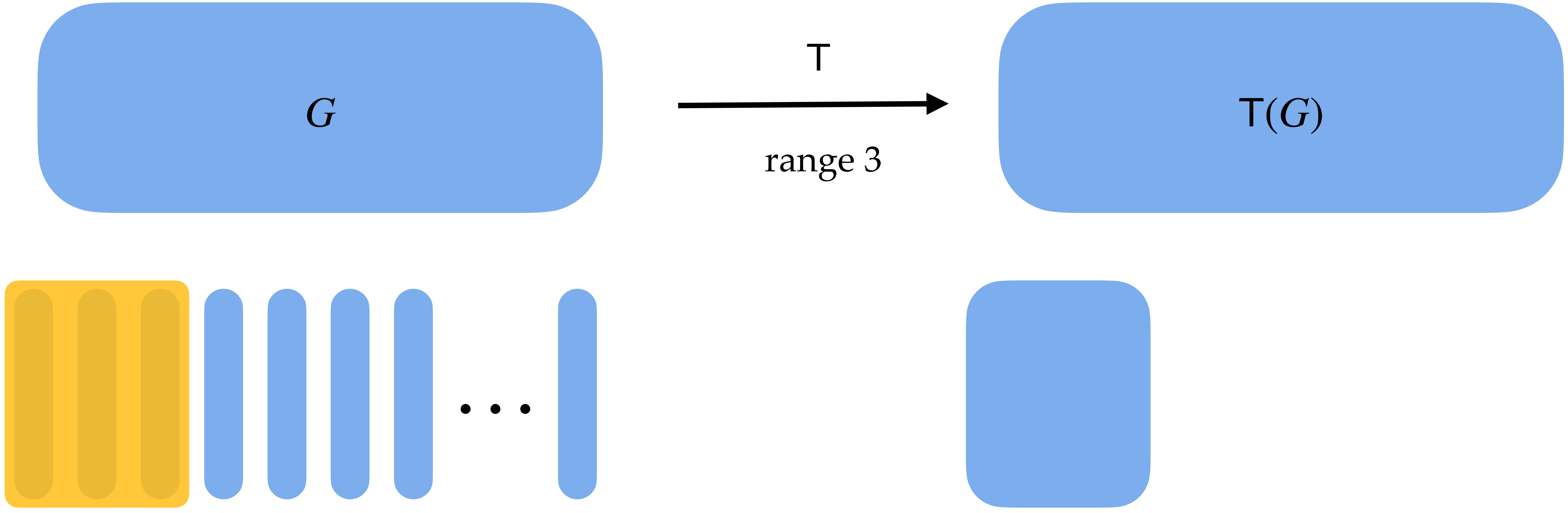

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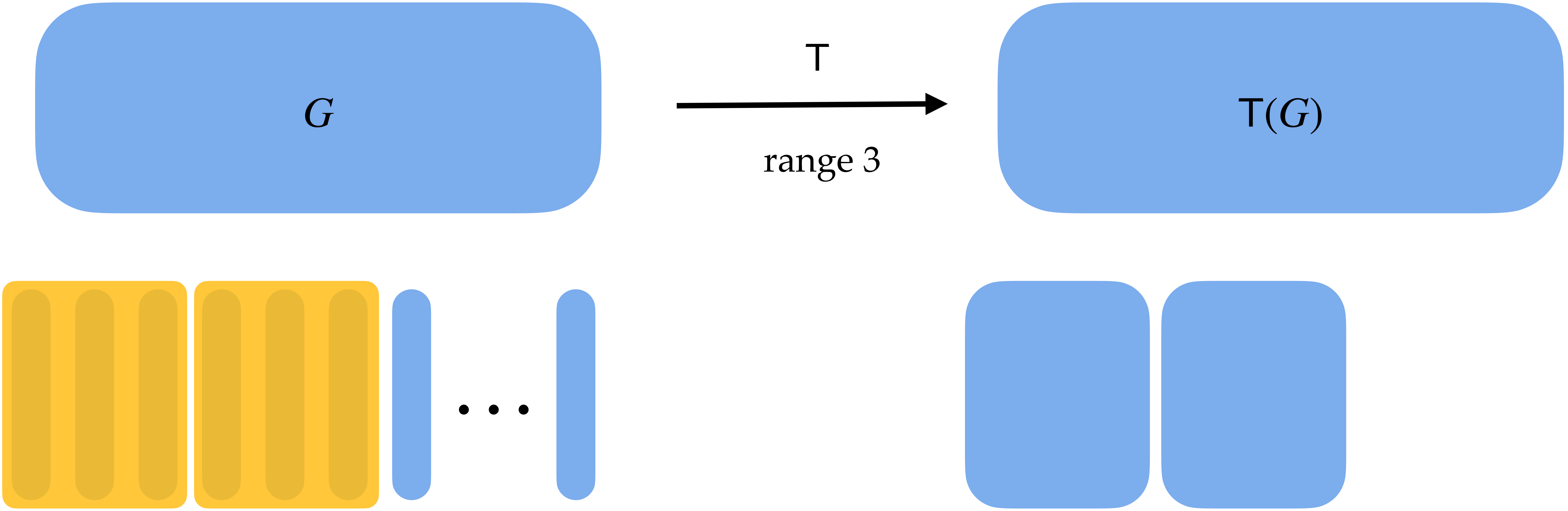

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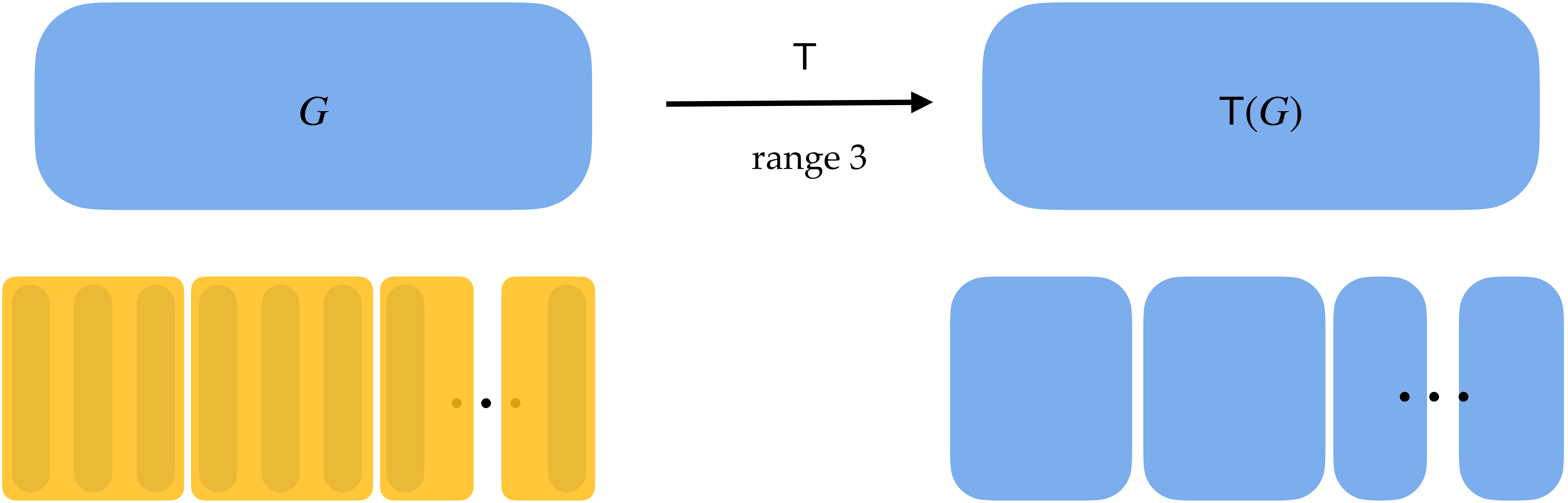

