Introduction to Games in Logic

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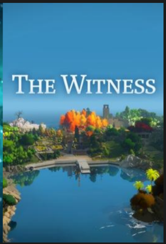
May, 2026

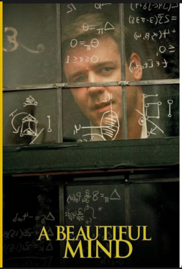
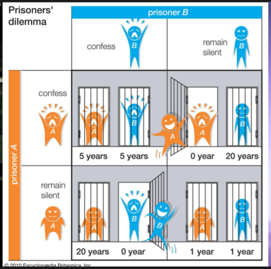
Games?



Fallout
NEW VEGAS

LIFE IS STRANGE

OXYGEN
NOT INCLUDED

THE WITNESS


A BEAUTIFUL
MIND

Prisoners' dilemma
prisoner B
confess
remain silent
prisoner A
confess
remain silent
5 years
5 years
0 year
20 years
20 years
0 year
1 year
1 year


Games in Logic

Truth

Semantic Game

$\mathcal{A} \models \varphi$?

Ownership of a winning strategy

Consistency

Model Existence Game

$\exists \mathcal{A} (\mathcal{A} \models T)$?

Separation

Comparison Game

$\exists \varphi (\mathcal{A} \models \varphi \wedge \mathcal{B} \not\models \varphi)$?

Model Theory Cheatsheet

A vocabulary L is a set contains

- predicate symbols $P, Q, R, \ldots$, and
- constant symbols $c, d, e\ldots$.

A term t is a variable or a constant.

A sentence is a formula with no free variables.

An atomic formula is

- $t_1 = t_2$, or
- $R(t_1, t_2, \ldots, t_n)$.

An $\mathbf{L}$-model is a tuple $\mathcal{M} := (M, P^{\mathcal{M}}, \ldots, c^{\mathcal{M}}, \ldots)$ where

- M is the universe,
- $P^{\mathcal{M}} \subseteq M^{\#P}$, and
- $c^{\mathcal{M}} \in M$.

An assignment s maps a finite set of variables to elements of M. Extend s to terms by setting $s(c) := c^{\mathcal{M}}$.

Model satisfaction for atomic formulas is defined as follows.

- $\mathcal{M} \vDash_s t_1 = t_2$ if $s(t_1) = s(t_2)$, and
- $\mathcal{M} \vDash_s R(t_1, \ldots, t_n)$ if $(s(t_1), \ldots, s(t_n)) \in R^{\mathcal{M}}$.

Example: Graphs

Given a vocabulary $L = \{E\}$, a graph $G := (V, E)$ is an L-model satisfying

- for all $v \in V$, $(v, v) \in E$, and
- for all $u, v \in V$, if $(u, v) \in E$, then $(v, u) \in E$.

Semantic Game

Semantic Game for First Order Logic

$\varphi :=$ Atm | ¬Atm | $\psi \wedge \theta$ | $\psi \vee \theta$ | $\forall x\, \psi$ | $\exists x\, \psi$.
Fix a vocabulary L, an L-model $\mathcal{M}$ and an L-sentence φ^*. SG$(\mathcal{M}, \varphi)$ goes as follows.

- Two players: Verifier  and Falsifier .
- Positions: (φ, s) with subformula φ and assignment s.
- Starting position: $(\varphi^*, \emptyset)$.

Semantic Game for First Order Logic (ctd.)

Moves:

- $(\psi_0 \wedge \psi_1, s)$:  chooses $i \in \{0,1\}$ and the game goes to (ψ_i, s).
- $(\psi_0 \vee \psi_1, s)$:  chooses $i \in \{0,1\}$ and the game goes to (ψ_i, s).
- $(\forall \psi, s)$:  chooses $a \in \mathcal{M}$ and the game goes to $(\psi, s \cup \{x \mapsto a\})$
- $(\exists \psi, s)$:  chooses $a \in \mathcal{M}$ and the game goes to $(\psi, s \cup \{x \mapsto a\})$

Winning conditions:

- $(\varphi_{\text{atm}}, s)$:  wins if $\mathcal{M} \models_s \varphi_{\text{atm}}$; otherwise,  wins.
- $(\neg\varphi_{\text{atm}}, s)$:  wins if $\mathcal{M} \models_s \varphi_{\text{atm}}$; otherwise,  wins.

