# ~~C~~MSO-transducing tree-like graph decompositions

Rutger Campbell

Based on work with: Bruno Guillon, Mamadou Kanté, Noleen Köhler, Eunjung Kim, Sang-il Oum

Q: When is a property recognizable with a tree-automaton?

Recognizable with a tree-automaton.

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E.g. 2-colourability for series-parallel graphs:

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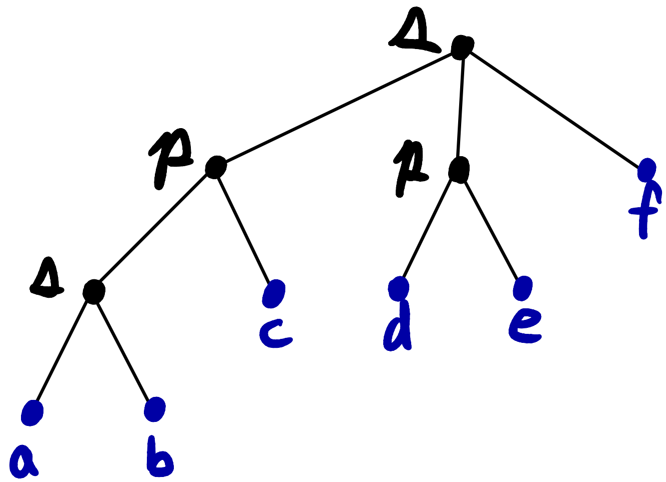


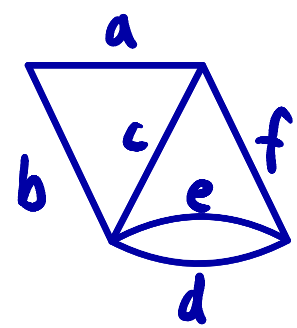

# Recognizable with a tree-automaton.

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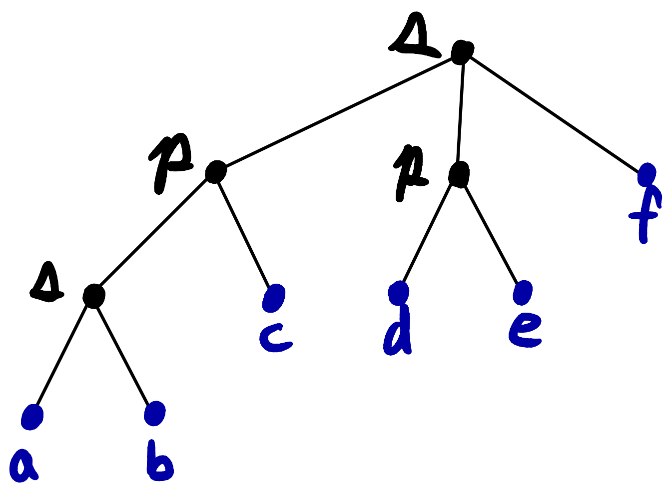


