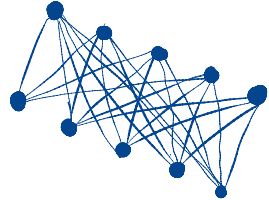
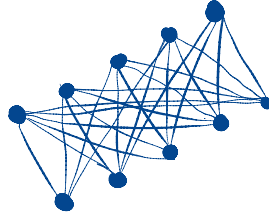
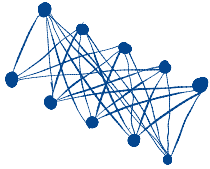
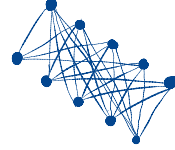
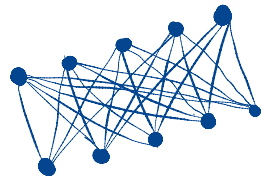


On Polynomial Degree-Boundedness



Linda Cook
UTRECHT UNI.



joint work with:

Bordeaux/ENS LYON

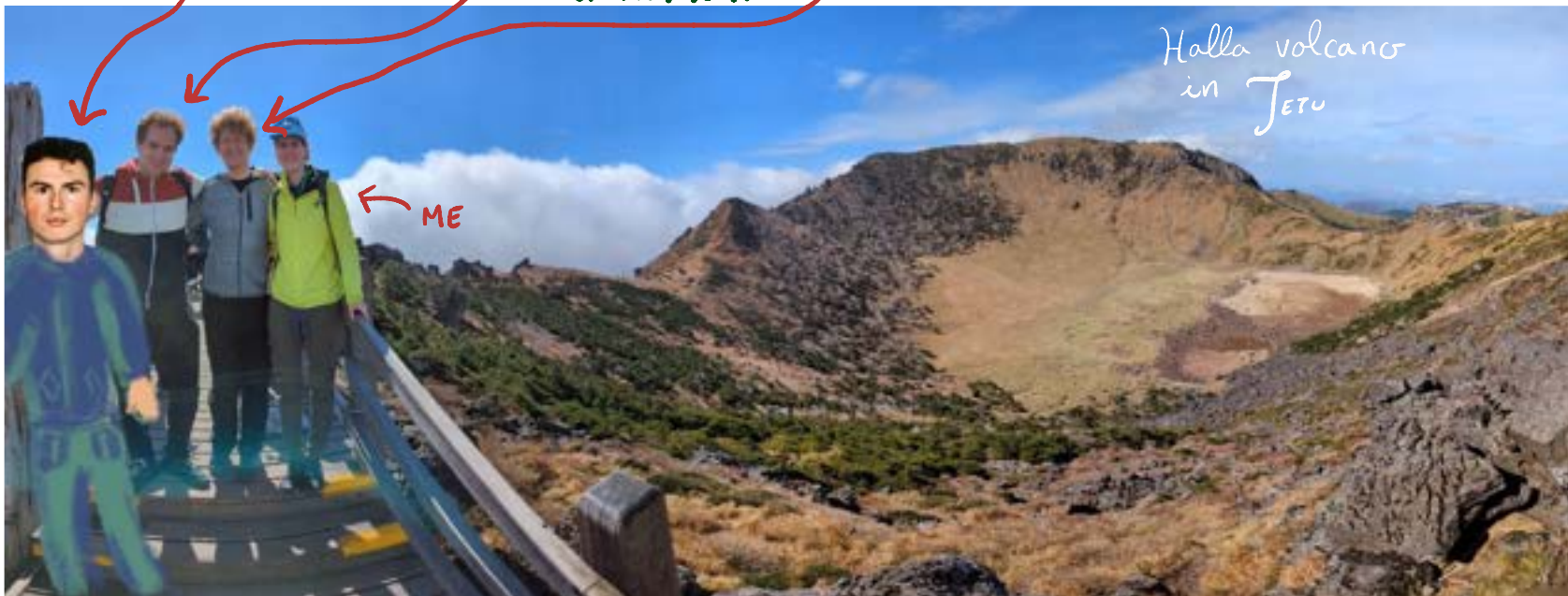
Romain Bourneuf, Matija Bucic, James Davies