Semantic Game for First Order Logic (ctd.)

Theorem

$\mathcal{M} \models \varphi$ if and only if  has a winning strategy in $\mathrm{SG}(\mathcal{M}, \varphi)$.

Key ideas:

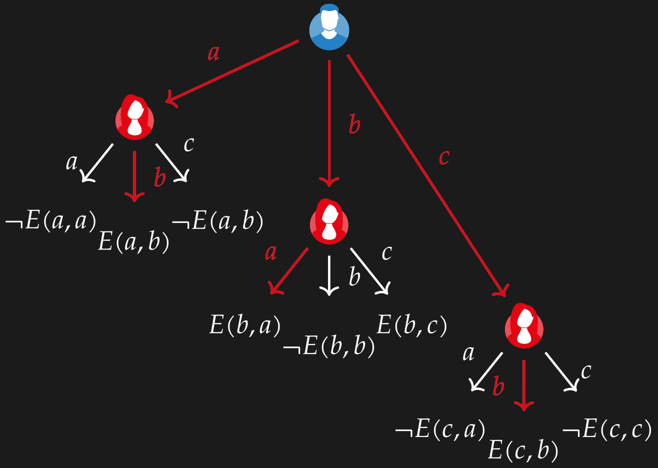
- Interprete the meaning of quantifiers and connectives as strategic choices of the players.
- Use the two-player competitive setting to interprete the duality of the quantifier/connective pair.

Example (SG)

φ: $\forall x \exists y\, E(x,y)$

(no independent vertex)

$\mathcal{M}$:



Comparison Game

Ehrenfeucht-Fraïssé Game

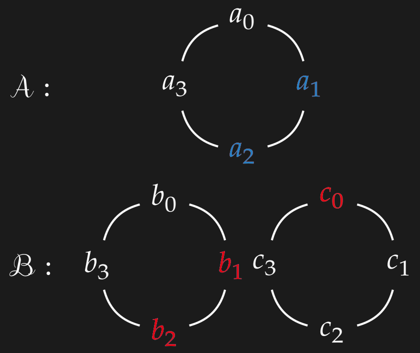
Let $\mathcal{A}$ and $\mathcal{B}$ be two L-models. EF$_m(\mathcal{A}, \mathcal{B})$ goes as follows. In each step,

-  chooses an element from either of the models, and
-  then chooses an element from the other model.

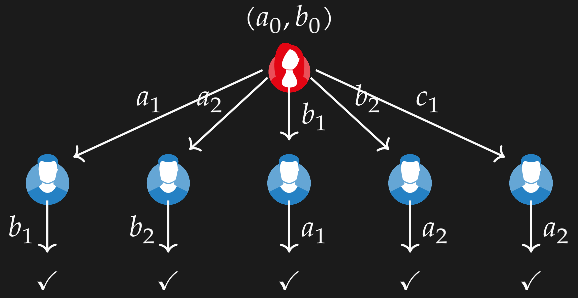
This goes on for m steps and gives us $a_1, a_2, \ldots, a_m \in A^m$ and $b_1, b_2, \ldots, b_m \in B^m$.
Define $\overline{a} : x_i \mapsto a_i$ and $\overline{b} : x_i \mapsto b_i$

-  wins if there is an atomic sentences $\varphi(x_1, x_2, \ldots x_m)$ such that $\mathcal{A} \vDash_{\overline{a}} \varphi$ and $\mathcal{B} \nvDash_{\overline{b}} \varphi$ or vice versa. Otherwise,  wins.

