Spectral Graph Theory and Algorithms, Summer 2017 22 June 2017

Lecture 5

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1 Graph Laplacians and Cuts

We now see that we can get some easy bounds on various types of cuts in graphs by using the eigenvalues of the Laplacian.

Definition 1 *If $|V|$ is even, let $b(G)$ be the smallest bisection of G; that is*

$$b(G) = min_{S \subset V : |S| = |V-S|} |\delta(S)|,$$

where $\delta(S)$ is the set of edges with one endpoint in S and the other endpoint in $V-S$. We say the edges in $\delta(S)$ are the edges in the cut *defined by S.*

Claim 1

$$\frac{n}{4}\lambda_2(G) \leq b(G).$$

Proof: Let $\bar{S}$ be an optimal bisection. Let $x\ in \{-1,+1\}^n$ s.t.

$$x(i) = \begin{cases} -1, & \text{if } i \in \bar{S} \\ +1, & \text{otherwise} \end{cases}.$$

Recall that

$$\lambda_2 = min_{z \in \Re^n, \langle z, e \rangle = 0} \frac{z^T L_G z}{z^T z}.$$

Note that $\langle x, e \rangle = 0$ since half of the entries of x are -1 and half are $+1$. Therefore,

$$\lambda_2 \leq \frac{x^T L_G x}{x^T x} = \sum_{(i,j) \in E} \frac{(x(i) - x(j))^2}{n} = \frac{1}{n} \cdot 4|\delta(\bar{S})| = \frac{4}{n} b(G).$$

□

We now turn to the largest eigenvalue of the Laplacian, and show that it has a connection to large cuts in the graph.

Definition 2 *Let $mc(G)$ be the maximum cut in the graph, so that*

$$mc(G) = max_{S \subseteq V} |\delta(S)|.$$

Then using the same idea as the proof above, we can show the following.

[0]This lecture is derived from Mohar and Poljak, *Eigenvalues in Combinatorial Optimization*, Sections 2.1 and 2.4; Cvetković, Rowlinson, and Simić, *An Introduction to the Theory of Graph Spectra*, Sections 7.1 and 7.2; and Godsil and Royle, *Algebraic Graph Theory*, Section 13.2.. Scribes notes were writen by Rahmtin Rotabi and Faisal Alkaabneh.

Claim 2

$$mc(G) \leq \frac{n}{4}\lambda_n(L_G).$$

Proof: Let $\bar{S}$ be a maximum cut and

$$x(i) = \begin{cases} -1, & \text{if } i \in \bar{S} \\ +1, & \text{otherwise} \end{cases}.$$

Then,

$$\lambda_n = \max_{z\in\Re^n} \frac{z^T L_G z}{z^T z} \geq \frac{x^T L_G x}{x^T x} = \sum_{(i,j)\in E} \frac{(x(i)-x(j))^2}{n} = \frac{4|\delta(\bar{S})|}{n} = \frac{4}{n} mc(G).$$

□

In fact, we can modify the bound above to give a tighter bound on the maximum cut.

Claim 3

$$mc(G) \leq \frac{n}{4} \min_{u:\langle u,e\rangle=0} \lambda_n(L_G + diag(u)),$$

where $diag(u)$ is a diagonal matrix that $diag(u)(i,i) = u(i)$.

Proof: Following the same definition of x as above, we get that

$$\begin{aligned}
\lambda_n(L_G + diag(u)) &= \max_{z\in\Re^n} \frac{z^T(L_G + diag(u))z}{z^T z} \\
&\geq \frac{x^T L_G x + x^T diag(u)) x}{x^T x} \\
&= \frac{4mc(G) + \sum_{i\in V} u(i)x(i)^2}{n} \\
&= \frac{4mc(G)}{n},
\end{aligned}$$

since $x^2(i) = 1$ for all $i \in V$, and $\sum_{i\in V} u(i) = \langle u, e\rangle = 0$. □

