Lecture 5

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The main topic of this course has been the unexpected connection between graph theory and spectral theory: we've seen the relationships between graph spectra and connectivity, bipartiteness, independent sets, colorings, trees, cuts, flows, and others. Today we see another connection with a standard concept in graph theory: planarity.

In many branches of mathematics, it is frequently helpful to visualize the information and find properties and relationships by seeing them rather than by deriving them. In graph theory, it is particularly nice if the arcs and vertices of interest are able to be neatly laid out for viewing. This is captured in the concept of a *planar* graph. However, a graph may not necessarily have such a nice visual form. In this lecture, we introduce that notion of planarity, ask how to tell if a graph is planar, and investigate the connections between this property and spectrum of matrices associated with the graph.

1 Introduction to Planarity

Let's begin by defining the term planar (somewhat informally; a standard graph theory textbook will have a more rigorous definition).

Definition 1 (Planar) *A graph $G = (V, E)$ is planar if*

- *for each vertex $i \in V$ there exists a point $x_i \in \mathbb{R}^2$*
- *for each edge $(i, j) \in E$ there exists a curve between x_i and x_j that intersects no other curve*

Conceptually, observe that if a graph is planar it means that it can be drawn in the plane without any edges intersecting. We call this collection of points and curves the *planar embedding* of G. The term *plane graph* is used to refer to the graph and the collection of points and curves together. The plane graph divides the $\mathbb{R}^2$ plane into regions called the *faces* of the graph. This includes the *external face*, which is the face formed by the outermost curves.

A planar graph G is *maximal* if adding any edge e to G makes $G + e$ non-planar. For any maximal planar graph, any face must be a triangle, since otherwise we could add an edge. A graph H is a *minor* of G if we can obtain H from G by some sequence of deleting and/or contracting edges.

Now, recall that K_5 is the complete graph with five vertices and $K_{3,3}$ is the complete bipartite graph with 3 vertices on each side. Additionally, recall that 3-vertex-connected means that up to any two vertices can be removed from a graph and it is still connected. These concepts are used in the following foundational planarity theorems. These proofs are somewhat outside the scope of the course and thus not given in this lecture.

Theorem 1 (Kuratowski 1930, Wagner 1937) *A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

Theorem 2 *Any maximal planar graph is 3-vertex-connected.*

[0]This lecture is derived from Godsil and Royle, Sections 1.8, 13.9-13.11; Van der Holst 1995 (http://oai.cwi.nl/oai/asset/2232/2232A.pdf); and Lau's 2012 notes, Week 2, http://appsrv.cse.cuhk.edu.hk/~chi/csc5160/notes/L02.pdf. The scribes were Andrew Daw and Sam Gutekunst.

2 Generalized Laplacian Matrix

We now switch gears and introduce a generalization of the Laplacian matrix concept that we have used frequently in the course. The formal definition, given below, is very similar to the original definition; however, there is no condition on the diagonal elements of the matrix.

Definition 2 (Generalized Laplacian) *A generalized Laplacian of graph G is a symmetric matrix $M = (m_{ij}) \in \mathbb{R}^{n\times n}$ such that*

$$\begin{aligned} &m_{ij} < 0 \text{ if } (i,j) \in E \\ &m_{ij} = 0 \text{ if } (i,j) \notin E \text{ and } i \neq j \end{aligned}$$

By following the proof of the Perron-Frobenius theorem from Lecture 3, we can assume that if G is connected then the eigenvalues λ_i are such that $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$ and the eigenvectors x_i are such that $x_1 > 0$.

Since the generalized Laplacian has no condition on the diagonal, then $M - \lambda I$ is a generalized Laplacian if M is. Thus, we can apply a linear shift to all of the eigenvalues of a generalized Laplacian matrix and still have a generalized Laplacian. So, we will assume that λ_1 is the unique negative eigenvalue and that $\lambda_2 = 0$. We now introduce another concept so that we can make full use of these concepts.

Definition 3 (Kernel & Co-Rank) *The kernel of M is*

$$\ker(M) = \{x \in \mathbb{R}^n : Mx = 0\}.$$

The co-rank of M is $\dim(\ker(M))$.

For G connected and generalized Laplacian M such that $\lambda_2(M) = 0$, the co-rank is the multiplicity of 0 as an eigenvalue.

Definition 4 (Colin de Verière invariant) *The* Colin de Verdière invariant *$\mu(G)$ is the largest corank of a generalized Laplacian such that:*

1. *M has exactly 1 negative eigenvalue*
2. *There is no $X = (x_{ij}) \in \mathbb{R}^{n\times n}$ such that $X \neq 0$, $MX = 0$, $x_{ii} = 0$ for all $i \in V$, and $x_{ij} = 0$ if $m_{ij} \neq 0$. (This is referred to as the Strong Arnold Property. Not to be confused with this strong Arnold.)*

3 Planarity via Generalized Laplacians

We can now begin to bring the two previous and seemingly unrelated topics together.