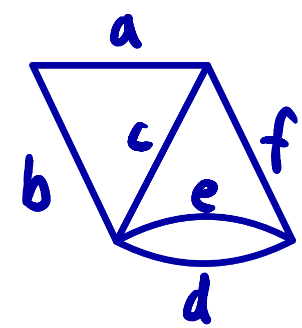


**Automaton:**

Leaves: take set of colourings as states

Parallel nodes ($p$): $c_1 - - - c_2,\ c_1 - - - c_2 \mapsto c_1 - - - c_2$

otherwise $\mapsto \perp$

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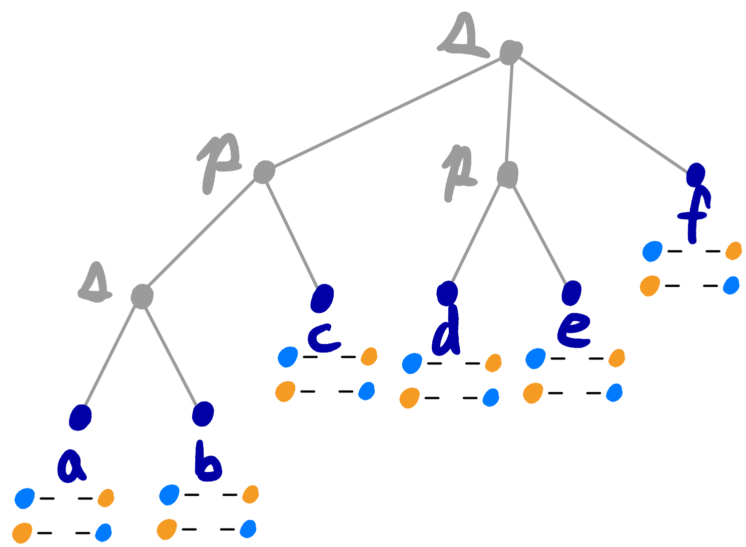


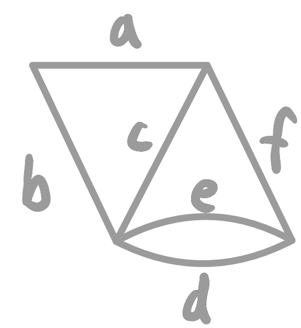


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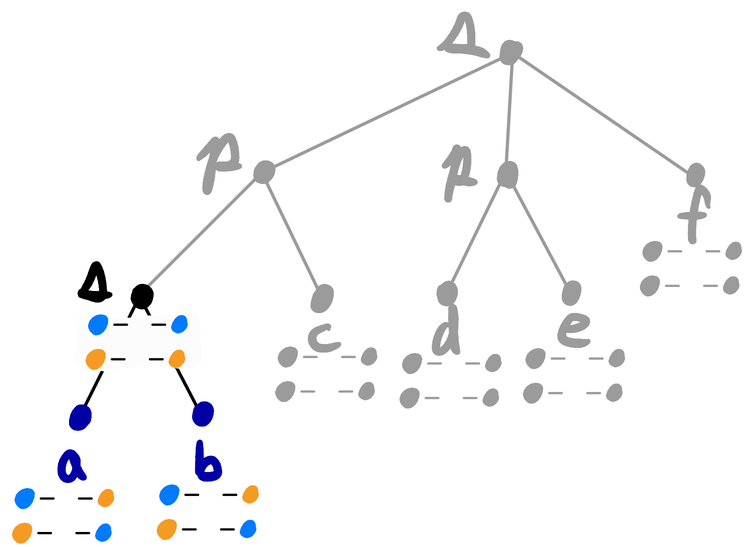


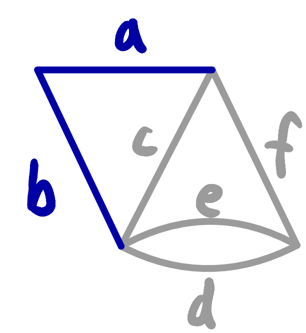


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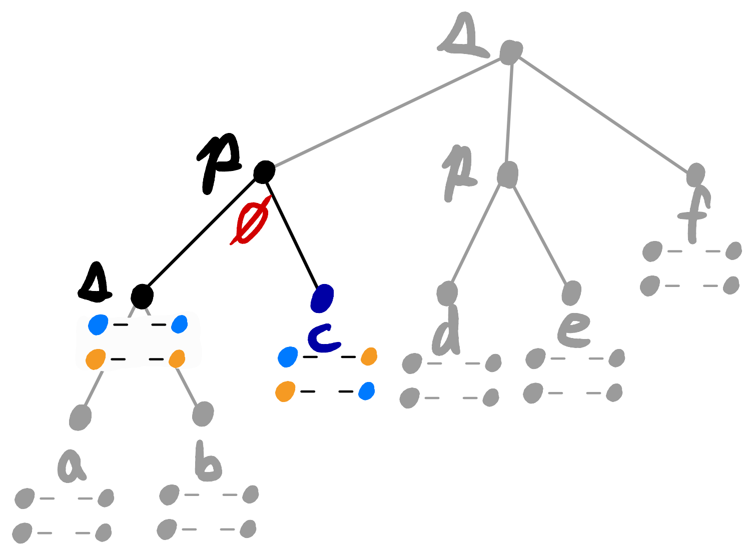


