Life and Work of Alfred Tarski

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"*Alfred Tarski is one of the great figures in the history of logic, along with Aristotle, Gottlob Frege, and Kurt Gödel. His work covers an astonishing range of subjects: set theory, measure theory, topology, geometry, classical and universal algebra, algebraic logic, various branches of formal logic and metamathematics, philosophy, even economics.*"

—-Steven Givant in
Unifying Threads in Alfred Tarski's Work

Alfred Tarski, 1968

Outline

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Biography

Biography: Early Work

- 1901: born Alfred Tajtelbaum in Warsaw.
- 1918: entered Warsaw University to study **biology**, then switched to **mathematics** and **logic** after solving and publishing an open problem from Stanisław Leśniewski's set theory seminar.
- 1924: doctorate under Stanisław Leśniewski. Dissertation: **On the Primitive Term of Logistic**. Same year:

Banach-Tarski Paradox

It is possible to cut a solid ball in 3-dimensional space into finitely many pieces, and reassemble them by rigid movements into two balls of the same size as the original.

Stanisław Leśniewski

Biography: New Directions

He mostly worked in **axiomatic set theory** right after his dissertation with a focus on **cardinal arithmetics**. In the late 1920s, Tarski became much more interested in **algebra**, **geometry** and **metamathematics**.

Reasons for the shift in interest:

- It seemed to be that the most fundamental problems in set theory appear to be unsolvable.
- He actively participated in the Warsaw seminars in mathematical logic led by Jan Łukasiewicz starting from 1926.
- He became a high school mathematics teacher for financial reasons.

Jan Łukasiewicz

Biography: Life in the U.S.

- In August 1939, Tarski left Poland for a congress in the U.S. Germany invaded Poland while he was traveling, stranding him in the U.S.
- He taught at Harvard, City College of New York, Princeton before being appointed Lecturer at UC Berkeley in 1942.
- 1946: promoted to full professor at Berkeley and reunited with his wife and two children after seven years of separation due to the war.
- 1958: set up the Group in Logic and the Methodology of Science at Berkeley, which he led until retirement in 1968.
- After retirement, he taught for another five years and continued research until his death in 1983.

Maria and Alfred Tarski

Quantifier Elimination

Quantifier Elimination I

- In the mid-1920s, Tarski developed systems for **first-order algebra** and **geometry**.
- He wanted to know which of the traditional algebraic and geometric notions were **expressibe** in his systems and which of the traditional theorems where **provable**.
- The method he used to investigate these questions was **quantifier elimination**.
- To eliminate quantifiers for a theory T, one defines a set of **special**, easily understood formulas and shows inductively that every formula is T-equivalent to a **Boolean combination** of special ones.

Example (Dense Linear Orders)

Assuming the theory of the dense linear orders, $\exists x.\, a < x \wedge x < b$ and $a < b$ are equivalent.

Quantifier Elimination II

If a theory admits an elimination procedure, then the procedure can be used to analyze **decidability**, the **definable notions**, the **provable sentences** and the **complete extensions** of the theory.

- 1926–1928: The geometry of the line and the algebra of real addition are **complete**.
- same time: A classification of the complete theories of discrete linear orderings.
- 1928: Supervised Master's student Mojzesz Presburger, who found an elimination procedure for the additive theory of the natural numbers.

Mojzesz Presburger, 1923

Definability, Satisfaction and Truth

Defining the definables

- In the late 1920s, to better formulate his findings, Tarski was compelled to give a definition to what it means for a concept to be **definable**.
- At the time, ”*mathematicians, in general, do not like to deal with the notion of definability; their attitude toward this notion is one of distrust and reserve*”
- In his paper ’On the definable sets of real numbers’ (1929), Tarski gave the now standard metamathematical definition of definability.
- A set, binary relation, etc., is definable if it is **selected** by a formula.

Example

A set is first-order definable in a structure if it consists of all elements of the structure that **satisfy** a first-order formula with one free variable.

Definability, Satisfaction and Truth

- The definition of **definability** rests upon the notion of **satisfaction**, and defining satisfaction led Tarski to define **truth**.
- When defining these notions, Tarski defined them with respect to a specific model ($\mathcal{M}$ with universe M) in mind that goes with the language, e.g the standard model of natural numbers together with FOL.

formula $\varphi(x)$ defines the set $\{a \in M \mid \varphi(a)\}$ → assignment s satisfies φ → sentence φ true if any s satisfies φ

Is there a formula $\mathrm{True}(x)$ that defines truth,
i.e. for any sentence φ, $\mathrm{True}(\ulcorner\varphi\urcorner)$ iff φ?
(Convention T, $\ulcorner\cdot\urcorner$ Gödel number)

Undefinability of Truth

Tarski distinguished the **object language** L from the **metalanguage** L' in which we talk about L. Let L be strong enough to arithmetize its own syntax.

