Spectral Graph Theory and Algorithms, Summer 2017 28 June 2017

Lecture 8

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1 Electrical flow

Consider an undirected graph $G = (V, E)$ such that each edge (i, j) has resistance $r(i, j)$ (or conductance $c(i, j) = 1/r(i, j)$). A *current flow* $f(i, j)$ is one that obeys both

- Kirchoff's current law (KCL) (i.e. flow conservation)

$$\text{flow into node } i = \text{flow leaving node } i$$

- Ohm's Law: there exist potentials $p(i)$, for all $i \in V$, such that

$$f(i,j) = \frac{p(i) - p(j)}{r(i,j)} = c(i,j)(p(i) - p(j)).$$

Then notice that if Ohm's Law is obeyed, then it must be the case that

$$f(j,i) = \frac{p(j) - p(i)}{r(i,j)} = -\frac{p(i) - p(j)}{r(i,j)} = -f(i,j),$$

so the flow from j to i is the negative of the flow from i to j. We call this condition *skew symmetry.*

Since G is undirected, we'll try use (i, j) when we want direction (e.g. $f(i, j)$ indicates the flow from i to j), and use $\{i, j\}$ when we don't (e.g. summing over $\{i, j\} \in E$). Notice there are also some exceptions, say, resistance $r(i, j)$, which is inherently undirected, and we assume this is clear from the context. It will also be useful to assume each edge is oriented arbitrary; we denote this set of edges as $\vec{E}$.

There is another law called Kirchoff's potential/voltage law (KPL or KVL), which states that

$$\sum_{(i,j) \in C} f(i,j) r(i,j) = 0, \text{for any directed cycle } C.$$

We prove that KPL is equivalent to Ohm's Law. It is occasionally useful for us to assume that the current flow is defined by KCL and KPL rather than KCL and Ohm's Law.

Theorem 1 *KPL is equivalent to Ohm's Law.*

[0]This lecture was derived in part from a survey of Arora, Hazan, and Kale 2012 `http://theoryofcomputing.org/articles/v008a006/v008a006.pdf`. Scribes for these lectures were Yuhang Ma and Yingjie Bi.

Proof: If there exists potentials satisfying Ohm's Law, then for any directed cycle C,

$$\sum_{(i,j)\in C} f(i,j)r(i,j) = \sum_{(i,j)\in C} (p(i) - p(j)) = 0.$$

For the other direction, pick some spanning tree T with root r. Let P_i be the directed path in T from i to r. Then we can create the *tree-defined* potentials:

$$p(i) = \sum_{(k,l)\in P_i} f(k,l)r(k,l), \forall i \in V.$$

Notice that these are exactly the potentials we get by assuming that Ohm's Law is obeyed for each edge $\{i,j\} \in T$, so that Ohm's Law obeyed for all $\{i,j\} \in T$.

For any edge $(i,j) \in E - T$, let P_j^R be the r-i path in T (the superscript R denotes the reverse). Then let C be the directed cycle formed by the directed path P_i, the P_j^R, and the arc (j,i). Then we have that

$$\begin{aligned}
p(i) - p(j) &= \sum_{(k,l)\in P_i} f(k,l)r(k,l) - \sum_{(k,l)\in P_j} f(i,l)r(k,l) \\
&= \sum_{(k,l)\in P_i \cup P_j^R} f(k,l)r(k,l) \\
&= \sum_{(k,l)\in C} f(k,l)r(k,l) - f(j,i)r(j,i) \\
&= 0 - f(j,i)r(j,i) = f(i,j)r(i,j),
\end{aligned}$$

where the last equality holds by skew symmetry. □

Notice that it is very easy to find a current flow and potentials obeying KCL and Ohm's Law: $f = 0$ and $p = 0$. To make things more interesting, we start to think about supplying and demanding current from the circuit. Let $b(i)$ be current supplied to i, where $b(i) > 0$ if it is a supply, and $b(i) < 0$ if it is a demand. Then

$$\begin{aligned}
b(i) &= \sum_{j:\{i,j\}\in E} f(i,j), && \text{by KCL} \\
&= \sum_{j:\{i,j\}\in E} c(i,j)(p(i) - p(j)), && \text{by Ohm's Law.}
\end{aligned}$$

If $b(s) = 1, b(t) = -1$ for some $s, t \in V$, we say f is an s-t *electrical flow.* An example is shown in Figure 1.