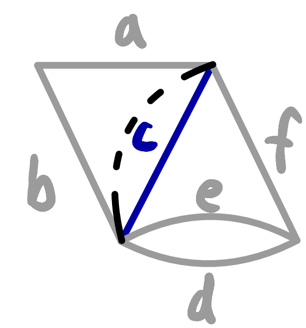


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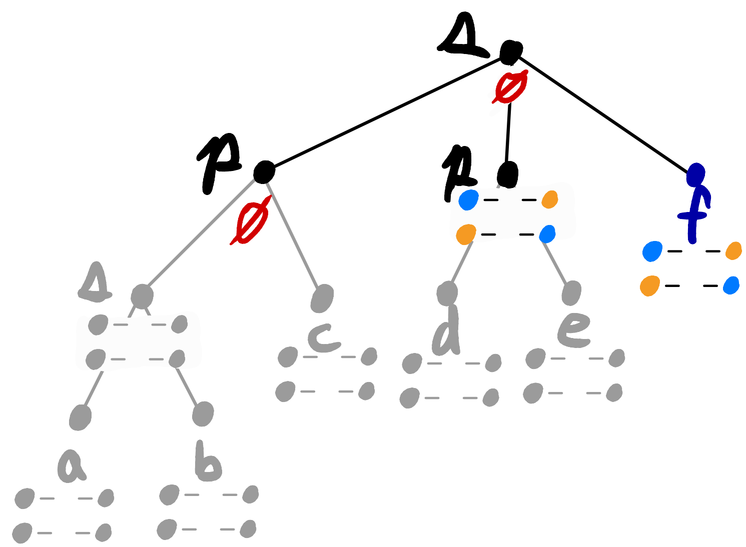


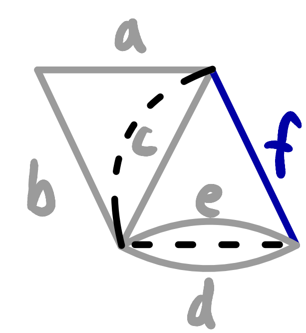


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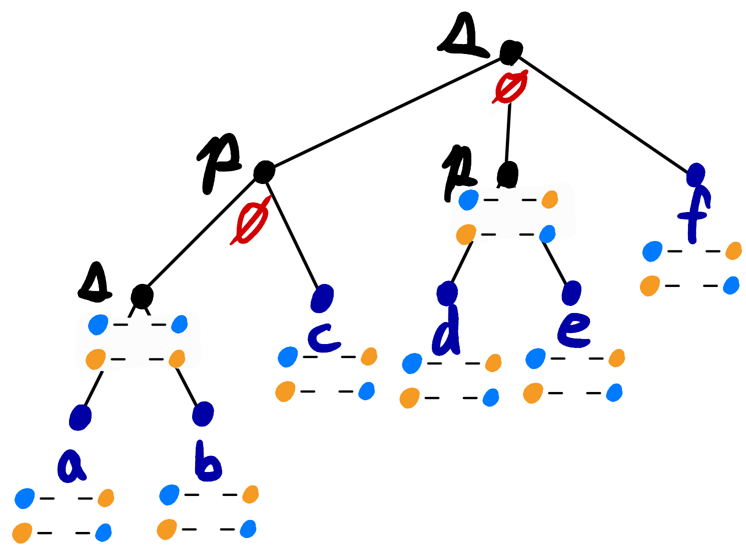


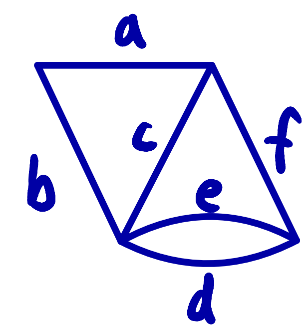


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Q: When is a property definable in MSO?

