Separability properties of monadically dependent graph classes

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Motivation

FO model checking over $\mathcal{C}$:

GIVEN: A FO-sentence ϕ and a graph $G \in \mathcal{C}$.
PARAMETER: $|\phi|$.
DECIDE: $G \models \phi$.

Conjecture

A hereditary class admits FPT first-order model checking iff it is monadically dependent.

where: monadic dependence = does not FO-transduce the class of all graphs

Motivation

The conjecture has been confirmed for:

- $K_{t,t}$-free classes (where mon. dependence = nowhere density, [GKS17])
- edge-stable classes (where mon. dependence = mon. stability, [DEM+24])
- ordered classes (where mon. dependence = bdd twin-width, [BGOdM+22])

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...we have insufficient combinatorial understanding of monadic dependence.

Characterisations of nowhere density

Originally defined in terms of the exclusion of shallow minors, nowhere dense classes have been shown to admit various equivalent characterisations:

- forbidden subgraphs
- quasi-bounded bounded treedepth covers
- Splitter game
- sparse neighbourhood covers
- neighbourhood complexity
- uniform quasi-wideness
- "local separability"
- ...

Separability in sparse classes

Lemma

For every G and $\varepsilon > 0$ there exists $k := k(\mathrm{tw}(G), \varepsilon)$ such that every connected component of G contains fewer than $\varepsilon \cdot |G|$ vertices after deleting at most k vertices.

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Theorem (Nesetril, Ossona de Mendez, 2016)

Let $\mathcal{C}$ be nowhere dense. Then for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k := k(\mathcal{C}, r, \varepsilon)$ such that for every $G \in \mathcal{C}$ there is some $S \subseteq V(G)$ with $|S| \leq k$ satisfying:

$$\frac{|\mathrm{Ball}^r_{G \setminus S}(v)|}{|G|} < \varepsilon, \quad \textit{for all } v \in V(G)$$

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Theorem (Nesetril, Ossona de Mendez, 2016)

A class $\mathcal{C}$ is nowhere dense if and only if for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k := k(\mathcal{C}, r, \varepsilon)$ such that for every $G \in \mathcal{C}$ and every $A \subseteq V(G)$ there is some $S \subseteq A$ with $|S| \leq k$ satisfying:

$$\frac{|\mathrm{Ball}^r_{G[A \setminus S]}(v)|}{|A|} < \varepsilon, \quad \text{for all } v \in V(G)$$

Separability in dense classes

Lemma

For every G and $\varepsilon > 0$ there exists $k := k(\mathrm{cw}(G), \varepsilon)$ and a k-flip G' of G such that every connected component of G' contains fewer than $\varepsilon \cdot |G|$ vertices.

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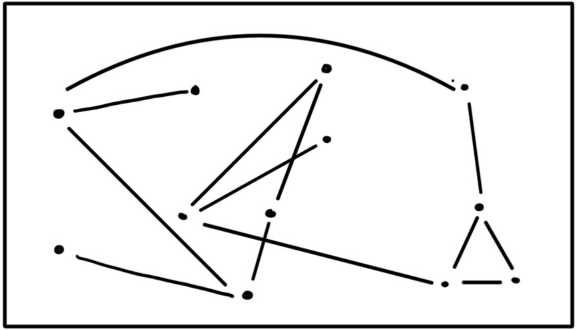

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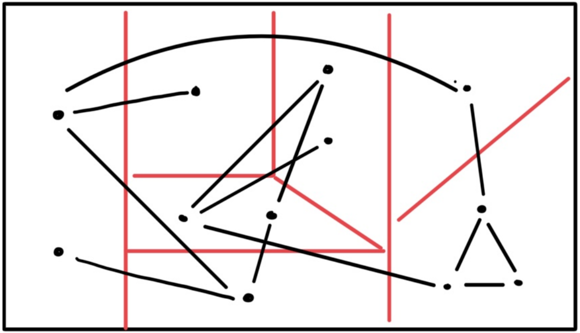

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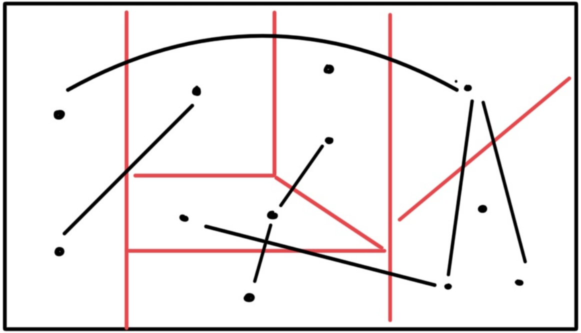

Separability in dense classes

Theorem (BBEGMPPT, 2025)