Theorem 3 (Colin de Verdière 1990) *The following hold:*

- *G is a collection of paths iff $\mu(G) \leq 1$.*
- *G is outerplanar (all vertices are on the external face) iff $\mu(G) \leq 2$.*
- *G is planar iff $\mu(G) \leq 3$.*

The most challenging part of proving the final relationship is showing that if G is not planar then $\mu(G) > 3$. To do the proof, we make use of the fact that $\mu(K_{3,3}) = \mu(K_4) = 4$ and the following theorem.

Theorem 4 (Colin de Verdière 1990) *If H is a minor of G then $\mu(H) \leq \mu(G)$.*

The easier direction of that proof is showing that if G is planar then $\mu(G) \leq 3$. We will now work towards proving this via a proof given by Van der Holst in 1995. Let the support of x be denoted by $\text{supp}(x) = \{i \in V : x(i) \neq 0\}$, $\text{supp}^+(x) = \{i \in V : x(i) > 0\}$, and $\text{supp}^-(x) = \{i \in V : x(i) < 0\}$. We start by stating and proving a series of lemmas.

Lemma 5 *Suppose $x \in \ker(M)$ where M is a generalized Laplacian matrix. Then, if $i \notin \text{supp}(x)$ then either all of the neighbors of i are not in $\text{supp}(x)$ or i has neighbors in both $\text{supp}^+(x)$ and $\text{supp}^-(x)$.*

Proof: If $Mx = 0$ then $(Mx)(i) = 0$ for each i. Then,

$$0 = (Mx)(i) = \sum_{j:(i,j)\in E} m_{ij}x(j) + m_{ii}x(i) = \sum_{j:(i,j)\in E} m_{ij}x(j)$$

since $i \notin \text{supp}(x)$. Since $m_{ij} < 0$ for all j such that $(i,j) \in E$, either all $x(j) = 0$ or some are positive and others are negative. $\square$

Lemma 6 *For $x \in \ker(M)$ for M a generalized Laplacian, $x \neq 0$, G connected, then $\text{supp}^+(x) \neq \emptyset$ and $\text{supp}^-(x) \neq \emptyset$.*

Proof: If $x \in \ker(M)$, then it is in the span of the eigenvectors that have eigenvalue 0. These eigenvectors are orthogonal to the one eigenvector z of negative eigenvalue, and by Perron-Frobenius, we can assume that $z > 0$. Thus $x^T z = 0$ as well. So for $x^T z = 0$, with $x \neq 0$, it must be the case that x has both positive and negative entries. $\square$

For the next lemma, we need to define another term.

Definition 5 (Minimal Support) *A vector x is said to have minimal support if $x \neq 0$ and for every $y \neq 0$ and $y \in \ker(M)$ with $\text{supp}(y) \subseteq \text{supp}(x)$ implies that $\text{supp}(x) = \text{supp}(y)$.*

Additionally, let's introduce some notation. Let $M[I,J]$ be the submatrix with rows from the index set I and columns from index set J.

Lemma 7 *Let G be a connected graph, M be a generalized Laplacian with only one negative eigenvalue. Let $x \in \ker(M)$ have minimal support. Then the graph induced by $\text{supp}^+(x)$ (or by $\text{supp}^-(x)$) is connected.*

Proof: We will use a proof by contradiction strategy. Suppose that the graph induced by the positive support of x is not connected. Suppose that I and J are a partition of $\text{supp}^+(x)$ such that I and J are not connected. We will now show that we can find $y \in \ker(M)$ and $y \neq 0$ but $\text{supp}(y) \subset \text{supp}(x)$.

Let $K = [n] - I - J$, so that K contains exactly the elements of supp^- and those not in the support; thus the entries $[n]$ is partitioned by I, J, and K. Since $Mx = 0$ and $M[I,J] = M[J,I] = 0$ (because I and J are not connected), we have

$$M[I,I]x[I] + M[I,K]x[K] = 0 \tag{1}$$

$$M[J,J]x[J] + M[J,K]x[K] = 0 \tag{2}$$

Let z be the eigenvector corresponding to the single negative eigenvalue. By Perron-Frobenius, we know that $z > 0$. Now, define α as

$$\alpha = \frac{z[I]^T x[I]}{z[J]^T x[J]}$$

and note that $\alpha > 0$ because all the terms in it are positive. Now, define y such that

$$y = \begin{cases} x(i) & \forall i \in I \\ -\alpha x(i) & \forall i \in J \\ 0 & \text{else} \end{cases}$$