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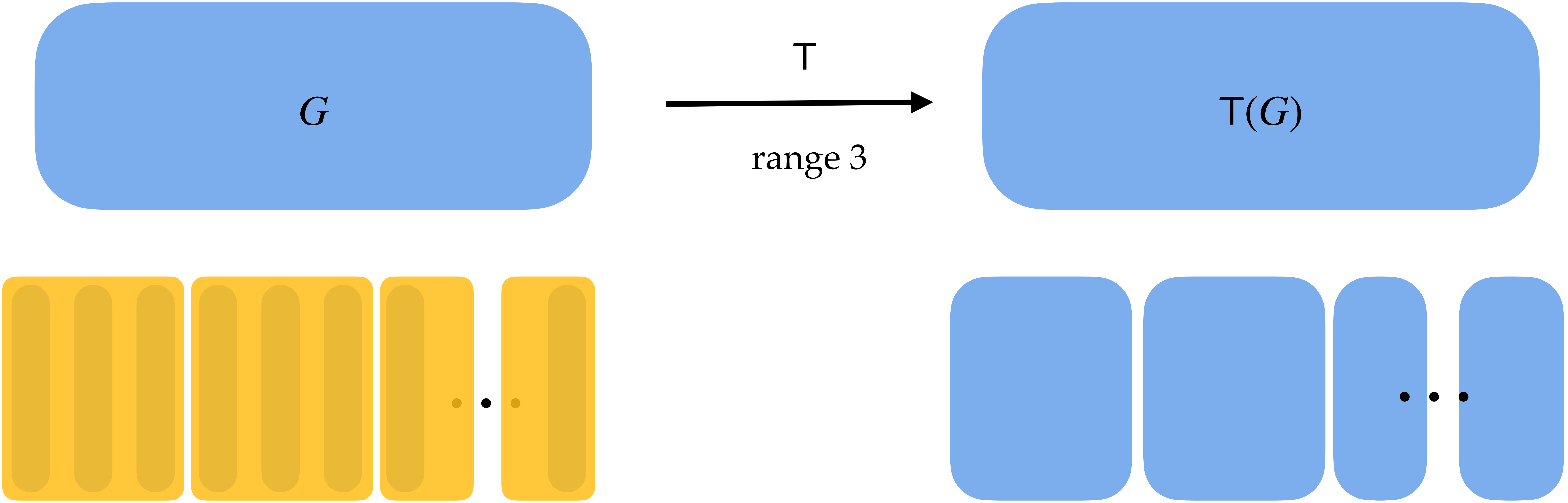

Need to also argue that each slice has small clique-width.

Slice decompositions and planar graphs

Theorem

Let T be a transduction of bounded range. Then T(*Planar*) admits slice decompositions.

Proof idea:

(I) Show that the class of planar graphs admits slice decompositions. ✓

(II) Show that if a class C admits slice decompositions, then $\mathsf{T}(C)$ admits slice decompositions for any transduction T of bounded range. ✓

Theorem

There is no first-order transduction that produces the class of all 3-dimensional grids from the class of planar graphs.

Proof plan:

- Focus on transductions of **bounded range** first:

  (i) Show that the class $\mathsf{T}(\mathit{Planar})$ has **slice decompositions** for every transduction T of bounded range.

  (ii) Show that 3D-grids do not have slice decompositions

  From (i) and (ii) we have that: 3D-grids $\subsetneq \mathsf{T}(\mathit{Planar})$

- Extend the result to full transductions.

Slice decompositions and planar graphs

Lemma

The class of 3D-grids does not admit slice decompositions.

Slice decompositions and planar graphs

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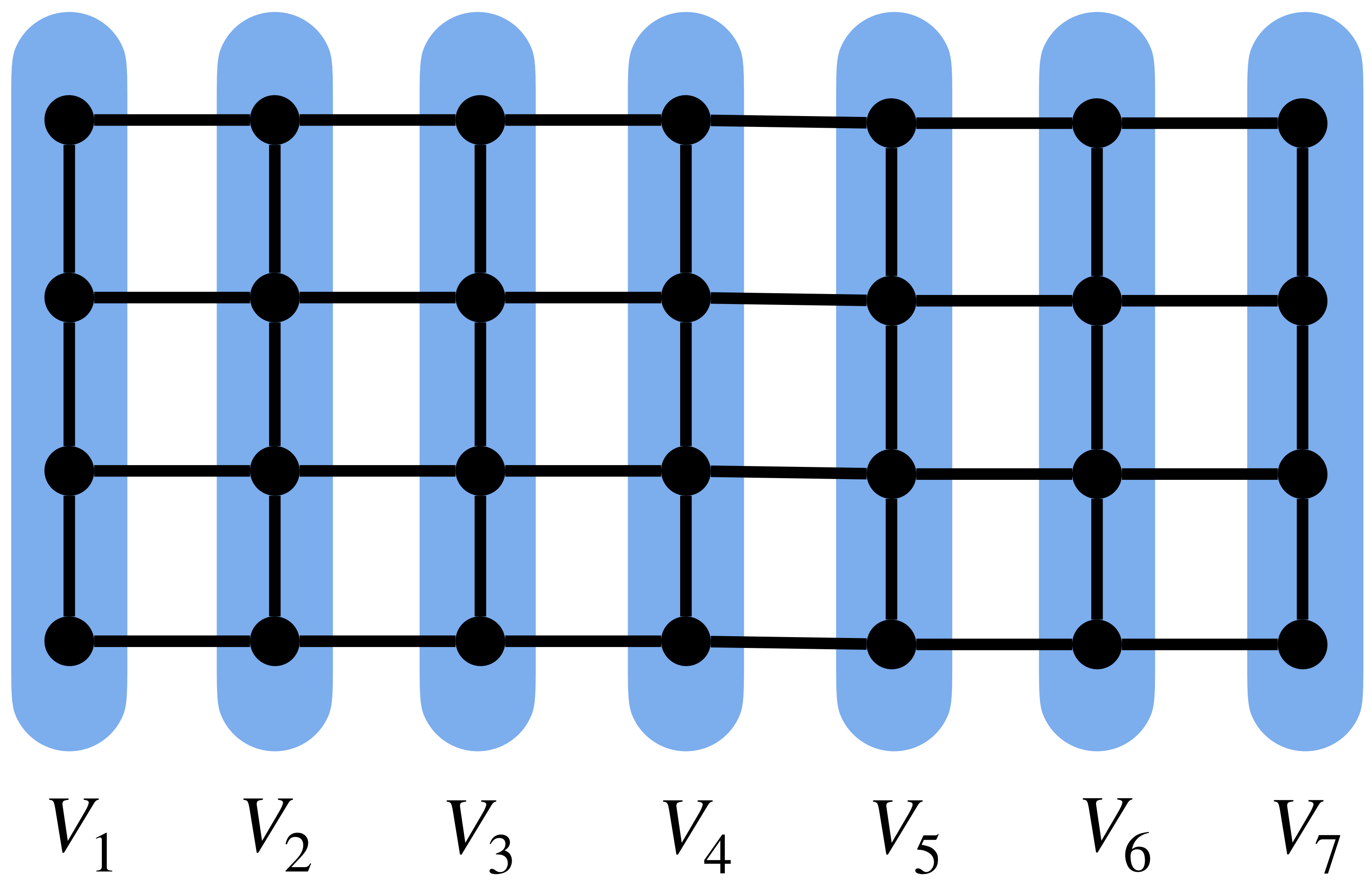

Will not work for 3D-grids.

Slice decompositions and planar graphs

Lemma

The class of 3D-grids does not admit slice decompositions.

Proof: Easy consequence of a result of Berger, Dvořák and Norine.

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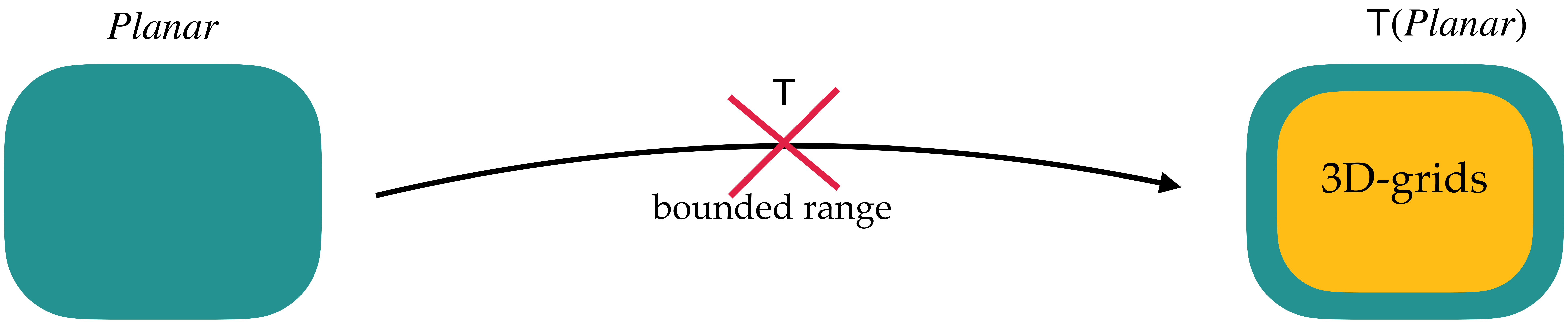