Undefinability (Definability) of Truth

- When $L = L'$, then $\text{True}(x)$ is not definable. The argument follows Gödel's diagonalization method.
- If L' includes second-order logic over L and PA, then $\text{True}(x)$ can be defined.

- This gives a rigorous response to certain semantic paradoxes. The problem with "This sentence is false" is that a language is trying to contain its own truth predicate unrestrictedly. The object-language/metalanguage distinction **prevents** this **self-reference**.
- These definitions are **frontrunners** of the later **model-theoretic notions**.

Algebraic Logic

Boolean Algebra and Algebraic Logic

- In the 1930s, Tarski started working on Boolean algebra and algebraic logic, partly due to the seminar of the Warsaw logicians.
- Algebraic logic is the study of logic by translating logical systems into algebraic structures.
- A **Boolean algebra** is a structure

$$(B, \wedge, \vee, \neg, 0, 1)$$

such that, for all $a, b, c \in B$:

$$\begin{aligned} a \wedge b &= b \wedge a, & a \wedge 1 &= a, & \neg(a \wedge b) &= \neg a \vee \neg b, \\ (a \wedge b) \wedge c &= a \wedge (b \wedge c), & \neg\neg a &= a, & 0 &= \neg 1. \end{aligned}$$

- A concrete example is $(\mathcal{P}(A), \cap, \cup, \cdot^C, \emptyset, A)$ where A is a set.

Lindenbaum-Tarski Algebra I

For a logic L, define

$$\varphi \equiv_L \psi \quad \text{iff} \quad L \vdash \varphi \leftrightarrow \psi.$$

The **Lindenbaum-Tarski algebra** is the quotient algebra over all formulas of L modulo provable equivalence, whose elements are equivalence classes $[\varphi]$.

- For **classical propositional logic**, the quotient algebra is a **Boolean algebra**, where the induced operations are as follows:

$$\begin{aligned} [\varphi] \wedge [\psi] &= [\varphi \wedge \psi], & [\varphi] \vee [\psi] &= [\varphi \vee \psi], \\ \neg[\varphi] &= [\neg\varphi], & 0 &= [\bot], \\ 1 &= [\top]. \end{aligned}$$

Adolf Lindenbaum

Lindenbaum-Tarski Algebra II

- For **normal modal logic**, the quotient algebra is a **Boolean algebra with operators**.
- In the investigation of the concept of **a deductive system**, Tarski noticed there is a correspondence between **ideals** in a Boolean algebra and deductive systems in his **calculus of theories**. The latter was later shown to coincide with Brouwer and Heyting's **intuitionistic logic** and the corresponding quotient algebra is **Heyting algebra**.
- To provide a quotient algebra and the related algebraic framework for **first-order logic**, Tarski and his collaborators developed the theory of **cylindric algebra**.

Representation Theorem for Boolean Algebra

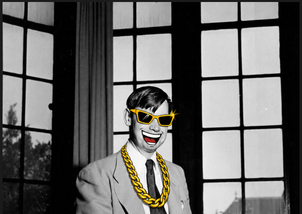
An **abstract** algebra is any structure that satisfies all defining equations. A **concrete** algebra arises from abstraction from concrete mathematical entities (e.g., Boolean algebra induced by the set operations over the powerset).
Do they **coincide**?

Tarski, 1935

A complete atomic Boolean algebra is isomorphic to a powerset algebra.

Stone, 1936

A Boolean algebra is isomorphic to an algebra of sets.



Relation Algebra

- Around 1939 and 1940, ambitious to develop an algebraic version of **all mathematics** and inspired by De Morgan, Peirce and Schöder, Tarski set out to formulate a **calculus of binary relations**, which later becomes what we now know as relation algebra.
- A **relation algebra** is a structure

$$(R, \wedge, \vee, \neg, 0, 1, ;, \smile, 1')$$

  where $(R, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra, ; is relative composition, $\smile$ is converse, and $1'$ is the identity relation. The theory is similarly axiomatized by a set of equations.
- He proves that the calculus is undecidable and the proof shows precisely how to re-formulate set theory in the calculus.

Representation Theorem for Relation Algebra I

- Tarski asked the analogous question of representation for relation algebra: Is every relation algebra isomorphic to a concrete algebra of binary relations?
- He tried Stone's method, but this only gave him a representation of relation algebras as algebras of sets instead of binary relations.
- He communicated this to his former student Bjarni Jónsson, who realized that the proof is valid in a more general context, and came up with **Boolean algebra with operators**, the algebra for **modal logic**.

Bjarni Jónsson, circa 1946

Representation Theorem for Relation Algebra II

- In 1950, Roger Lyndon gave an **unrepresentable** relation algebra.
- In 1952, Tarski showed that the class of **representable** relation algebra is axiomatizable by a set of equations, a conclusion from a combination of earlier results, including his famous **preservation theorem**.

Preservation Theorem

If a class of algebras can be axiomatized by a set of first-order formulas, the class of its subalgebras can be axiomatized by a set of quantifier-free formulas.

Roger Lyndon

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Alfred Tarski, 1930s