This is all very interesting, but what does it have to do with spectral graph theory? Suppose for a given b that we want to find the corresponding potentials p for the resulting electrical flow. Consider the weighted Laplacian with conductances as weights, i.e.

$$L_G = \sum_{\{i,j\}\in E} c(i,j)(e_i - e_j)(e_i - e_j)^T.$$

We claim that the potentials we want actually satisfies $L_G p = b$.

Claim 2 *$L_G p = b$.*

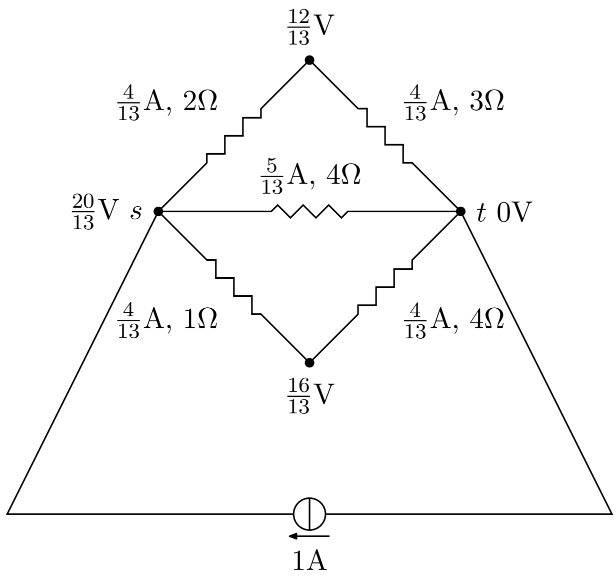

Figure 1: Example of a flow in an electrical network. We put in one unit of current (in amps) at s and remove one at t. The nodes show the potentials (in volts), and each edge has the current passing through it, followed by its resistance.

Proof: Note that

$$\begin{aligned} L_G p &= \sum_{\{i,j\}\in E} c(i,j)(e_i - e_j)(e_i - e_j)^T p \\ &= \sum_{\{i,j\}\in E} c(i,j)(e_i - e_j)(p_i - p_j), \end{aligned}$$

so that by flow conservation

$$b(i) = \sum_{j:\{i,j\}\in E} f(i,j) = \sum_{j:\{i,j\}\in E} c(i,j)(p(i) - p(j)) = L_G p(i).$$

Hence, it follows that $L_G p = b$. □

Thus we get the potentials p by solving $L_G p = b$ for p. However, notice that L_G is singular (its smallest eigenvalue $\lambda_1 = 0$), and we cannot use L_G^{-1} directly. But we can use pseudoinverse instead. Recall

$$L_G^\dagger = \sum_{i:\lambda_i \neq 0} \frac{1}{\lambda_i} x_i x_i^T,$$

where $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are the eigenvalues and $x_1 = \frac{e}{\sqrt{n}}, \ldots, x_n$ are associated orthonormal eigenvectors.

Claim 3 $p = L_G^\dagger b$.

Proof: In order for there to be a solution for p, we need $b^T e = 0$ (we assume G is connected $\Rightarrow \lambda_2 > 0$); that is b is orthogonal to the nullspace of L_G. And this is a natural physical condition: it says that the the amount of current supplied must be equal to the amount of current demanded, and there is overall conservation of current.

Notice that $x_1, \dots, x_n$ forms a basis, and we can write $b = \sum_2^n \alpha_i x_i$, since $b^T x_1 = 0$. Then

$$\begin{aligned}
L_G p &= L_G L_G^\dagger b \\
&= \left(\sum_{i=2}^n \lambda_i x_i x_i^T\right)\left(\sum_{i=2}^n \frac{1}{\lambda_i} x_i x_i^T\right) b \\
&= \left(\sum_{i=2}^n x_i x_i^T\right)\left(\sum_{i=2}^n \alpha_i x_i\right) = \sum_{i=2}^n \alpha_i x_i = b.
\end{aligned}$$

□

The main reason this topic is of interest to us (and the main reason we have this course) is the following theorem shown by Spielman and Teng about a decade ago.

Theorem 4 (Spielman and Teng, 2004) *$L_G p = b$ can be solved for p (approximately) in $\tilde{O}(m)$ time.*

The significance of the paper is that we can solve this linear system in time nearly linear in the number of edges of the graph (i.e. essentially the number of nonzeroes of the Laplacian); this is useful since in many cases graphs really are sparse. There has since been a significant amount of followup work improving this result (including a simple solver we will see Friday) and finding various applications of it (which we will see tomorrow).