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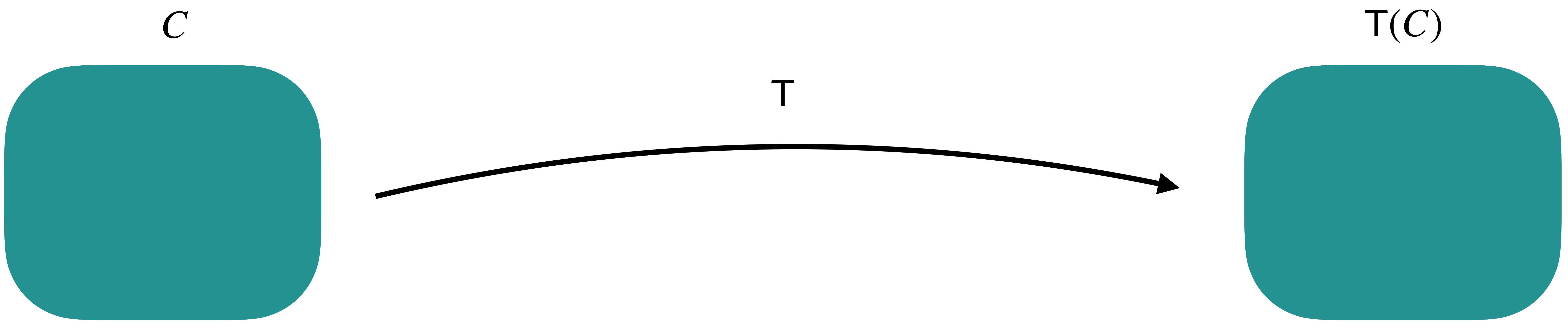
Decomposing transductions



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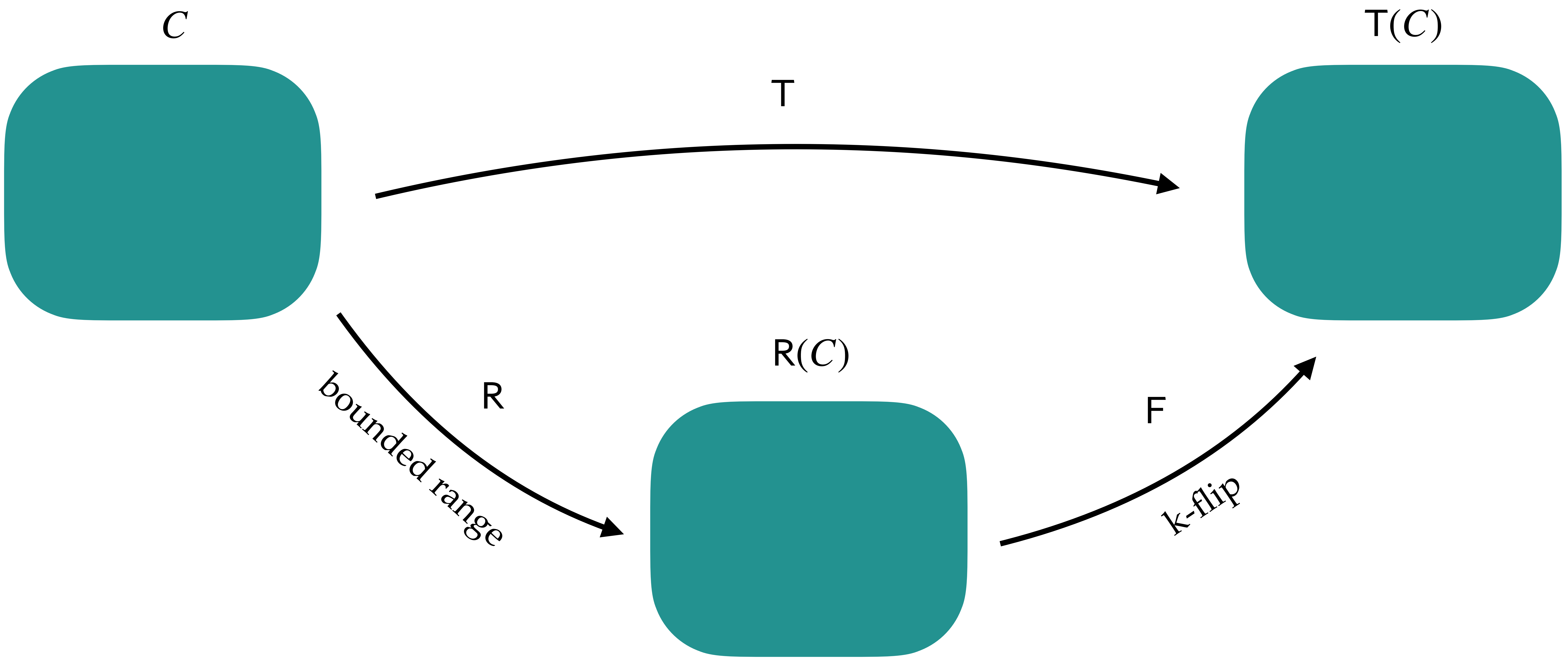

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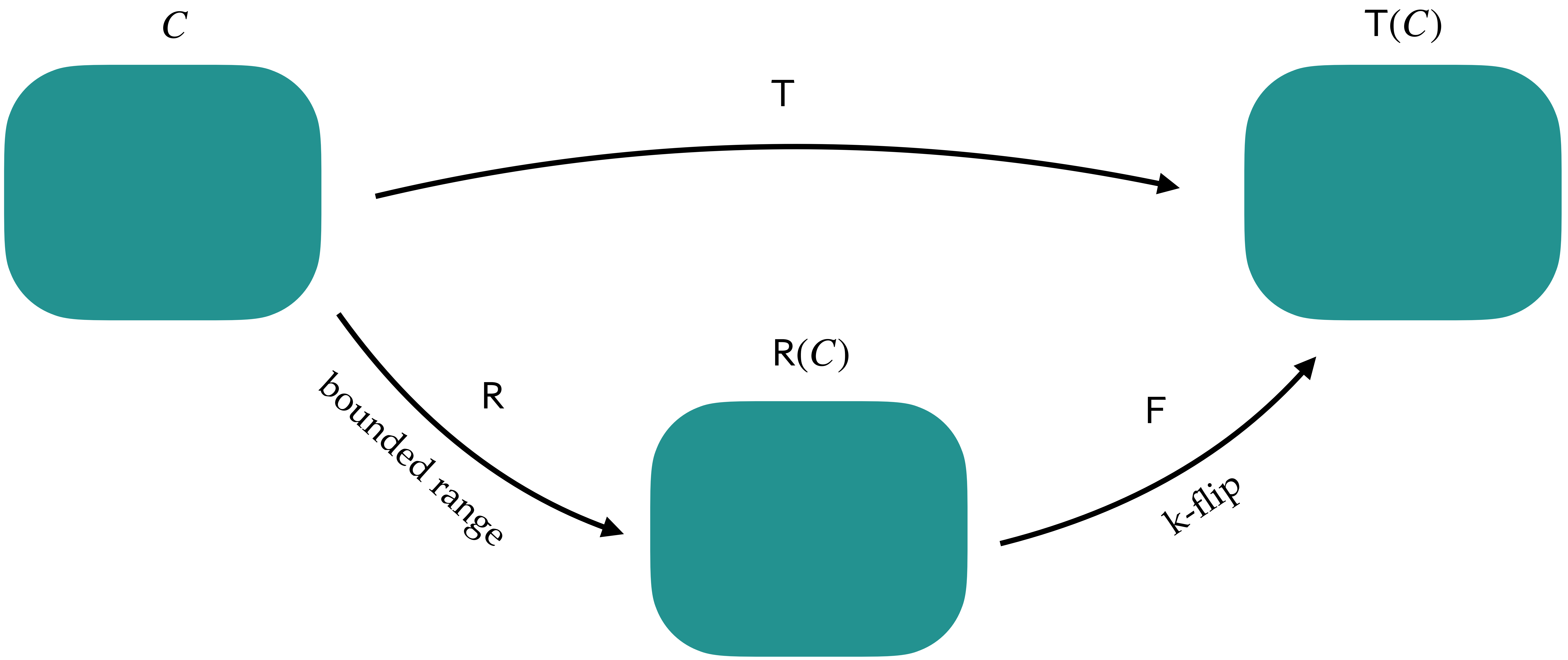