U. of VIENNA!

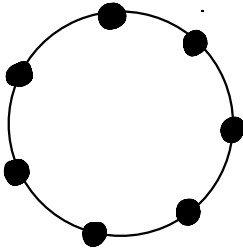
Leipzig

Halla volcano
in JETU

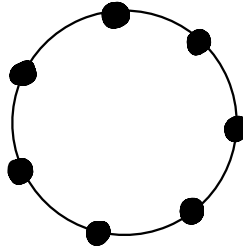
ME



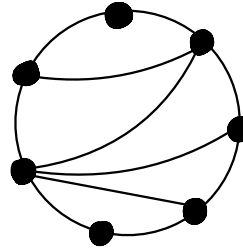
Subgraph: CAN DELETE VERTICES AND EDGES
Induced Subgraph: CAN DELETE VERTICES



C_7



has an
induced C_7



No induced
cycle of length ≥ 4

$\forall H \exists d_H$ s.t. IF G HAS AVERAGE DEGREE $\geq d_H$ THEN
 G CONTAINS H AS A _____.



SUBGRAPH



$\exists G$ WITH ARBITRARILY
BIG GIRTH AND AVG. DEGREE

SUBDIVISION



MADER

INDUCED SUBDIVISION

$\forall H \exists d_H$ s.t. IF G HAS AVERAGE DEGREE $\geq d_H$ THEN
 G CONTAINS H AS A _____.



SUBGRAPH



$\exists G$ WITH ARBITRARILY
 BIG GIRTH AND AVG. DEGREE

SUBDIVISION



MADER

INDUCED SUBDIVISION



IF G HAS AVERAGE DEGREE $\geq f_H$ THEN
 G CONTAINS H AS A INDUCED SUBDIVISION.

$\forall H \exists d_H$ s.t. IF G HAS AVERAGE DEGREE $\geq d_H$ THEN
 G CONTAINS H AS A _____.



SUBGRAPH



$\exists G$ WITH ARBITRARILY
 BIG GIRTH AND AVG. DEGREE

SUBDIVISION



MADER

INDUCED SUBDIVISION



$\forall H \exists f_H$ s.t. IF G HAS NO $K_{t,t}$ SUBGRAPH AND

IF G HAS AVERAGE DEGREE $\geq f_H(t)$ THEN

G CONTAINS H AS A INDUCED SUBDIVISION.

$\forall H \exists d_H$ s.t. IF G HAS AVERAGE DEGREE $\geq d_H$ THEN
 G CONTAINS H AS A _____.



SUBGRAPH



$\exists G$ WITH ARBITRARILY
BIG GIRTH AND AVG. DEGREE

SUBDIVISION



MADER

INDUCED SUBDIVISION



THM KUHN & OTTHUS 2004

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IF G HAS AVERAGE DEGREE $\geq f_H(t)$ THEN
 G CONTAINS H AS A INDUCED SUBDIVISION.

KUHN OTHUS
2004

$$f_H \in O(2^{2^t})$$

DU, GIRÃO
HUNTER
MC CARTY
SCOTT '23

$$f_H \in O(2^t)$$

CONJECTURE

BONAMY, BOUSQUET
PIŁIPCZUK*, RZĄŻEWSKI
THOMASSÉ, WALCZAK
2020

THEOREM

$$f_H \in \text{poly}(t)!$$

(Bourneuf, Bucić, C., Davies 2023)
(ALSO Girão & Hunter 2023)

YES IF H IS PATH OR CYCLE

BONAMY BOUSQUET
PIŁIPCZUK RZĄŻEWSKI
WALCZAK

YES IF H IS A TREE

SCOTT, SEYMOUR
& SPIRKL

THM KUHN & OTHUS 2004

$\forall H \exists f_H$ s.t. IF G HAS NO $K_{t,t}$ SUBGRAPH AND
IF G HAS AVERAGE DEGREE $\geq f_H(t)$ THEN
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*MICHAŁ

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*MICHAŁ

APPLICATION TO COLORING

WHAT MAKES χ BIG?

$$\forall G \quad \omega(G) \leq \chi(G)$$

~~$\exists f \forall G \quad f(\omega(G)) \leq \chi(G)?$~~ NO!