Now we claim that $\text{supp}(y) \subset \text{supp}(x)$: Note that

$$z^T y = z[I]^T x[I] - \alpha z[J]^T x[J] = 0.$$

We also have that

$$\begin{aligned} y^T M y &= y[I]^T M[I,I] y[I] + y[J]^T M[J,J] y[J] \\ &= x[I]^T M[I,I] x[I] + \alpha^2 x[J]^T M[J,J] x[J] \\ &= -x[I]^T M[I,K] x[K] - \alpha^2 x[J]^T M[J,K] x[K] \leq 0, \end{aligned}$$

where the last equality uses (1) and (2), and the final inequality follows because $x[I], x[J] > 0$, $x[K] < 0$, $M[I,K], M[J,K] \leq 0$ and $\alpha > 0$.

So we have that $z^T y = 0$ and $y^T M y \leq 0$. Because y is orthogonal to the only eigenvector of negative eigenvalue, it must also be the case that $y^T M y \geq 0$. Thus $My = 0$ which means that $y \in \ker(M)$, implying that x does not have minimal support. This is the desired contradiction. □

We can now prove the following.

Theorem 8 *If G is planar and 3-vertex-connected, then $\mu(G) \leq 3$.*

Proof: Let's suppose G is planar but $\mu(G) > 3$. We show that we can find a $K_{3,3}$ minor in G.

Choose some plane embedding of G and choose a face. Let u, v, w be on the face. Because $\dim(\ker(M)) \geq 4$, there exists $x \in \ker(M)$ such that $u, v, w \notin \text{supp}(x)$. Assume that x has minimal support.

Now let's add another point s into the face, and add edges from s to u, v, w. Pick $p \in \text{supp}(x)$. Since G is 3-vertex-connected, there exist vertex disjoint paths from p to u, v, w. Let a, b, c be the first vertices on these paths such that they are not in the support of x and all subsequent vertices are not in the support of x (see Figure 3). We know that $a, b, c \notin \text{supp}(x)$ have neighbors in the support of x, and thus by Lemma 5 we know that they must have neighbors both in $\text{supp}^+(x)$ and $\text{supp}^-(x)$ (see Figure 3).

Now, let's contract $\text{supp}^+(x)$ and $\text{supp}^-(x)$ to single vertices s^+ and s^-, respectively (see Figure 3). Now, let's also then contract the $a - u$, $b - v$, and $c - w$ paths. This leaves us with $\{s, s^+, s^-\}$ and $\{a, b, c\}$ forming the $K_{3,3}$ complete graph, and so we have completed the proof. □

With a little twist via the closing corollary, we can complete the proof of that direction of any G.

Corollary 9 *Since $\mu(G)$ only increases when adding edges, any maxmimal planar graph is 3-vertex-connected. Thus, we can make any G maximally planar, so $\mu(G) \leq 3$ for any planar G.*

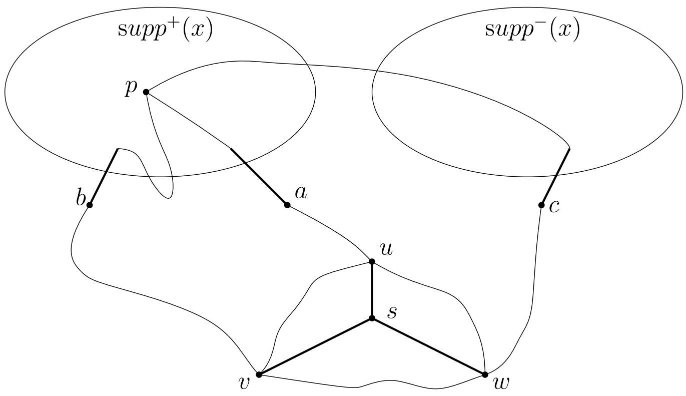


Figure 1: Vertex disjoint paths from p to u, v, w on a single face, with s added to the face.

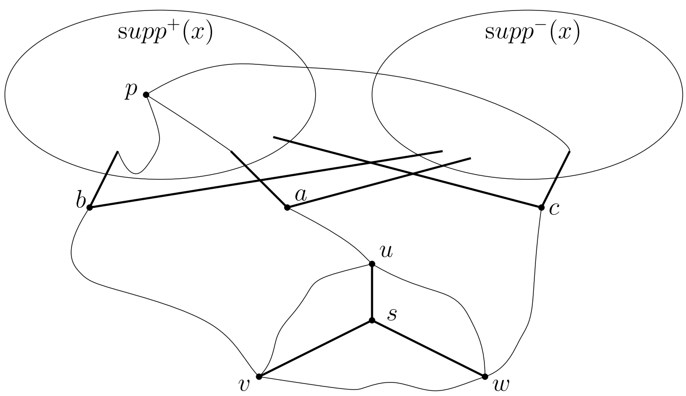


Figure 2: a, b, c must have neighbors in both $\text{supp}^+(x)$ and $\text{supp}^-(x)$.