This bound on the eigenvalue has a connection to other well-known bounds on the maximum cut problem. For a given vector u such that $\langle u, e\rangle = 0$, let $\lambda = \lambda_n(L_G + u)$. Define $\gamma(i) = \lambda - (u(i) + d(i))$ for all $i \in V$, where $d(i)$ is the degree of i in G. Then for adjacency matrix A, we have that

$$A + diag(\gamma) = \lambda I - (L_G + u).$$

Then we can see that $A + diag(\gamma) \succeq 0$ since for any $x \in \Re^n$,

$$\begin{aligned}
x^T(A + diag(\gamma))x &= x^T(\lambda I - (L_G + u))x \\
&= \lambda x^T x - x^T(L_G + u)x \\
&\geq x^T(L_G + u)x - x^T(L_G + u)x \\
&= 0,
\end{aligned}$$

where the inequality follows since $\lambda \geq x^T(L_G + u)x/x^Tx$. Then we observe that

$$\begin{aligned}\frac{n}{4}\lambda &= \frac{1}{4}\sum_{i\in V}(\gamma(i) + u(i) + d(i)) \\ &= \frac{1}{4}\sum_{i\in V}\gamma(i) + \frac{1}{4}\sum_{i\in V}d(i) \\ &= \frac{1}{4}\sum_{i\in V}\gamma(i) + \frac{1}{2}|E|.\end{aligned}$$

Then finding a u to minimize $\frac{n}{4}\min_{u:\langle u,e\rangle=0}\lambda_n(L_G + diag(u))$ turns out to be equivalent to finding a γ to minimize

$$\frac{1}{4}\sum_{i\in V}\gamma(i) + \frac{1}{2}|E|,$$

subject to

$$A + diag(\gamma) \succeq 0.$$

This is a *semidefinite program*, and it has a dual semidefinite program of maximizing

$$\frac{1}{2}\sum_{(i,j)\in E}(1 - x_{ij})$$

subject to

$$x_{ii} = 1 \text{ for all } i \in V, \qquad X = (x_{ij}) \succeq 0.$$

This semidefinite program is used in a .878-approximation algorithm for the maximum cut problem due to Goemans and W. Thus one can show that the eigenvalue bound is a strong one; we also have that

$$mc(G) \geq .878 \cdot \frac{n}{4}\min_{u:\langle u,e\rangle=0}\lambda_n(L_G + diag(u)).$$

2 The Matrix-Tree Theorem

We continue to see the usefulness of the graph Laplacian, and its connection to yet another standard concept in graph theory, that of a spanning tree. Let $A[i]$ be the matrix A with its i^{th} column and row removed. We will give two different proofs of the following.

Theorem 4 (Kirchhoff's Matrix-Tree Theorem) $\det(L_G[i])$ *gives the number of spanning trees in G (for any i).*

In order to do the first proof, we need to use the following fact.

Fact 1 *Let E_{ii} be a matrix with 1 in the $(i,i)^{th}$ entry and 0s elsewhere. Then*

$$\det(A + E_{ii}) = \det(A) + \det(A[i]).$$

If you think about a determinant as being the sum over all permutations of the products of the entries corresponding to the permutation, the fact makes sense: we've increased the (i,i) entry, a_{ii}, to $(a_{ii}+1)$, and we can think about each permutation that uses the (i,i) entry either multiplying by a_{ii} (in which case we just get $\det(A)$ or by the 1, in which case, we get the sum over all the permutations that avoid the ith row and column, or $\det(A[i])$.

Proof of Theorem 4: Our first proof will be by induction on the number of vertices and edges of graph G.

