Solving Partial Dominating Set and Related Problems Using Twin-Width

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Partial vertex covers and dominating sets

- PARTIAL VERTEX COVER: are there k vertices that cover **at least** t edges?
- PARTIAL DOMINATING SET: are there k vertices that dominate **at least** t vertices?

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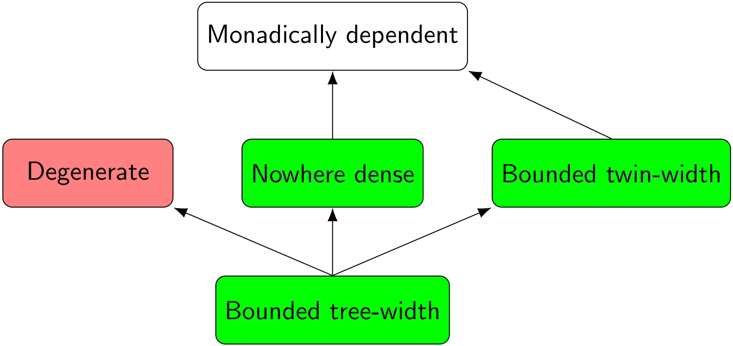
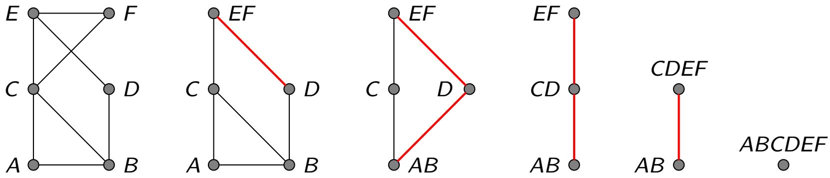
- On degenerate graphs, DOMINATING SET is FPT but PARTIAL DS is W[1]-hard (Golovach et al. WG 2008).
- On nowhere-dense classes, PARTIAL DS can be solved in time $f(k)n^{1+\varepsilon}$ for every $\varepsilon > 0$. (Dreier et al. ESA 2023).



- Green = FPT. For twin-width, it is our result.
- Red = W[1]-hard.
- White = unknown.

What is twin-width?

Graph parameter introduced in 2020 by Bonnet et al.



Width of a contraction sequence = maximum red degree.

Twin-width = minimum width over all contraction sequences.

Our results

Theorem

Given an n-vertex graph G, a contraction sequence for G of width d, and integers k and t:

- PARTIAL VERTEX COVER *can be decided in time* $2^{\mathcal{O}(kd\log(kd))} \cdot n$.
- PARTIAL DOMINATING SET *can be decided in time* $2^{\mathcal{O}(kd\log(kd))} \cdot n$.

Note that the running times do not depend on t!

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- Can these two problems be expressed via some logic?
- Yes, using counting quantifiers!

Our results: $(\exists^* \sum \#QF)$-logic

Partial Dominating Set is expressed by the formula:

$$\exists x_1 \cdots \exists x_k \left(\#y \bigvee_{i=1}^{k} E(x_i, y) \vee x_i = y \right) \geq t$$

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We can only count vertices, so expressing Partial Vertex Cover is more complicated.

$$\exists x_1 \cdots \exists x_k \left(\sum_{i=1}^{k} \#y \, E(x_i, y) \wedge \bigwedge_{j \in [i-1]} y \neq x_j \right) \geq t$$

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Note that the FO formula following # must be quantifier-free!

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- Partial Dominating Set *can be decided in time*
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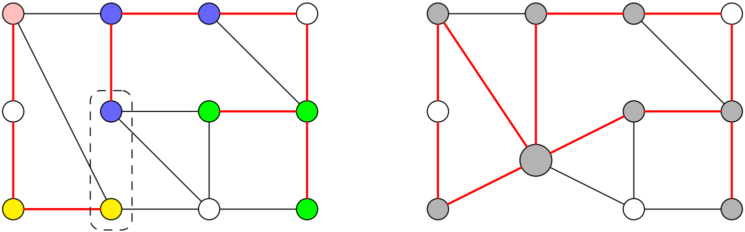
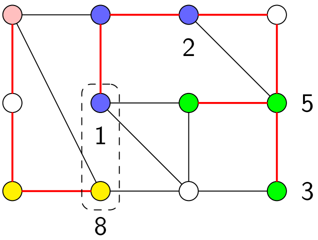
- $\exists^* \sum \#$*QF-model checking can be decided in time*
$$2^{\mathcal{O}(k^2 d \log(d))} \cdot n.$$

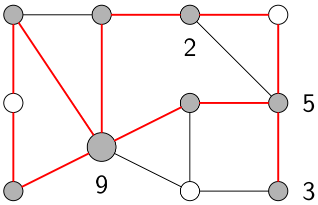
Notice the worse running time for the general algorithm.