Definable with Monadic Second-Order (MSO) logic.

$\forall, \exists, \neg, \&, \vee, \Rightarrow, =, \leq, \in,$ [other relations]

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Set system: $(S, \mathcal{F})$ : $\ldots$ , $\mathrm{set}(X)$

$\mathcal{F}$ in $2^S$

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E.g. $\exists X \forall u \forall v \, [u \sim v \Rightarrow (u \in X \,\&\, v \notin X) \vee (u \notin X \,\&\, v \in X)]$

Theorem (Büchi):

A property of strings is recognizable by a string automata if and only if it is MSO-definable.

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## Theorem (Thatcher-Wright):

A property of labelled binary trees is recognizable by a tree-automata if and only if it is MSO-definable.

Graphs with $u \sim v$ and $inc(v,e)$:
On bounded tree-width:

$$\text{Definable with a MSO sentence.}$$
$$\Big\Downarrow \text{ Courcelle}$$
$$\text{Recognizable with a tree-automata.}$$

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Bojańczyk, Pilipczyk $\Uparrow$ $\Downarrow$ Courcelle

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Definable with a MSO sentence.

$\Uparrow$ ? $\qquad\qquad$ $\Downarrow$ Courcelle

Recognizable with a tree-automata.

Set systems with set(X):

On bounded decomposition-width:

Definable with a MSO sentence.

?

Funk, Mayhew, Newman

Recognizable with a tree-automata.

# General strategy:

tree-automata — MSO sentence

Recognizable $\xleftarrow{\text{encode truth tables/types as states}}$ Definable

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tree-automata $\longmapsto$ MSO sentence

Recognizable　　Definable

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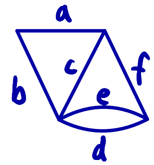
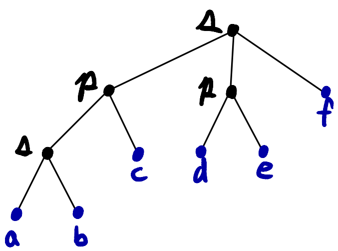



Objects with width $\leq k$

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Recognizable

Definable

# Transductions

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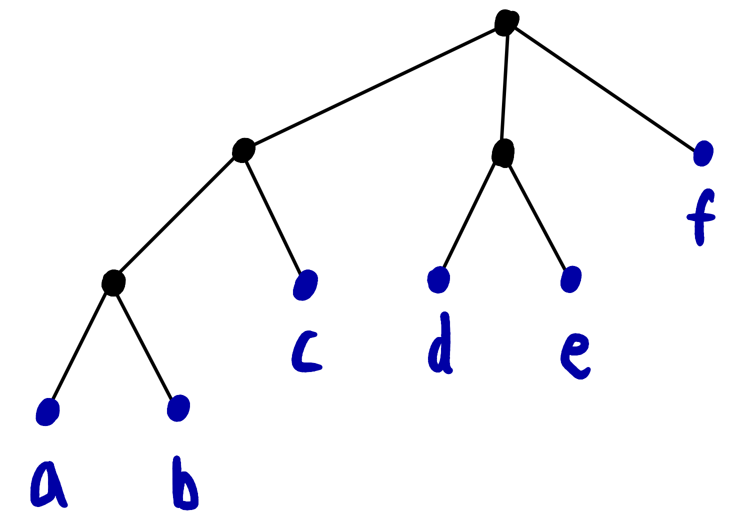
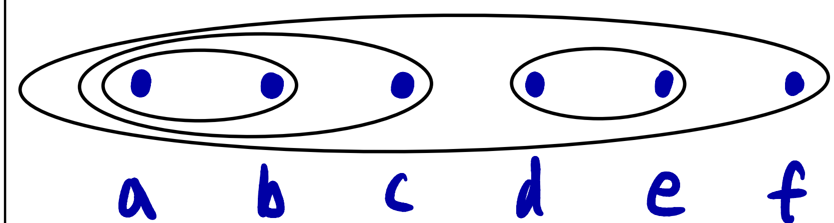