Before we move on to two more concepts we will later need, we note that sometimes it helps to write $f(i,j)$ for $(i,j) \in \vec{E}$ in matrix notation. Let $C \in \mathbb{R}^{m \times m}$ matrix with conductance $c(i,j)$ in diagonal, $B \in \mathbb{R}^{n \times w}$ with column $(i,j) \in \vec{E}$ equal to $(e_i - e_j)^T$, then

$$\begin{aligned}
f &= CB^T p, \\
L_G &= \sum_{\{i,j\} \in E} c(i,j)(e_i - e_j)(e_i - e_j)^T = BCB^T, \\
b &= L_G p = BCB^T p = Bf.
\end{aligned}$$

2 Effective Resistance and Energy

We now introduce the definitions for effective resistance and energy.

- The *effective resistance* $r_{\text{eff}}(i,j)$ between i and j is the potential drop between i and j induced by an i-j electrical flow. Essentially this quantity is the resistance between i and j if we replace the entire network by a single resistor. Since the potentials induced by an i-j electrical flow is the p such that $L_G p = e_i - e_j$, we have that $p = L_G^\dagger (e_i - e_j)$,

$$r_{\text{eff}}(i,j) = p(i) - p(j) = (e_i - e_j)^T p = (e_i - e_j)^T L_G^+ (e_i - e_j).$$

- Given current f, the *energy* dissipated by a resistance r is $f^2 r$. Thus, the energy

$\mathcal{E}(f)$ dissipated by our electrical network G with current flow f is

$$\begin{aligned}\mathcal{E}(f) &= \sum_{\{i,j\}\in E} f^2(i,j)r(i,j) = \sum_{\{i,j\}\in E} \frac{(p(i)-p(j))^2}{r(i,j)} \\ &= \sum_{\{i,j\}\in E} c(i,j)(p(i)-p(j))^2 = p^T L_G p, \text{ for associated potentials } p.\end{aligned}$$

For an s-t electrical flow, $b = e_s - e_t$. If p is the associated potential with $L_G p = e_s - e_t$, we have that

$$\mathcal{E}(f) = p^T L_G p = p^T(e_s - e_t) = p(s) - p(t) = r_{\text{eff}}(s,t).$$

We end by showing flows, potentials and energy are actually closely related with each other.

Lemma 5 *The electrical flow f is the unique minimizer of energy $\mathcal{E}(f)$ of system among all flows g satisfying KCL for b, i.e. flows g such that $Bg = b$.*

Proof: Set $h = f - g$, then for any $i \in V$,

$$\sum_{j:\{i,j\}\in E} h(i,j) = \sum_{j:\{i,j\}\in E} f(i,j) - \sum_{j:\{i,j\}\in E} g(i,j) = 0.$$

by flow conservation (KCL). Then

$$\begin{aligned}\mathcal{E}(g) &= \sum_{(i,j)\in\vec{E}} g^2(i,j)r(i,j) \\ &= \sum_{(i,j)\in\vec{E}} (f(i,j)+h(i,j))^2 r(i,j) \\ &= \sum_{(i,j)\in\vec{E}} f(i,j)^2 r(i,j) + 2\sum_{(i,j)\in\vec{E}} f(i,j)h(i,j)r(i,j) + \sum_{(i,j)\in\vec{E}} h(i,j)^2 r(i,j) \\ &= \mathcal{E}(f) + 0 + \sum_{(i,j)\in\vec{E}} h(i,j)^2 r(i,j) > \mathcal{E}(f), \text{ if } h \neq 0,\end{aligned}$$

where the fourth equality holds since

$$\sum_{(i,j)\in\vec{E}} f(i,j)h(i,j)r(i,j) = \sum_{(i,j)\in\vec{E}} (p(i)-p(j))h(i,j) = \sum_{i\in V} p(i) \sum_{j:\{i,j\}\in E} h(i,j) = 0,$$

by using the skew-symmetry of h. Therefore, we concludes that f is the unique minimizer of $\mathcal{E}(f)$. □

Lemma 6 *For a given b such that $b^T e = 0$, the potentials p for electrical flow f determined by b maximize $2x^T b - x^T L_G x$ over all $x \in \mathbb{R}^n$.*

Proof: Use calculus by setting $\nabla(2x^T b - x^T L_G x) = 2(b - L_G x) = 0$, i.e. $L_G x = b$. Thus, it directly follows that the potential p is the maximizer. □

Notice that by substituting x with optimal solution p,

$$2p^T b - p^T L_G p = 2p^T L_G p - p^T L_G p = p^T L_G p = \mathcal{E}(f).$$

In fact, the above two lemmas can be viewed as *dual* to each other, i.e. the primal and dual problems share the same optimal value, with flows and potential as their corresponding minimizer and maximizer respectively. We will be using this duality in the simple solver for $L_G p = b$ we will be seeing on Friday.