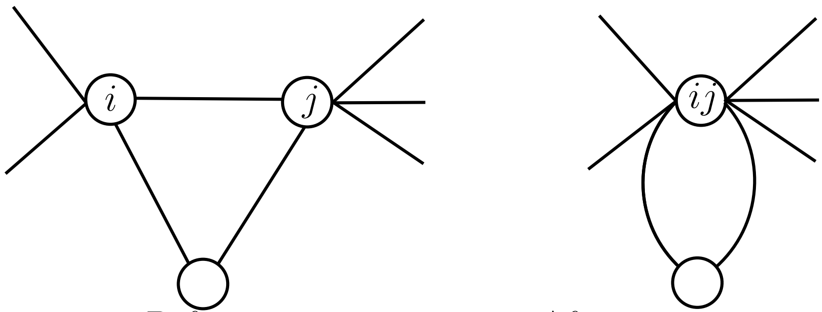
Base case: G is an empty graph of two vertices, then

$$L_G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so that $L_G[i] = [0]$ and $\det(L_G[i]) = 0$.

Inductive step: Suppose there exists $e = (i,j)$ incident in i. If there is not and i is an isolated vertex, then there are zeros along i^{th} row and column of L_G. Then $\det(L_G[i]) = \det(L_{G-i}) = 0 = \prod_{i=1}^{n} \lambda_i$ and, as we showed previously, $\lambda_1 = 0$ for any L_G. Note also that the number of spanning trees is 0 if i is isolated, so the theorem holds in this case.

Now we introduce some notations. Let $\tau(G)$ is the number of spanning trees in G, let $G-e$ be G with edge e removed, and G/e be G with edge e contracted. See below for an illustration of graph contraction.



Before contraction After contraction

For any spanning tree T, either $e \in T$ or $e \notin T$. We note that $\tau(G/e)$ gives the number of trees T with $e \in T$, while $\tau(G-e)$ gives the number of trees T with $e \notin T$. Thus

$$\tau(G) = \tau(G\backslash e) + \tau(G-e);$$

note that the first term is G with one fewer edge, while the second has one fewer vertex, and so these will serve as the basis of our induction.

First we try to relate L_G to L_{G-e}, and we observe that $L_G[i] = L_{G-e}[i] + E_{jj}$ (that is, if we remove edge e, then the only difference in the matrix $L_G[i]$ is that we have to correct for the change in degree of j). Then by the Fact 1

$$\begin{aligned} \det(L_G[i]) &= \det(L_{G-e} + E_{jj}) \\ &= \det(L_{G-e}[i]) + \det(L_{G-e}[i,j]) \\ &= \det(L_{G-e}[i]) + \det(L_G[i,j]), \end{aligned}$$

where by $L_G[i,j]$ we mean L_G with both the ith and jth rows and columns removed; the last equality follows since once we've removed both the ith and jth rows and columns there's no difference between L_G and L_{G-e} for $e = (i,j)$.

Now to relate L_G to $L_{G/e}$. Suppose we contract i onto j (so that $L_{G/e}$ has no row/column corresponding to i). Then $L_{G/e}[j] = L_G[i, j]$.

Thus we have that

$$\begin{aligned}\det(L_G[i]) &= \det(L_{G-e}[i]) + \det(L_{G/e}[j]) \\ &= \tau(G-e) + \tau(G/e) = \tau(G).\end{aligned}$$

where the second equation follows by induction; this completes the proof. □

For the second proof of the theorem, we need the following fact which explains how to take the determinant of the product of rectangular matrices.

Fact 2 (Cauchy-Binet Formula) *Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times n}$, for $m \geq n$. Let A_S (respectively B_S) be submatrices formed by taking the columns (respectively rows) indexed by $S \subseteq [m]$ of A (respectively B).*

Let $\binom{[m]}{n}$ be the set of all size n subsets of $[m]$. Then

$$\det(AB) = \sum_{S\in\binom{[m]}{n}} \det(A_S)\det(B_S).$$

Recall that $L_G = \sum_{(i,j)\in E}(e_i - e_j)(e_i - e_j)^T$. Thus we can write $L_G = BB^T$ where $B \in \mathbb{R}^{m\times n}$ has one column of B per edge (i, j), with the column $(e_i - e_j)$. Since we can write $L_G = BB^T$, this is yet another proof that L_G is positive semidefinite. Then if $B[i]$ denotes B with its i^{th} row omitted, then $L_G[i] = B[i]B[i]^T$. We let $B_S[i]$ denote $B[i]$ with just the columns of $S \subseteq E$.