Rooted trees: ..., desc(u,v)

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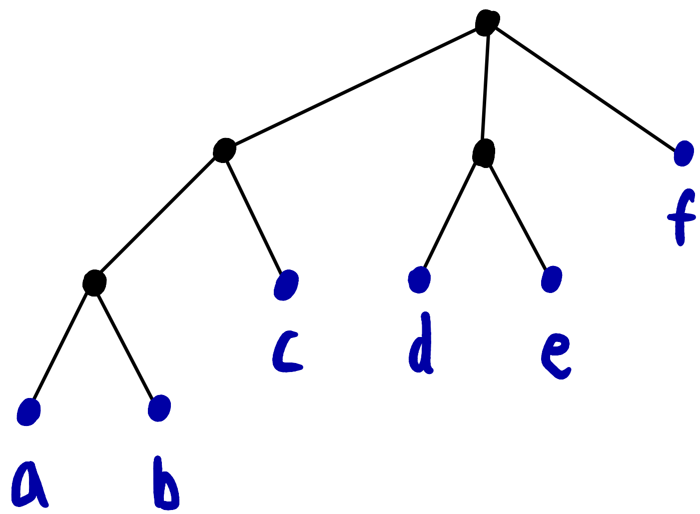


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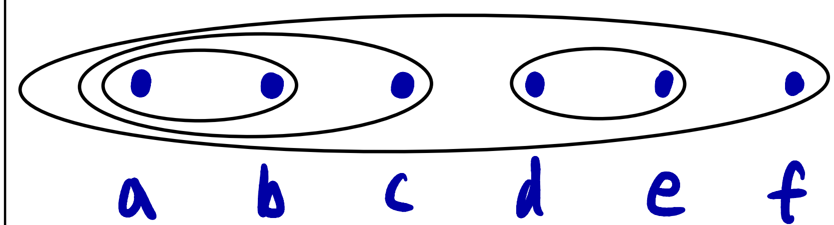
$\rightsquigarrow$

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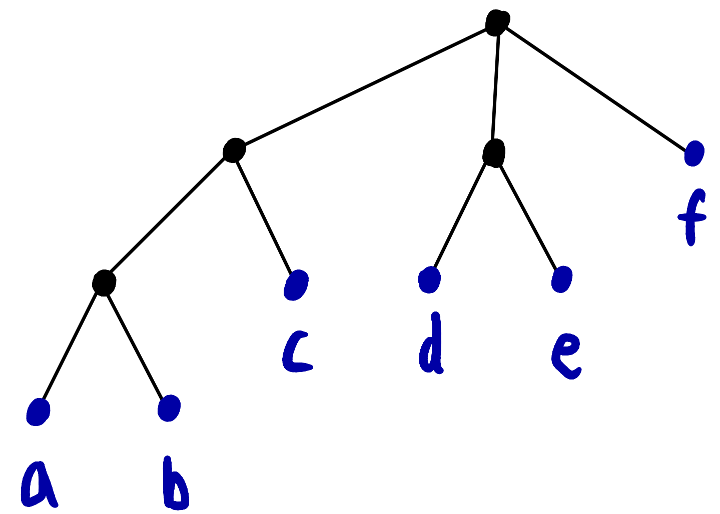


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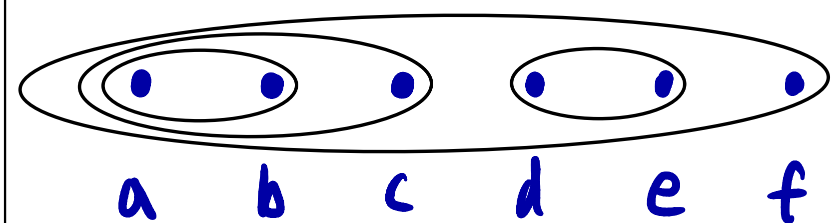