Example (EFG)



's winning strategy for $\mathrm{EF}_2(\mathcal{A}, \mathcal{B})$



Theorem

For any $m \in \mathbb{N}$, has a winning strategy in $\mathrm{EF}(C_k, C_k \sqcup C_k)$ for all $k > 2^m$.

Quantifier Rank

The quantifier rank of a formula is the maximum number of nested quantifiers in it:

- $\mathrm{qr}(\varphi_{\mathrm{atm}}) = \mathrm{qr}(\neg\varphi_{\mathrm{atm}}) \ = \ 0,$
- $\mathrm{qr}(\varphi \wedge \psi) = \mathrm{qr}(\varphi \vee \psi) = \max\{\mathrm{qr}(\varphi), \mathrm{qr}(\psi)\}$
- $\mathrm{qr}(\exists x \varphi) = \mathrm{qr}(\forall x \varphi) = \mathrm{qr}(\varphi) + 1$

If $\mathcal{A}$ and and $\mathcal{B}$ agree on all sentences of quantifier rank $\leq m$, we write $\mathcal{A} \equiv_m \mathcal{B}$.

Ehrenfeucht's Theorem

$\mathcal{A} \equiv_m \mathcal{B}$ if and only if  has a winning strategy in $\mathrm{EF}_m(\mathcal{A}, \mathcal{B})$.

Non-axiomatizability in FOL

A class of finite models K is axiomatizable if there is a sentence φ such that for any finite model $\mathcal{M}$, $\mathcal{M} \models \varphi$ iff $\mathcal{M} \in K$.

Non-axiomatizability

K is not axiomatizable in FOL if for each $m \in \mathbb{N}$ there is $\mathcal{A}_m \in K$ and $\mathcal{B}_m \notin K$ such that $\mathcal{A}_m \equiv_m \mathcal{B}_m$.

Corollary

K is not axiomatizable in FOL if for each $m \in \mathbb{N}$ there is $\mathcal{A}_m \in K$ and $\mathcal{B}_m \notin K$ such that  has a winning strategy in $\mathrm{EF}_m(\mathcal{A}_m, \mathcal{B}_m)$.

Theorem

The class of connected graphs is not axiomatizable in FOL.

Model Existence Game

Hintikka Sets

A Hintikka set H is a maximally consistent set of NNF sentences that satisfies that following condition.

= For every constant term t, $t = t \in H$.

= For every atomic $\varphi(x)$, if $\varphi(c), t = c \in H$, then $\varphi(t) \in H$.

∀ If $\varphi \wedge \psi \in H$, then $\varphi, \psi \in H$.

∃ If $\varphi \vee \psi \in H$, then $\varphi \in H$ or $\psi \in H$.

∀ If $\forall x\, \varphi(x) \in H$, then $\varphi(c) \in H$ for every constant c.

∃ If $\exists x\, \varphi(x) \in H$, then $\varphi(c) \in H$ for some constant c.

∃ For every constant term t, $t = c \in H$ for some constant c.

! No atomic φ such that, $\varphi, \neg\varphi \in H$.

Model Existence Game

Let T be a L-theory with constant symbols. MEG(T) goes as follows.

Start with $T = H$. The two players' turn alternates with beginning first.

- adds the sentences required by one of the conditions.
- wins immediately if this causes a violation of the ! condition.
- If the used condition is a $\exists$ condition, then chooses the disjunct or the constant. Otherwise, she does nothing.
- wins all infinite plays.

Theorem

T is consistent if and only if has a winning strategy in MEG(T).

More Games

Logic	Semantic Game	Comparison Game
Modal Logic	SG on Kripke models	Bisimulation Game
Modal μ-calculus	Parity Game	Infinite Bisimulatioin Game
Finite-variable FO (FO^k)	Restricted semantic game with k pebbles	k-pebble EF game
Infinitary Logic ($L_{\infty\omega}$)	Infinite Semantic Game	Infinite EF game
Monadic Second-Order Logic (MSO)	SG with moves over sets	MSO EF games
...	...	...

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