Let $\mathcal{C}$ be monadically dependent. Then for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k := k(\mathcal{C}, r, \varepsilon)$ such that for every $G \in \mathcal{C}$ there is some k-flip G' of G satisfying:

$$\frac{|\mathrm{Ball}^r_{G'}(v)|}{|V(G)|} < \varepsilon, \quad \text{for all } v \in V(G)$$

Separability in dense classes

Theorem (BBEGMPPT, 2025)

A class $\mathcal{C}$ is monadically dependent if and only if for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k := k(\mathcal{C}, r, \varepsilon)$ such that for every $G \in \mathcal{C}$ and every probability measure $\mu : \mathcal{P}(V(G)) \to [0,1]$ there is some k-flip G' of G satisfying:

$$\mu(\mathrm{Ball}^r_{G'}(v)) < \varepsilon, \quad \text{for all } v \in V(G)$$

Separability in dense classes

Theorem (BBEGMPPT, 2025)

A class $\mathcal{C}$ is monadically dependent *if and only if* *for every $r \in \mathbb{N}$ and $\varepsilon > 0$ there exists $k := k(\mathcal{C}, r, \varepsilon)$ such that for every $G \in \mathcal{C}$ and every* *probability measure $\mu : \mathcal{P}(V(G)) \to [0,1]$* *there is some k-flip G' of G satisfying:*

$$\mu(\mathrm{Ball}^r_{G'}(v)) < \varepsilon, \quad \text{for all } v \in V(G)$$

We call this property *flip-separability*.

Flip-separability $\implies$ monadic dependence

Lemma

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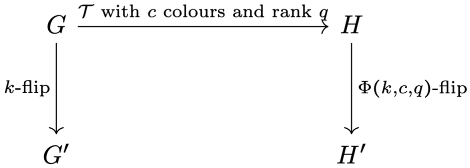

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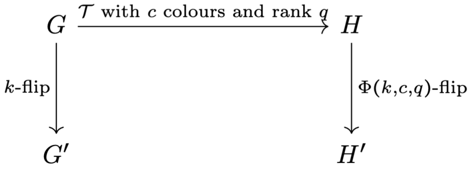

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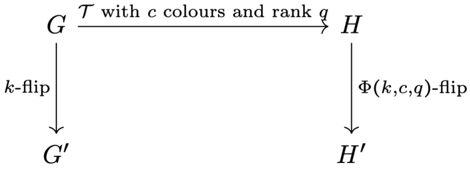

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Lemma

The class of all graphs is not flip-separable.

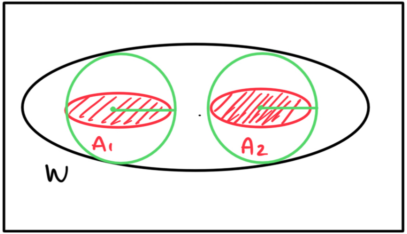
Proof: Any flip-separable class has bounded VC-dimension.

Flip-breakability

Theorem (Dreier, Mählmann, Toruńczyk, 2024)

Every monadically dependent graph class $\mathcal{C}$ is flip-breakable, i.e. for every $r \in \mathbb{N}$ there exists $t := t(\mathcal{C}, r)$ and a function $M_r : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{C}$, $m \in \mathbb{N}$, and set $W \subseteq V(G)$ of size at least $M_r(m)$, there are disjoint subsets $A_1 \subseteq W, A_2 \subseteq W$ each of size at least m and a t-definable flip H of G satisfying

$$\mathrm{Ball}^r_H(A_1) \cap \mathrm{Ball}^r_H(A_2) = \emptyset$$



Monadic dependence $\Longrightarrow$ flip-separability

Proof idea:

- Fix $r \in \mathbb{N}$ and $\varepsilon > 0$. Let $G \in \mathcal{C}$ and $S \subseteq V(G)$ be of minimal size such that there is S-flip of G witnessing that all the r-balls are ε-small. We know that $S = V(G)$ works;

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- Proceed by contradiction: if k is big enough then we can apply breakability $p := \lceil \frac{1}{\varepsilon} \rceil$ many times on S to get sets $A_1, \ldots, A_p$ pairwise far apart is some F-flip, where $|F|$ is bounded;

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- In this F-flip, one of the A_i's must have large measure;