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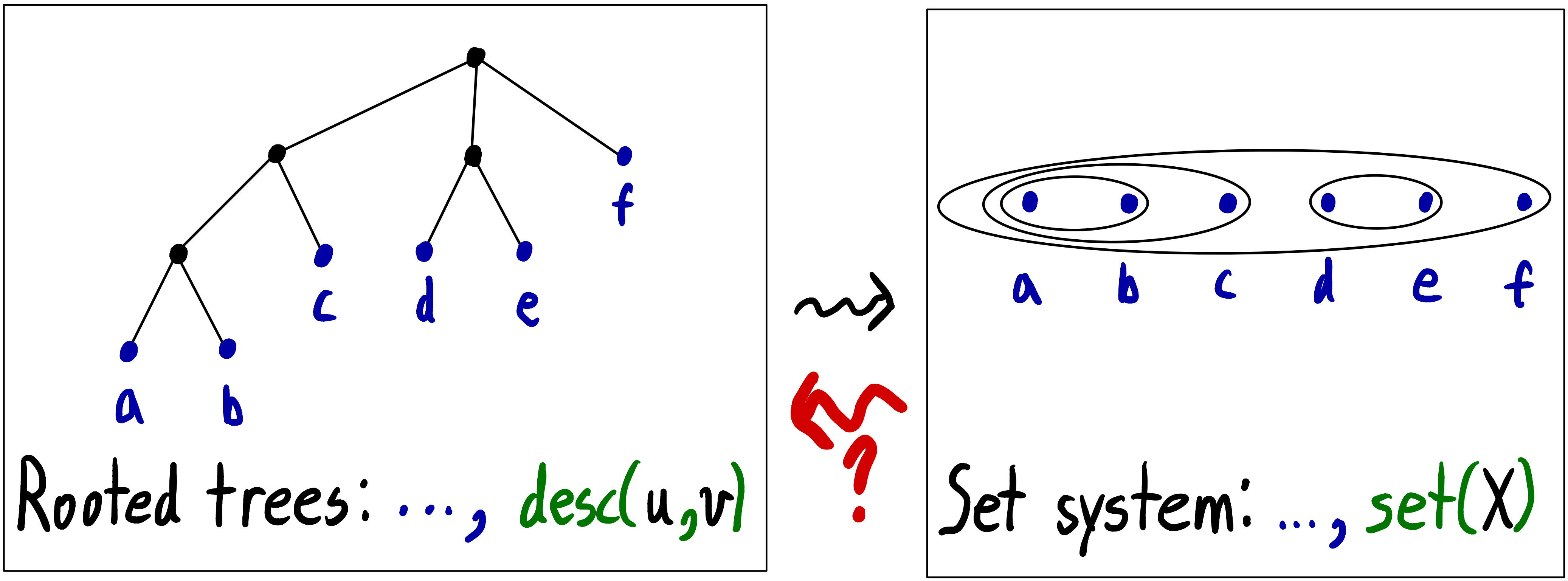


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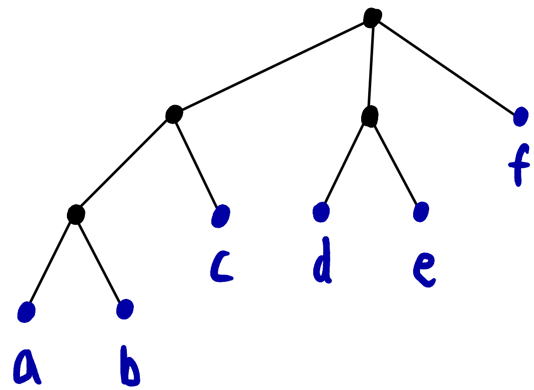