THM(ERDŐS)

$\forall k, l \exists G$ WITH NO CYCLE $\leq k$
and $\chi(G) \geq l$

WE CALL A HEREDITARY CLASS OF GRAPHS $\mathcal{C}$

POLYNOMIALLY χ -BOUNDED IF $\exists f \forall G \in \mathcal{C} \quad f(\omega(G)) \leq \chi(G)$
IF f IS A POLYNOMIAL

~~CONJECTURE~~ SCOTT '97: $\forall H$ FORBIDDING H AS
AN INDUCED SUBDIVISION IS χ -BOUNDED.

FALSE!

TRUE WHEN: H is a tree, cycle, banana tree or a complete bipartite graph

FALSE FOR E.G. $H =$



PAWLIK, KOZIK, KRAWCZYK
LASON, MILEK, TROTTER, WALCZAK

SCOTT
1997

CHUDNOVSKY

SCOTT & SEYMOUR '17

SCOTT &

SEYMOUR '20

KUHN

graph

OSTHUS '04

BBCD '23
GH '23

POLYNOMIALLY χ -BOUNDED !

Main New Pod: Induced Extremal Numbers

$$ex(n, H) = \max \{ |E(G)| \text{ s.t. } |V(G)| = n \text{ \& } G \text{ has no } H \text{ subgraph} \}$$

$$ex(n, J, H\text{-ind}) = \max \{ |E(G)| \text{ s.t. } |V(G)| = n \text{ \& } G \text{ has no } H \text{ induced subgraph} \text{ \& no } J \text{ as a subgraph} \}$$

Erdős-Stone-Simonovits

$$\forall H \quad ex(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right) \binom{n}{2} + o(n^2)$$

WHEN H IS BIPARTITE
THIS IS
 $ex(n, H) = o(n^2)$

Korari-Sós-Turán 1954

$$\forall s \quad ex(n, K_{s,s}) \leq c_s n^{2-\frac{1}{s}}$$

asymptotically the same

WHAT ABOUT INDUCED?

$$\text{Lothar, Pimmons Zhou 2018: } \forall H \forall s \quad ex(n, H, K_{s,s}\text{-ind}) = O(n^{2-\frac{1}{s}})$$

Bourneuf, Bucić, C, Davies: $\forall$ BIPARTITE H

$$\exists \varepsilon_H \in (0, \frac{1}{2}], c_H > 1 \text{ s.t. } \underline{ex(n, K_{s,s}, H\text{-ind})} \leq c_H s^4 n^{2-\varepsilon_H}$$

SINCE THEN, MORE STUDY OF $\nearrow$

↓ Main New Tool

★ Bourneuf, Bucić, C. Davies: $\forall$ BIPARTITE H
 $\exists \varepsilon_H \in (0, \frac{1}{2}]$, $c_H > 1$ s.t. $\text{ex}(n, K_{S,S}, H\text{-ind}) \leq c_H s^4 n^{2-\varepsilon_H}$

PROOF SKETCH

① THM (LOH, PART, PIMMONS, ZHOU):

$\forall r, t \geq 2$, $\exists$ an $\varepsilon \in (0, 1]$, $c > 1$ s.t. $\text{ex}(n, K_{r,t}, K_{S,S}\text{-ind}) \leq c s^4 n^{2-\varepsilon}$

② LEMMA: If H bipartite satisfies ★ and $e \in E(H)$ then
 $H-e$ satisfies ★

① + ② $\Rightarrow$ ★

★ Bourneuf, Bucić, C. Davies: $\forall$ BIPARTITE H

★ $\exists \varepsilon_H \in (0, \frac{1}{2}]$, $c_H > 1$ s.t. $\text{ex}(n, K_{S,S}, H\text{-ind}) \leq c_H s^4 n^{2-\varepsilon_H}$

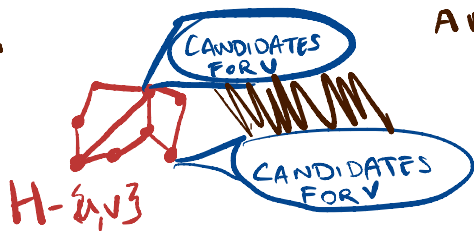
② LEMMA: If H bipartite satisfies ★ and $e \in E(H)$ then $H-e$ satisfies ★

Proof SKETCH: let $e = \{u, v\}$. Suppose NOT.