3 The Multiplicative Weights Update Algorithm

In this section, we will introduce a very useful algorithm that has been repeatedly discovered by different people in many fields. At first glance, this algorithm seems have no relationship with the main subject of our course. However, tomorrow we will demonstrate a fast algorithm for maximum flows combining electrical flows and the idea here.

Assume at each time step, there are N choices for a decision to be made. At time t, we will gain an unknown value $v_t(i) \in [0,1]$ for making decision i, and the values of all decisions will be revealed after the decision is made. Surprisingly, there is a simple strategy which guarantees to do as well as the best fixed decision over the time.

In the multiplicative weights update algorithm (Algorithm 1), a weight is maintained for each decision i. If we let $w_t(i)$ be weight for decision i at start of time step t and

$$W_t = \sum_{i=1}^{N} w_t(i),$$

then decision i is chosen with probability proportional to $w_t(i)$, so the probability we make decision i at time t is $p_t(i) = w_t(i)/W_t$. After making the decision, the weights will be updated so as to be larger for decisions that get larger values. When the algorithm terminates, the expected value gained by the algorithm is

$$\sum_{t=1}^{T} \sum_{i=1}^{N} p_t(i) v_t(i).$$

Algorithm 1: Multiplicative Weights

$w_1(i) \leftarrow 1, \ \forall i = 1, \ldots, N$
for $t \leftarrow 1$ **to** T **do**
  Pick i with probability $p_t(i)$, get value $v_t(i)$
  $w_{t+1}(i) \leftarrow (1 + \epsilon v_t(i)) w_t(i), \ \forall i = 1, \ldots, N$
end

Theorem 7 *Assume $\epsilon \leq 1/2$, then for any j,*

$$\sum_{t=1}^{T} \sum_{i=1}^{N} p_t(i) v_t(i) \geq (1-\epsilon) \sum_{t=1}^{T} v_t(j) - \frac{1}{\epsilon} \ln N.$$

Proof: The proof idea is to find both the upper and lower bound on W_{T+1}. Note that

$$\begin{aligned}
W_{t+1} = \sum_{i=1}^{N} w_{t+1}(i) &= \sum_{i=1}^{N} w_t(i)(1+\epsilon v_t(i)) \\
&= W_t + \epsilon W_t \sum_{i=1}^{N} p_t(i) v_t(i) \\
&= W_t \left(1 + \epsilon \sum_{i=1}^{N} p_t(i) v_t(i)\right) \\
&\leq W_t \exp\left(\epsilon \sum_{i=1}^{N} p_t(i) v_t(i)\right).
\end{aligned}$$

Therefore,

$$W_{T+1} \leq W_1 \exp\left(\epsilon \sum_{t=1}^{T} \sum_{i=1}^{N} p_t(i) v_t(i)\right) = N \exp\left(\epsilon \sum_{t=1}^{T} \sum_{i=1}^{N} p_t(i) v_t(i)\right).$$

On the other hand, for any given j,

$$W_{T+1} \geq w_{T+1}(j) = \prod_{t=1}^{T} (1+\epsilon v_t(j)) \geq (1+\epsilon)^{\sum_{t=1}^{T} v_t(j)},$$

using the result $1+\epsilon x \geq (1+\epsilon)^x$ for $x \in [0,1]$.

Combining the above two inequalities, we get

$$(1+\epsilon)^{\sum_{t=1}^{T} v_t(j)} \leq N \exp\left(\epsilon \sum_{t=1}^{T} \sum_{i=1}^{N} p_t(i) v_t(i)\right).$$

Taking the logarithm of each side,

$$\ln(1+\epsilon) \sum_{t=1}^{T} v_t(j) \leq \ln N + \epsilon \sum_{t=1}^{T} \sum_{i=1}^{N} p_t(i) v_t(i),$$

which implies

$$\sum_{t=1}^{T} \sum_{i=1}^{N} p_t(i) v_t(i) \geq \frac{\ln(1+\epsilon)}{\epsilon} \sum_{t=1}^{T} v_t(j) - \frac{1}{\epsilon} \ln N \geq (1-\epsilon) \sum_{t=1}^{T} v_t(j) - \frac{1}{\epsilon} \ln N.$$

Here in the last step we are using the inequality $\ln(1+x) \geq x - x^2$ for $x \leq 1/2$. $\square$