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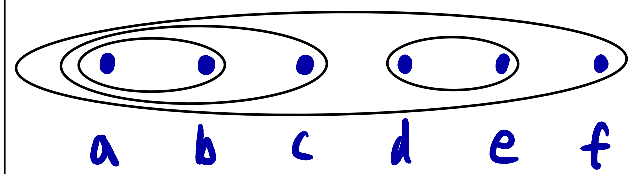
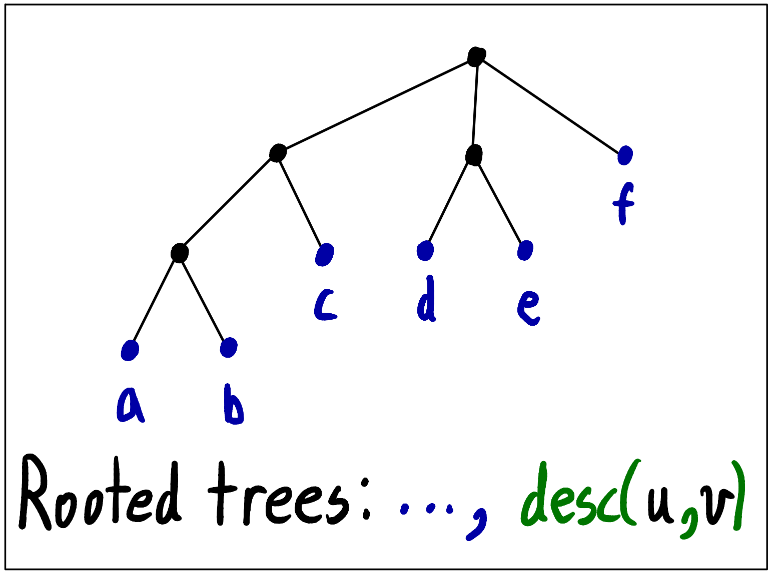
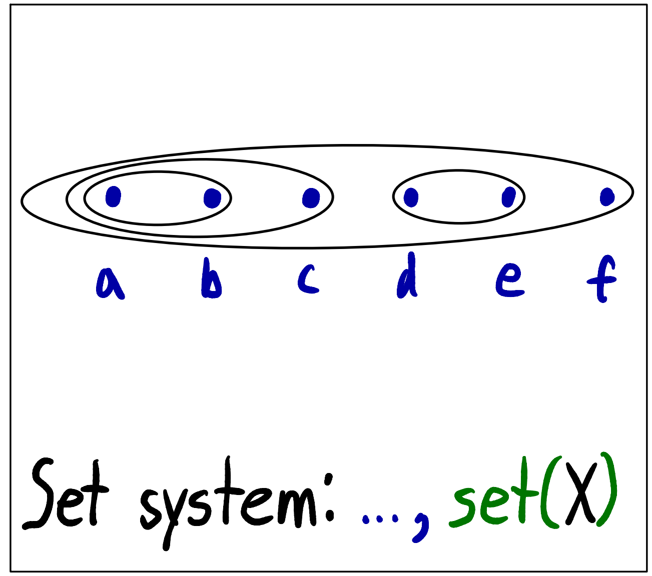



Set system: …, set(X)

**Theorem (C., Guillon, Kanté, Kim, Köhler):**
There is a $C_2MSO$-transduction from laminar set systems to their laminar trees.

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**Theorem(C., Köhler):**
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**weakly bipartitive system** is a pair $(U, \mathcal{B})$, where $\mathcal{B}$ is a collection of bipartitions of $U$ with $\{\emptyset, U\} \notin \mathcal{B}$; $\{\{a\}, U-\{a\}\} \in \mathcal{B}$ for all $a \in U$; and if $\{X_1, X_2\}, \{Y_1, Y_2\} \in \mathcal{B}$, then $\{X_1 \cap Y_1, X_2 \cup Y_2\}, \{X_1 \cup Y_1, X_2 \cap Y_2\}, \{X_1 \cap Y_2, X_2 \cup Y_1\}, \{X_1 \cup Y_2, X_2 \cap Y_1\} \in \mathcal{B}$.

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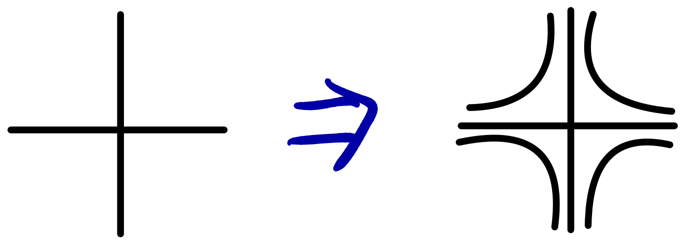
i.e. 