① WE SHOW IF G has a bit more than $c_H s^4 n^{2-\varepsilon_H}$ edges THEN IT HAS MANY ($\geq 2sn^{H-1}$) INDUCED COPIES OF H

② Suppose G has $\geq c_H s^4 n^{2-\frac{\varepsilon_H}{2|H|}}$ edges and no induced $H-e$ or $K_{S,S}$ -subgraph. SO THERE IS AN INDUCED EMBEDDING OF $H - \{u, v\}$ THAT CAN BE EXTENDED TO AN EMBEDDING IN SEVERAL ($\geq 2sn$) WAYS.

AND WE CAN ENSURE THAT $\geq 2s$ candidates for u $\geq 2s$ candidates for v



a Contradiction! 

Open Problems

CONJECTURE: SCOTT+SEYMOUR '17

$\forall H \exists f_H$ s.t. IF G HAS NO $K_{t,t}$ SUBGRAPH AND
IF G HAS ~~AVERAGE DEGREE~~ $\chi(G) \geq f_H(t)$ THEN
 G CONTAINS H AS AN INDUCED **ODD** SUBDIVISION.

CONJECTURE (HUNTER, MILOJEVIĆ, SUDAKOV, TOMON '25)

$\forall$ BIPARTITE H , $\text{ex}(n, K_{s,s}, H\text{-induced}) \leq C_H(s) \cdot \text{ex}(n, H)$

SEE ALSO SURVEY BY McCARTY + DU

AND NOW for AN

An Overview of our PROOF



*THM MAIN (Bourneuf, Bucić, C., DAVIES 2023)

$\forall$ Graph $H \exists$ a polynomial f s.t. $\forall G$ w/ average degree $\geq p(s)$ then G has a $K_{s,s}$ -subgraph or H as an induced subdivision.

PROOF SKETCH:

THM: Bourneuf,
Bucić, C., Davies

ENOUGH TO SHOW * FOR
BIPARTITE G
w/ CERTAIN "NICE" STRUCTURE

PROP: BB $\overline{C}$ D

$\forall$ Graph $H \exists$ a polynomial f s.t. $\forall G$ w/ max average degree $d \geq P(s)$ w/o a $K_{s,s}$ subgraph
Either

(a) C_4 -free bipartite induced sg w/ avg degree $\geq k$

(b) An induced very unbalanced bipartite subgraph w/ big
side regular

$$\hookrightarrow G^L[A'B'] \quad |A'| \geq 2^{d^{2\epsilon}} |B'| \quad |N(a) \cap B'| = \left\lfloor \frac{d^\epsilon}{s^4} \right\rfloor \quad \forall a \in A'$$

* **THE MAIN** (Bourneuf, Bucić, C., DAVIES 2023)

$\forall$ Graph $H \exists$ a polynomial f s.t. $\forall G$ w/ average degree $\geq p(s)$ then G has a $K_{s,s}$ -subgraph or H as an induced subdivision.

PROOF SKETCH:

PROP: BBCD

$\forall$ Graph $H \exists$ a polynomial f s.t. $\forall G$ w/ max average degree $\geq P(s)$ w/o a $K_{s,s}$ subgraph
Either

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$$\hookrightarrow G[A'B'] \quad |A'| \geq 2^{d^{2\epsilon}} |B'| \quad |N(a) \cap B'| = \left\lfloor \frac{d^\epsilon}{s^4} \right\rfloor \quad \forall a \in A'$$

* IS AN "EASY" CONSEQUENCE OF $\uparrow$

USES RESULTS
FROM THE
LITERATURE

*THM MAIN (Bourneuf, Bucić, C., DAVIES 2023)

$\forall$ Graph $H \exists$ a polynomial f s.t. $\forall G$ w/ average degree $\geq p(s)$ then G has a $K_{s,s}$ -subgraph or H as an induced subdivision.