k-flip: special transduction — at most k subset complementations

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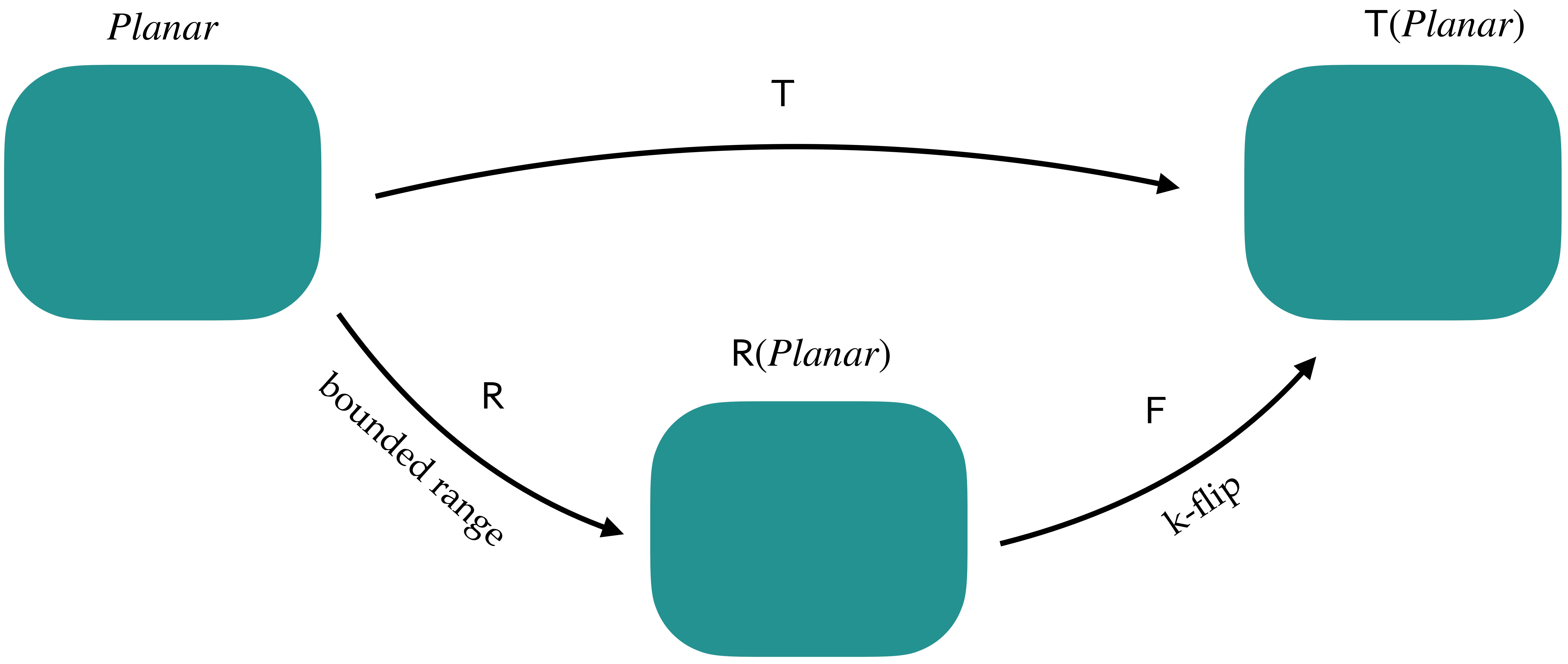

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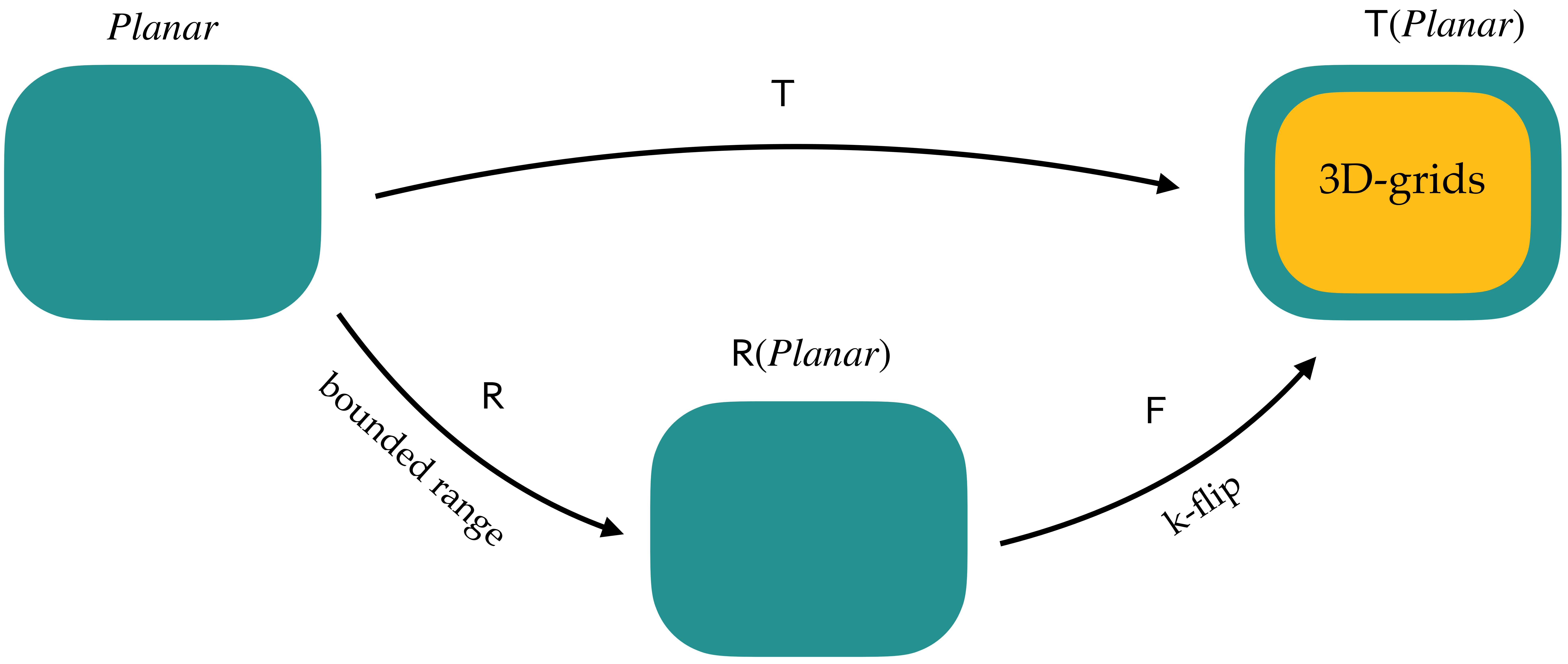

k-flip: special transduction — at most k subset complementations

Theorem

Let C be a graph class such that all 3D-grids are in $\mathsf{F}(C)$, i.e. every 3D-grid is a k-flip of some graph from C. Then there is a bounded range transduction S such that every 3D-grid is in $\mathsf{S}(C)$.