Proof idea for all our algorithms



- Dynamic programming along a contraction sequence.
- We compute an entry for each small red-connected set.
- To compute the entry for the grey set, we combine entries of the blue, yellow, green, and pink sets.
- Inspired by the algorithm for DOMINATING SET from (Twin-width III., Bonnet et al., ICALP 2021).

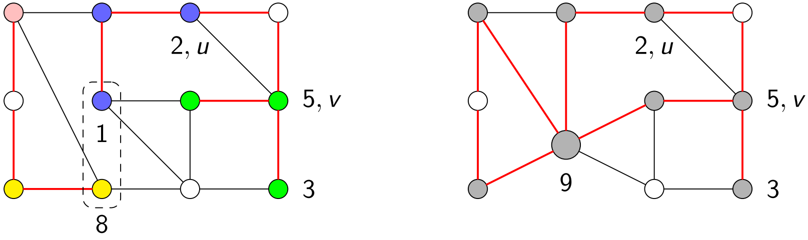




Entry (T, f):

- T = small red-connected set.
- f = the distribution of the covering vertices.

For each entry, we compute the maximum possible number of covered edges inside T.

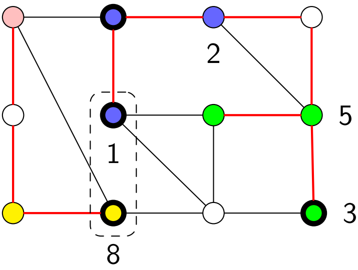


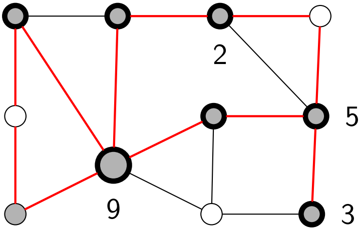
We need to add:

- Edges "inside" each previous entry.
- Edges "between" two entries.

For example, the edge uv implies $2 \cdot |\beta(v)| + 5 \cdot |\beta(u)| - 10$ covered edges "between" blue and green.

Partial Dominating Set

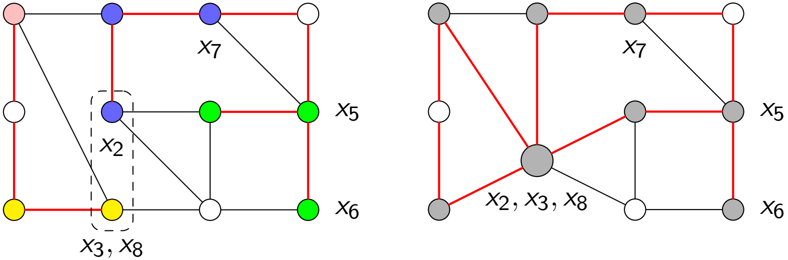




Entry (T, f, M). T and f as before. $M \subseteq T$, thick.

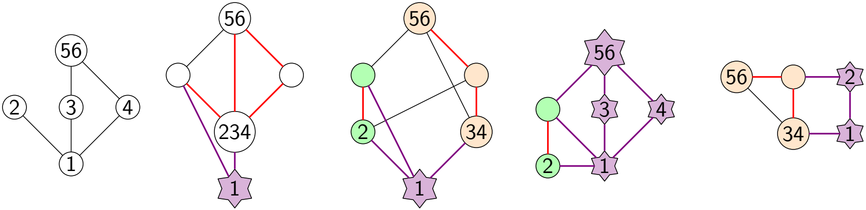
- Value of an entry = maximum number of vertices dominated in $\bigcup_{u \in M} \beta(u)$.
- Vertices fully dominated "from outside" are removed from M.
- This avoids double-counting.

Model checking $\exists x_1 \cdots \exists x_k \# y\, \psi(x_1, \ldots, x_k, y) \geq t$



- The profile needs to know which variable is realized where.
- However, to count the number of y's, we need to know also the edges to the variables realized in other profiles.
- For example, the green profile needs to know that x_5 and x_7 are adjacent.
- We solve this using *virtual* vertices.

Model checking $\exists x_1 \cdots \exists x_k \# y\, \psi(x_1, \ldots, x_k, y) \geq t$



Purple stars are virtual vertices.

- Left: connections between the variables. Notice $x_5 = x_6$ holds.
- Second: after a contraction.
- Middle: before a contraction.
- Right: the entries for green and orange profiles.

Conclusion

Summary:

1. FPT algorithms for Partial Vertex Cover and Partial Dominating Set on bounded twin-width.
2. Generalized to model checking $\exists x_1 \cdots \exists x_k \sum_i \#y\, \psi_i(x_1, \ldots, x_k, y) \geq t$ for quantifier-free ψ_i's.

Open problems:

- Dropping the requirement that ψ_i's are quantifier-free?
- Approximate model checking for more general counting logics?

Thank you for your attention!