PROOF SKETCH:

PROP: BBCD

$\forall$ Graph $H \exists$ a polynomial f s.t. $\forall G$ w/ max average degree $\geq P(s)$ w/o a $K_{s,s}$ subgraph
Either

(a) C_4 -free bipartite induced sg w/ avg degree $\geq k$

(b) An induced very unbalanced bipartite subgraph w/ big side regular

$$\hookrightarrow G[A'B'] \quad |A'| \geq 2^{d^{\frac{1}{s^4}}} |B'| \quad |N(a) \cap B'| = \left\lfloor \frac{d^{\frac{1}{s^4}}}{s^4} \right\rfloor \quad \forall a \in A'$$

IN CASE (a): WE WIN BY APPLYING A RESULT

Lem: Kühn-Osthus

$\forall$ graph $H \exists k$ s.t. $\forall G$ w/o C_4 subgraph $\frac{1}{k}$ induced subdivision of H
 $\text{avg deg}(G) \leq k$

IN CASE (b): WE WIN BY:

(1) LEM(KUHN OSTHUS) A VERY UNBALANCED BIPARTITE GRAPH WITH BIG SIDE REGULAR-ISH
CONTAINS EITHER A LARGE $K_{\frac{1}{\epsilon}, \frac{1}{\epsilon}}$ or an induced 1-subdivision of some graph w/ large avg degree

WIN

Thm (Bollabás-Thomason, Komlós-Szemerédi)
EVERY GRAPH OF LARGE ENOUGH AVG DEGREE
HAS A BIG K_n -subdivision.

$\hookrightarrow$ WIN

PROP. BB_{CD}

$\forall$ Graph $H \exists$ a polynomial f s.t. $\forall G$ w/ max average degree $\geq P(s)$ w/o a $K_{s,s}$ subgraph
Either

- (a) C_4 -free bipartite induced sg w/ avg degree $\geq k$
 - (b) An induced very unbalanced bipartite subgraph w/ big side regular
- or

PROOF SKETCH:

$\forall \text{ Graph } H \exists \text{ a polynomial } f \text{ s.t. } \forall G \text{ w/ max average degree } d \geq P(s) \text{ w/o a } K_{s,s} \text{ subgraph}$
 Either

- (a) C_4 -free bipartite induced sg w/ avg degree $\geq k$ or
 (b) An induced very unbalanced bipartite subgraph w/ big side regular

PROOF SKETCH:

Lemma: Du, Girão, Hunter, McCarty Scott)

G is n -vertex graph d -degenerate + avg degree d
 then

(b) G has very unbalanced bipartite subgraph
 w/ constant fraction of edges.

OR

(a) ALMOST REGULAR
 INDUCED SUBGRAPH WITH
 AVG DEGREE ONLY "POLYNOMIALLY"
 SMALLER

IN CASE B

OBTAIN (b) using ★

IN CASE (a) WE REDUCE TO GIRTH 5
 WITH ONLY POLY DECREASE IN AVG DEGREE
 USING ★

THEN APPLY:

Lemma: Kwan, Letzter, Sudakov, Tran

$\exists C$ s.t. every Δ -free
 g w/ avg degree d has
 an induced bipartite subgraph
 w/ avg degree $\frac{C \log d}{\log \log d}$

→ OBTAIN (a)

MAIN NEW TOOL INDUCED KÓVÁRI-SÓS-TURÁN

★ Bourneuf, Bucié, C. Daries: $\forall$ BIPARTITE H

$\exists \varepsilon_H \in (0, \frac{1}{2}]$, $c_H > 1$ s.t. $\text{ex}(n, K_{s,s}, H\text{-ind}) \leq c_H s^4 n^{2-\varepsilon_H}$

Danke schön!

qujannamiik

متشكراً

ASANTE

хвала

Bedankt!

Köszönöm

MERCI!

Cảm ơn!

Thank you!

謝謝

شكراً لك

ΓΡΑCΙΑC!

ਧੰਨਵਾਦ

Hvala!

Ευχαριστώ

תודה

ཐུགས་ཀྱི་བཤེད་

Děkuji!

Dziękuję

감사

합니다