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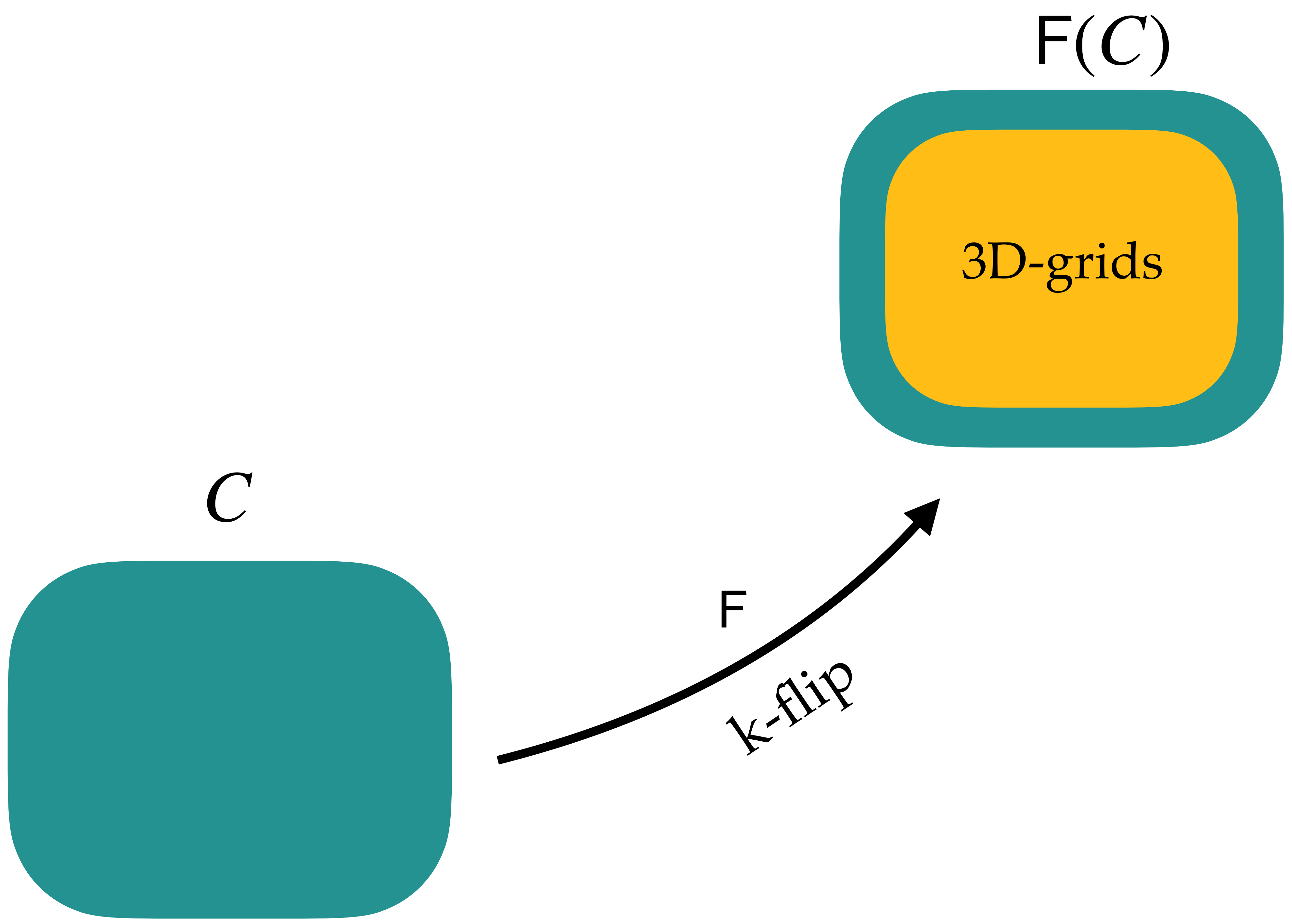

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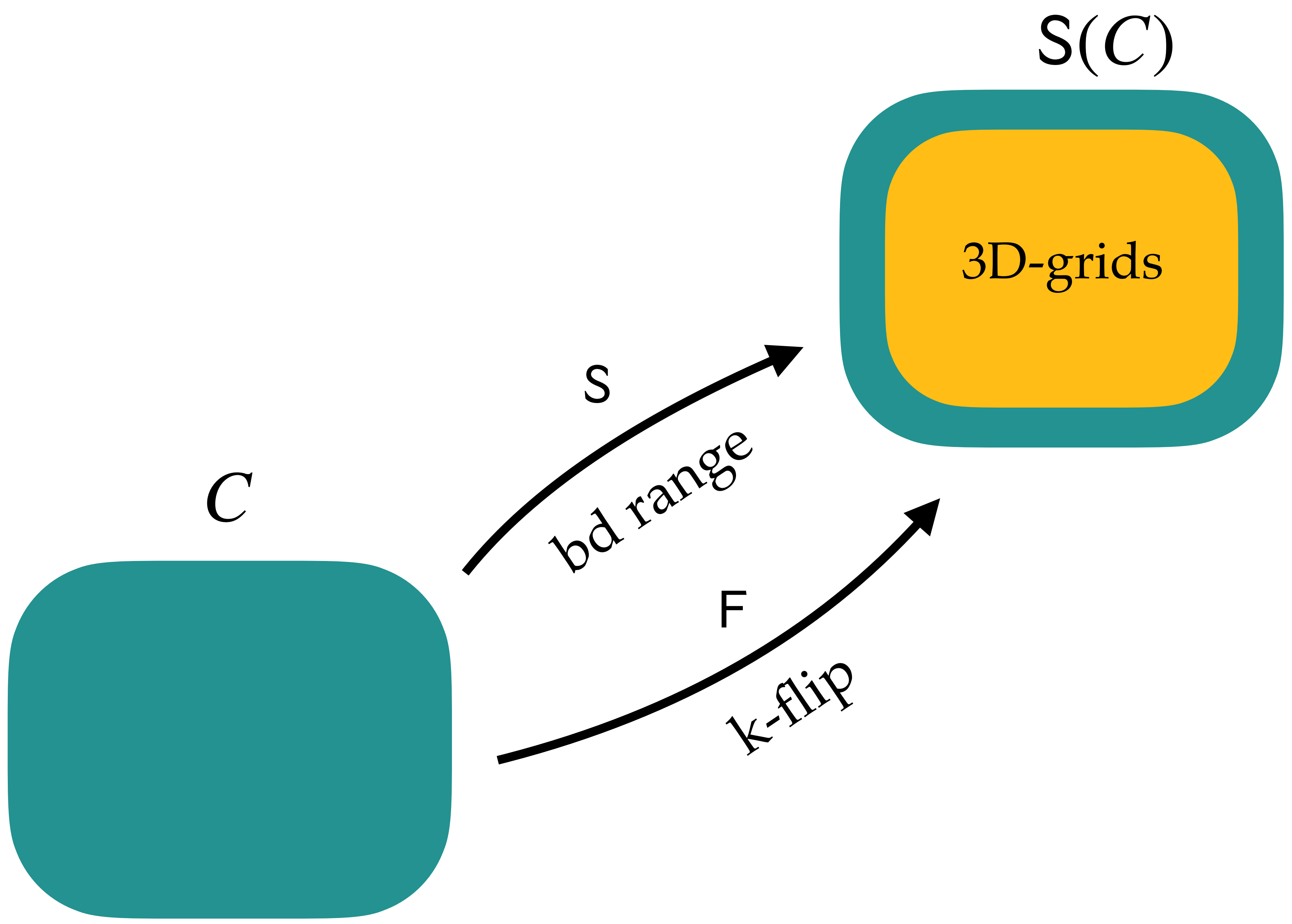

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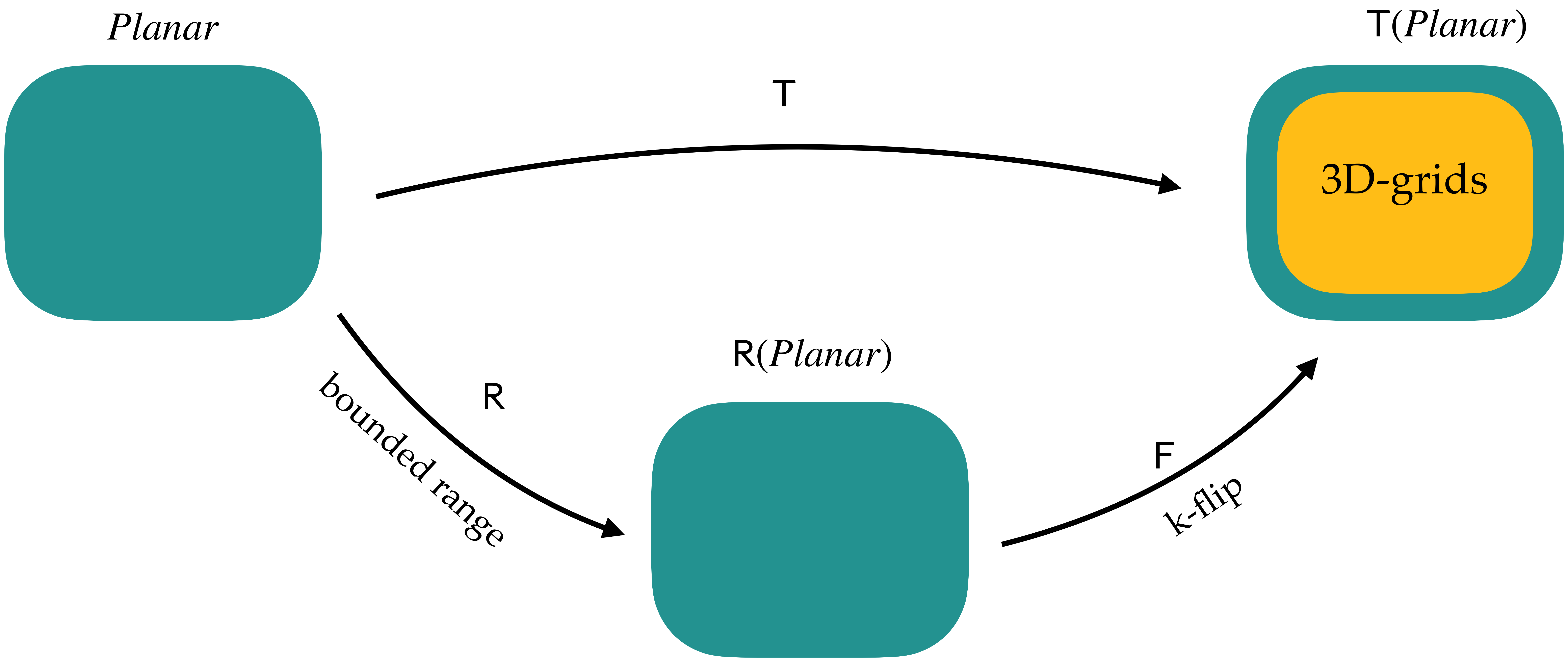