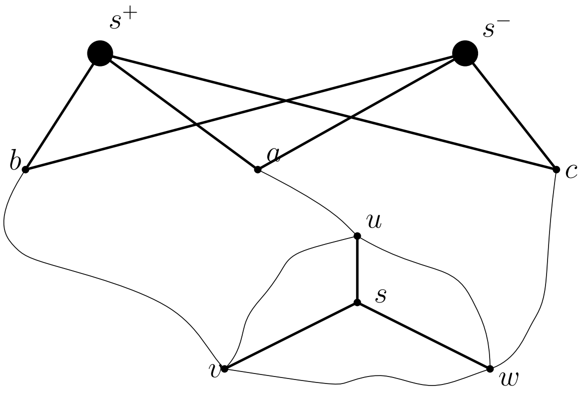


Figure 3: Contracting $\text{supp}^+(x)$ and $\text{supp}^-(x)$ to s^+ and s^- respectively.

4 More relationships between connectivity and the Laplacian

4.1 Flavors of Connectivity

Let $S \subset V$. Recall that $\delta(S)$ denotes the set of edges with exactly one endpoint in S, and define $\mathrm{vol}(S) \equiv \sum_{i \in S} d(i)$.

Definition 6 *The* **conductance** *of $S \subset V$ is*

$$\phi(S) \equiv \frac{|\delta(S)|}{\min\{\mathrm{vol}(S), \mathrm{vol}(V-S)\}}.$$

The **edge expansion** *of S is*

$$\alpha(S) \equiv \frac{|\delta(S)|}{|S|}, \qquad \text{for } |S| \le \frac{n}{2}.$$

The **sparsity** *of S is*

$$\rho(S) \equiv \frac{|\delta(S)|}{|S||V-S|}.$$

These measures are similar if G is d-regular (i.e., $d(i) = d$ for all $i \in V$). In this case,

$$\alpha(S) = d\phi(S), \qquad \frac{n}{2}\rho(S) \le \alpha(S) \le n\rho(S).$$

To see the first equality, e.g., notice that the volume of S is $d|S|$.

In general, notice that $0 \le \phi(S) \le 1$ for all $S \subset V$.

We're usually interested in finding the sets S that minimize these quantities over the entire graph.

Definition 7 *We define*

$$\phi(G) \equiv \min_{S \subset V} \phi(S), \qquad \alpha(G) \equiv \min_{S \subset V : |S| \le \frac{n}{2}} \alpha(S), \qquad \rho(G) \equiv \min_{S \subset V} \rho(S).$$

We call a graph G an *expander* if $\phi(G)$ (or $\alpha(G)$) is "large" (i.e. a constant[1]). Otherwise, we say that G has a sparse cut.

One algorithm for finding a sparse cut that works well in practice, but that lacks strong theoretical guarantees is called **spectral partitioning**. In the following, $\mathscr{L}$ is the "normalized Laplacian". We'll define later what that is, but it's related to our usual Laplacian L_G.

Algorithm 1: Spectral Partitioning

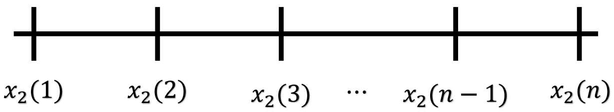
1. Compute x_2 of $\mathscr{L}$ (the eigenvector corresponding to $\lambda_2(\mathscr{L})$);
2. Sort V such that $x_2(1) \le \cdots \le x_2(n)$.;
3. Define the **sweep cuts** for $i = 1, ..., n-1$ by $S_i \equiv \{1, ..., i\}$.;
4. Return $\min_{i \in \{1,...,n-1\}} \phi(S_i)$.;

The following picture illustrates the idea of the algorithm; sweep cuts correspond to cuts between consecutive bars:

Cheeger's inequality provides some insight into why this algorithm works well.

Next week we will prove the following.

[1] One should then ask "A constant with respect to what?" Usually one defines families of graphs of increasing size as families of expanders, in which case we want the conductance or expansion constant with respect to the number of vertices.



Theorem 10 (Cheeger's Inequality) *Let λ_2 be the second smallest eigenvalue of $\mathscr{L}$. Then:*

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}.$$

The theorem proved by Jeff Cheeger actually has to do with manifolds and hypersurfaces; the theorem above is considered to be a discrete analog of Cheeger's original inequality. But the name has stuck.