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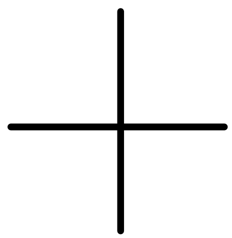
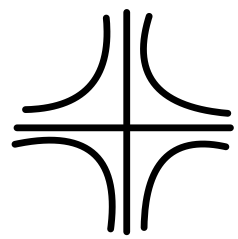
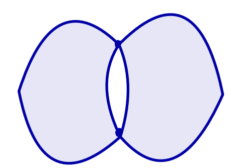
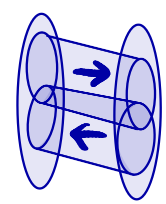
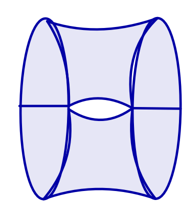
i.e.  $\Rightarrow$ 

2-vertex separation (partitioning $E(G)$): 

split (partitioning $V(G)$): 

bi-join (partitioning $V(G)$): 

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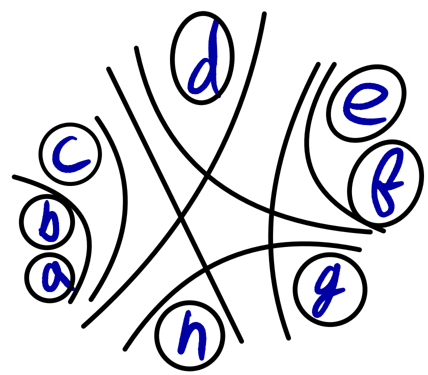

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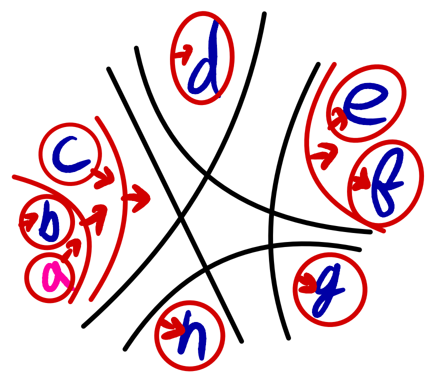

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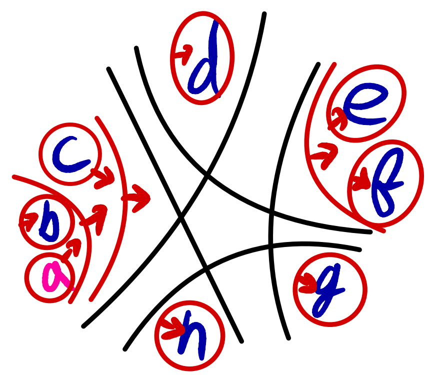


**Corollary (C., Guillon, Kanté, Kim, Köhler):**
Recogizability equals definability for:

- cographs (rankwidth 1)
- series-parallel graphs (branchwidth 2)
- bounded split-width

Theorem (C., Guillon, Kanté, Kim, Oum):

For $GF(q)$-representable matroids of linear branch-width $\leq k$, there is a MSO-transduction to a linear branch-decomposition of order $\leq f(q,k)$.

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For $GF(q)$-representable matroids of linear branch-width $\leq k$, there is a $MSO$-transduction to a linear branch-decomposition of order $\leq f(q,k)$.

**Corollary (Bojańczyk, Grohe, Pilipczyk):**
For graphs of linear rank-width $\leq k$, there is a $MSO$-transduction to a linear rank-decomposition of order $\leq f(k)$.

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**Corollary:**

For graphs of rank-width $\leq k$, there is a MSO-transduction to a rank-decomposition of order $\leq f(k)$.

Conjecture:

For stuctures where the relations are given by tree-automata with $k$ states, recognizability is equivalent to definability.

Thank You!