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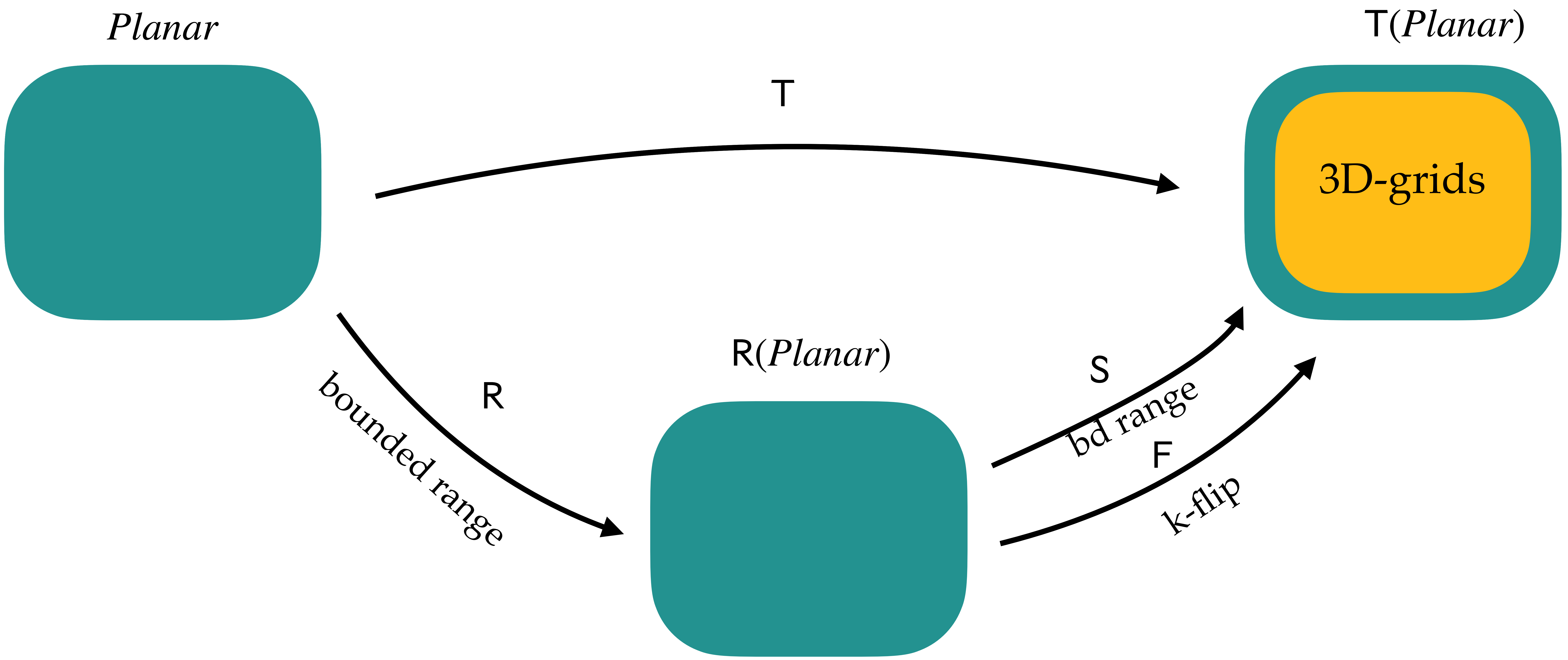

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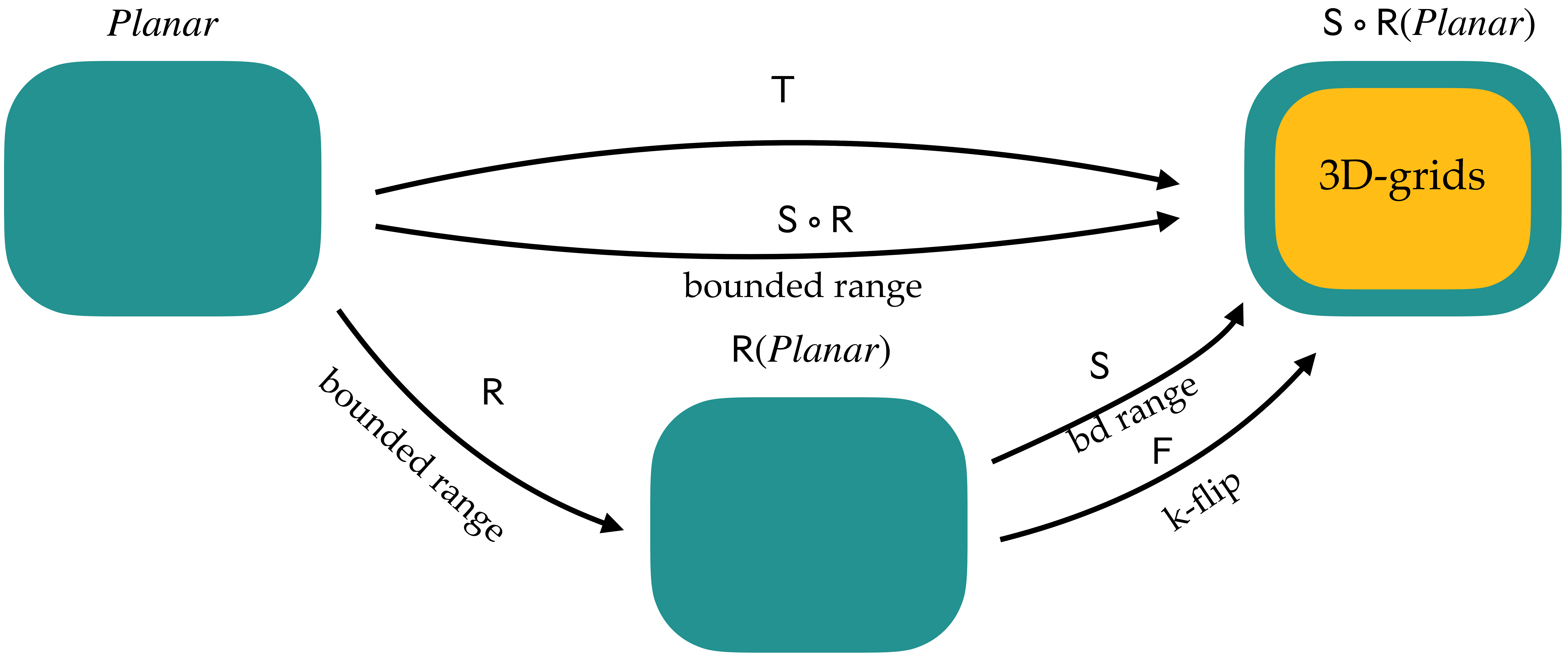