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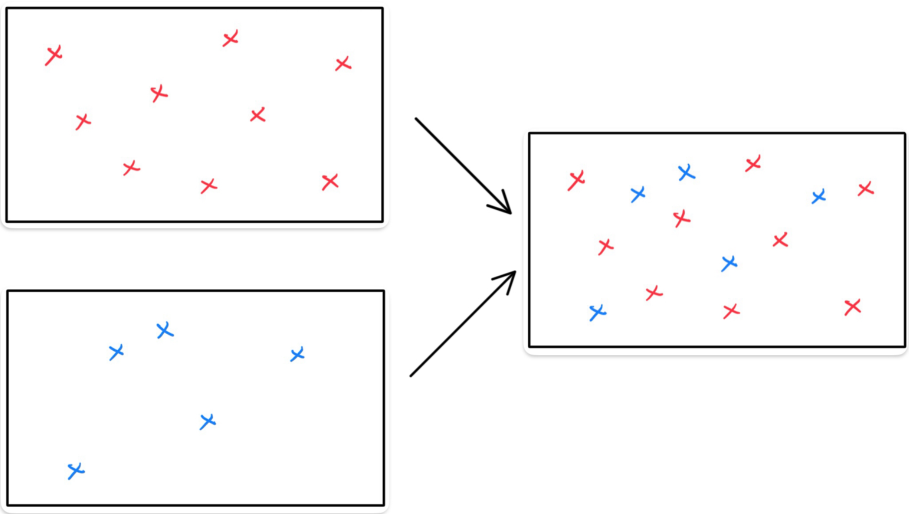
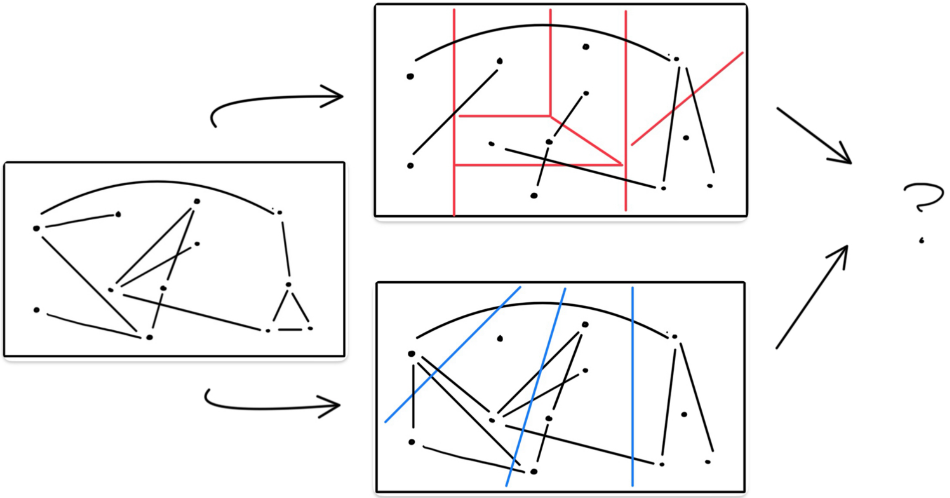
- In this F-flip, one of the A_i's must have large measure;
- Ideally we would want to remove A_i from S, but we should also add F back to it...

We want to somehow combine the information coming from multiple flips.

Aggregation of deletions



Aggregation of flips..?



Partition metric

For any $\mathcal{P}$ partition of $V(G)$ we define

$$\mathrm{dist}^{G}_{\mathcal{P}}(u, v) := \max\{\, \mathrm{dist}_{G'}(u, v) \colon G' \text{ is a } \mathcal{P}\text{-flip of } G\},$$

or equivalently

$$\mathrm{Ball}^{r}_{\mathcal{P}}(u) := \bigcap_{G' \text{ is a } \mathcal{P}\text{-flip of } G} \mathrm{Ball}^{r}_{G'}(u)$$

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Intuition: things are far apart in $\mathrm{dist}_{\mathcal{P}}$ iff this is witnessed by some $\mathcal{P}$-flip

Metric conversion

Sometimes we only care about distance, and sometimes we care about concrete structure.

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It turns out that these perspectives are interchangeable:

Theorem

For every graph G of VC-dimension d and a partition $\mathcal{P}$ of $V(G)$, there is a set of vertices S of size $\mathcal{O}(d \cdot |\mathcal{P}|^{2d+2})$ and an S-definable flip G' of G such that for every $r \in \mathbb{N}$, $v \in V(G)$

$$\mathrm{Ball}^r_{G'}(v) \subseteq \mathrm{Ball}^{30r}_{\mathcal{P}}(v).$$

Monadic dependence $\implies$ flip-separability

Proof idea:

- Fix $r \in \mathbb{N}$ and $\varepsilon > 0$. Let $G \in \mathcal{C}$ and $S \subseteq V(G)$ be of minimal size such that all the r-balls are ε-small in the S-metric. We know that $S = V(G)$ works;

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- In this F-flip, one of the A_i's must have large measure;
- Take $S' = (S \setminus A_i) \cup F$.

This does the trick, but it gets slightly technical...

Thank you!

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