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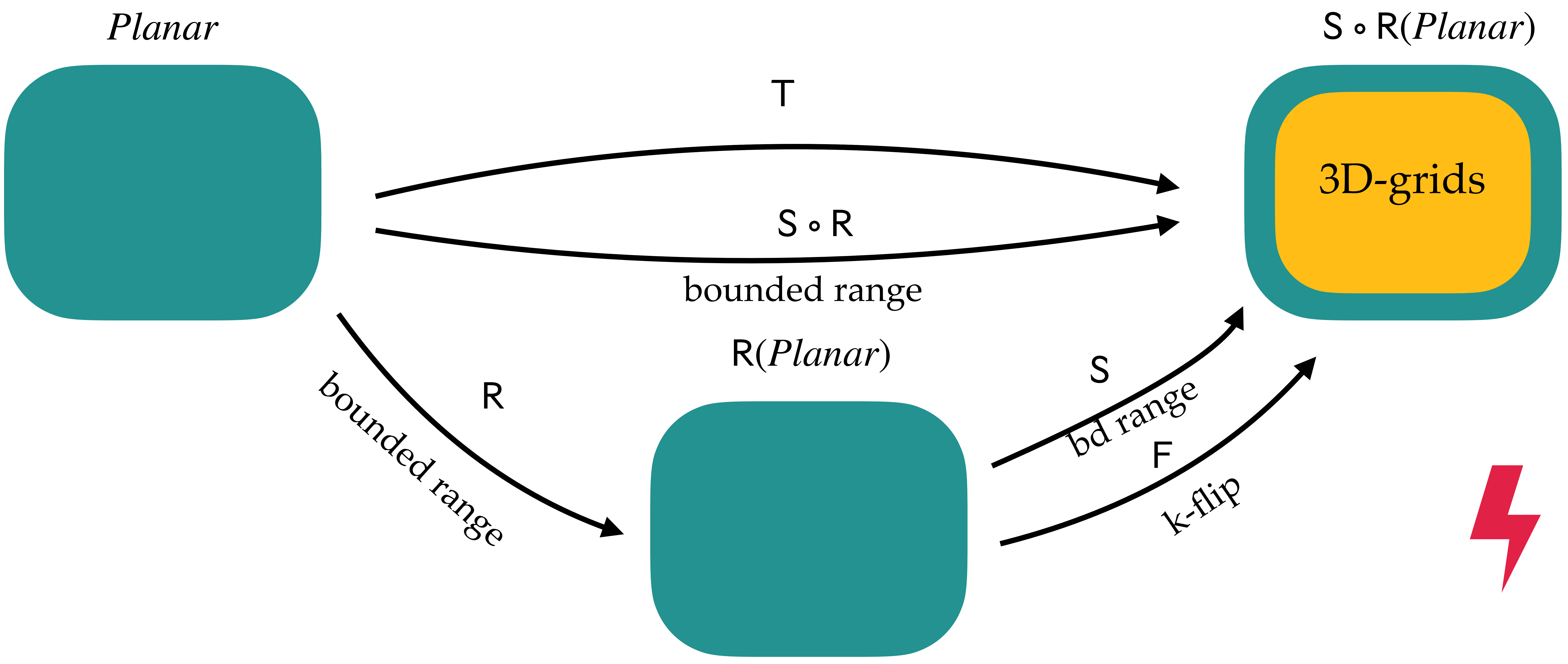

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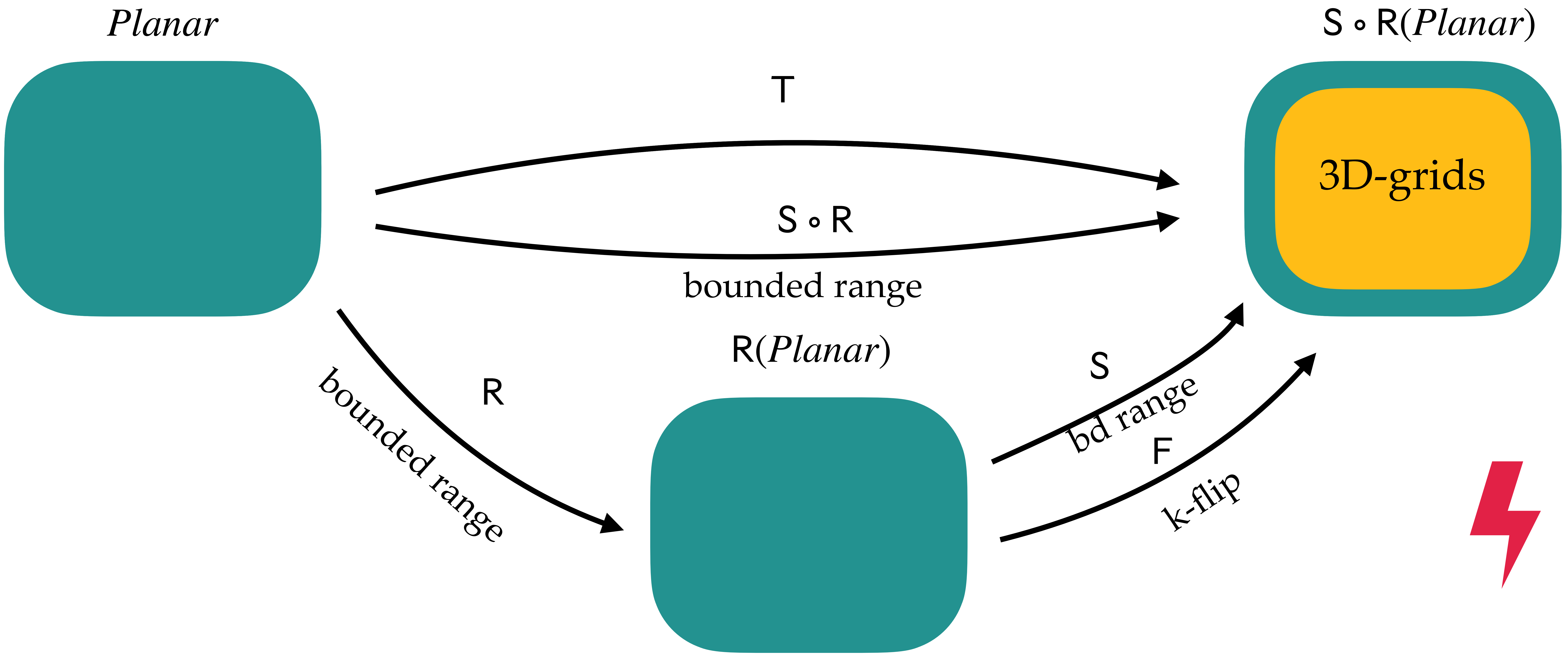

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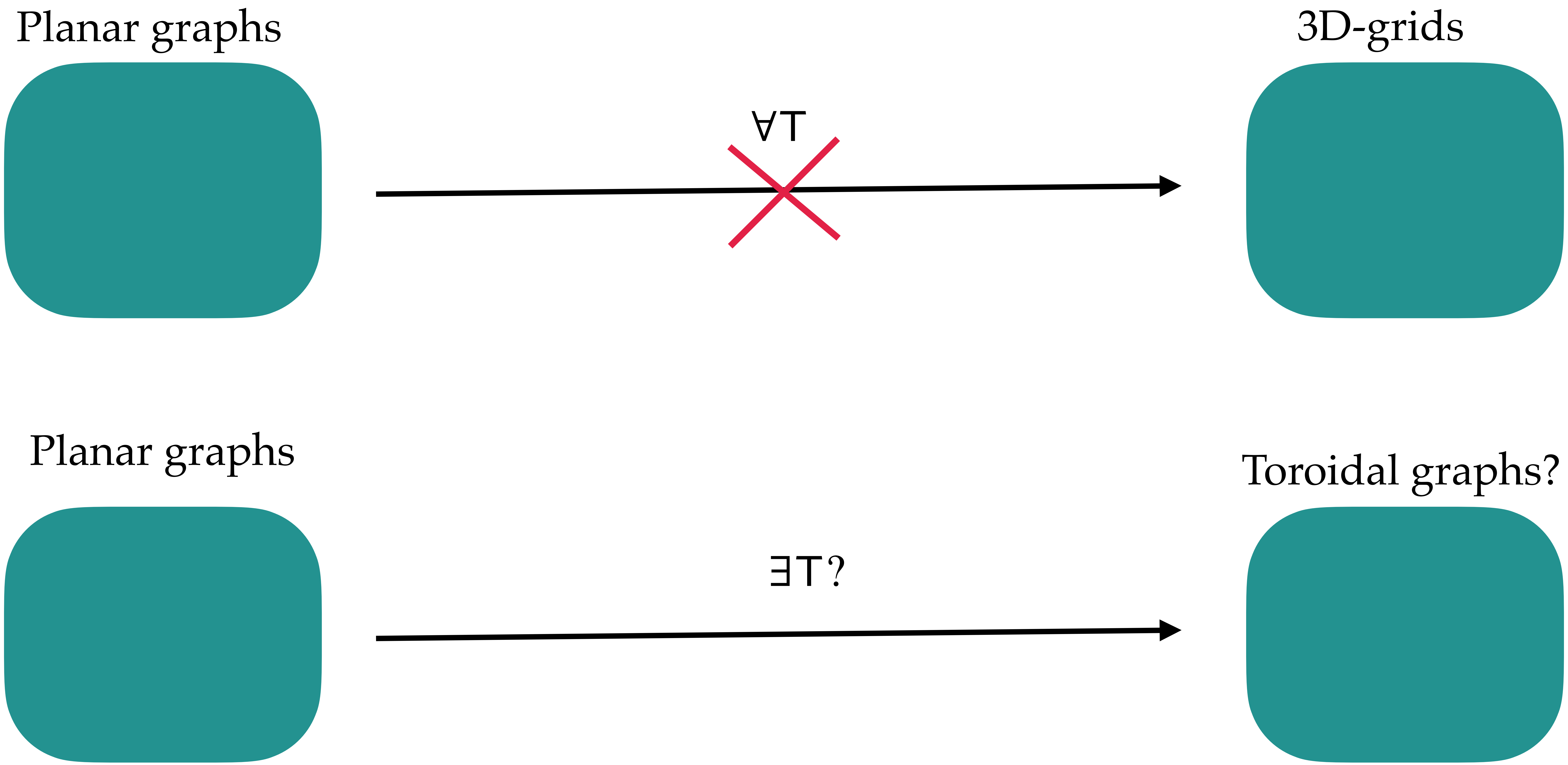
Thank you!



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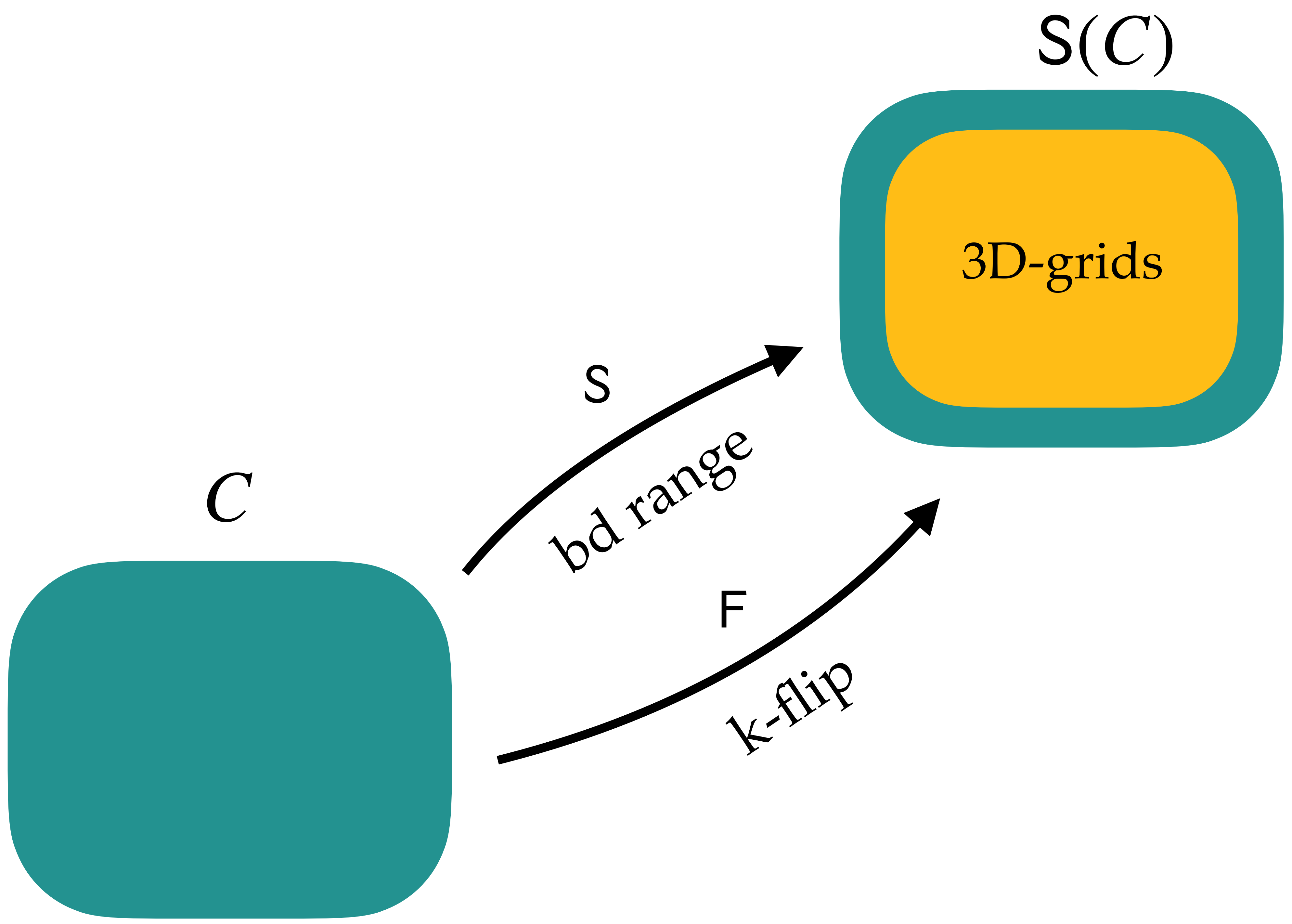

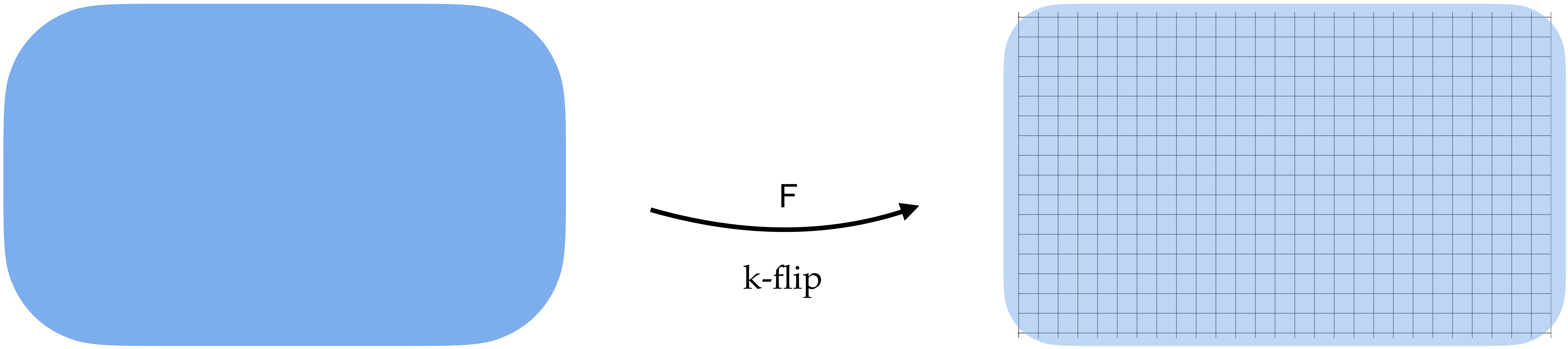
F
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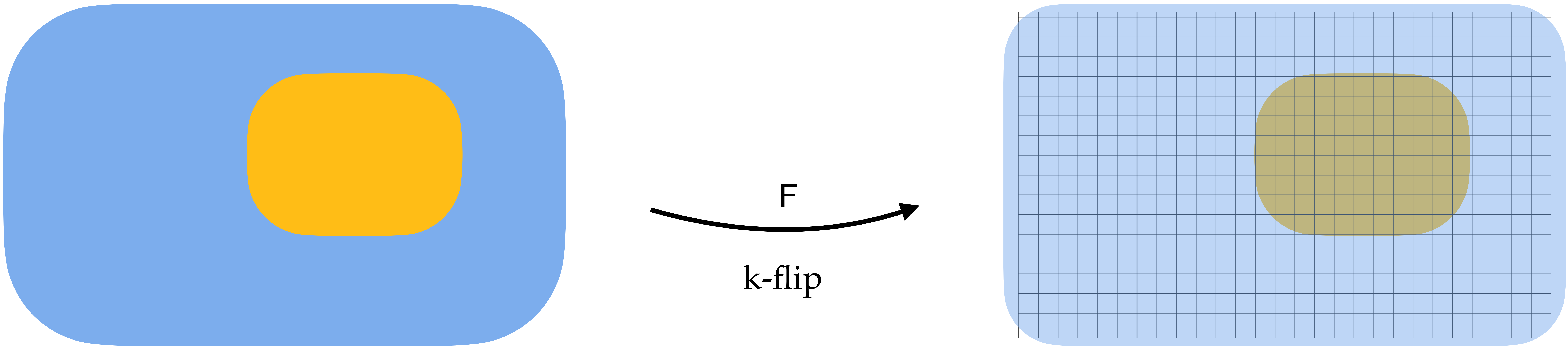
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