We need the following lemma, whose proof we defer for a moment.

Lemma 5 *For $S \subseteq E$, $|S| = n - 1$, $|det(B_S[i])| = 1$ if S is a spanning tree, 0 otherwise.*

The second proof of the matrix-tree theorem now becomes very short.

Proof of Theorem 4:

$$\begin{aligned}\det(L_G[i]) &= \det(B[i]B[i]^T) \\ &= \sum_{S\in\binom{E}{n-1}} (\det(B_S[i]))(\det(B_S[i])) \\ &= \tau(G),\end{aligned}$$

where the second equation follows by the Cauchy-Binet formula, and the third by Lemma 5. □

We can now turn to the proof of the lemma.

Proof of Lemma 5: Assume that the edges in $B_S[i]$ are "directed" however we want; that is, we can change the column corresponding to (i, j) from $e_i - e_j$ to $e_j - e_i$, since this only flips the sign of the determinant.

If $S \subseteq E$, $|S| = n - 1$, and S is not a spanning tree, then it must contain a cycle. We direct edges around the cycles. If we then sum the columns of $B_S[i]$ corresponding to the cycle, we obtain the 0 vector, which implies that the columns of $B_S[i]$ are linearly dependent, and thus $\det(B_S[i]) = 0$.

Now we suppose that S is a spanning tree; we prove the lemma statement by induction on n.

__**Base case**__ $n = 2$. Then

$$B_S = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

so that $B_S[i] = \pm 1$, and thus $\det(B_S[i] = 1)$.

__**Inductive case:**__ Suppose the lemma statement is true for graphs of size $n-1$. Let j leaf of the tree $j \neq i$. Let (k, j) be edge incident on j. We exchange rows/columns so that (k, j) is last column, and j is last row; this may flip sign of determinant, but that doesn't matter. Then

$$B_S[i] = \begin{array}{c} \begin{array}{cccc} & & & (k,j) \end{array} \\ \left[\begin{array}{ccc|c} & & & 0 \\ & & & 0 \\ & & & 1 \\ & & & 0 \\ \hline 0 & \dots & 0 & -1 \end{array}\right] \end{array}$$

Thus if we expand the determinant along the last row we get

$$|\det(B_S[i])| = |\det(B_{S-\{(k,j)\}}[i])| = 1.$$

The last equality follows by induction since $S - \{(k, j)\}$ is a tree on the vertex set without j, since we assumed that j is a leaf. $\square$

3 Consequences of the Matrix-Tree Theorem

Once we have the matrix-tree theorem, there are a number of interesting consequences, which we explore in this section. Given a square matrix $A \in \mathbb{R}^{n \times n}$, let A_{ij} be matrix without row i column j (so $A[i] = A_{ii}$). Let $C_{ij} = (-1)^{i+j} \det(A_{ij})$ be the i, j *cofactor* of A. Then we define the *adjugate* $\text{adj}(A)$ as the matrix with i, j entry C_{ji}. We will need the following fact.

Fact 3

$$A \,\text{adj}(A) = \det(A) I.$$

By the matrix-tree theorem, the (i, i) cofactor of L_G is equal to $\tau(G)$. But we can say something even stronger.

Theorem 6 *Every cofactor of L_G is $\tau(G)$, so that*

$$\text{adj}(L_G) = \tau(G) J.$$

Proof:

If G is not connected, then $\tau(G) = 0$ and $\lambda_2(L_G) = 0 = \lambda_1(L_G)$. So the rank of L_G rank is at most $n - 2$. Then $\det((L_G)_{ij}) = 0$, which implies that $\text{adj}(L_G) = 0$, as desired.

If G is connected, since $\det(L_G) = 0$, by the fact above $L_G \,\text{adj}(L_G) = 0$ (i.e. the zero matrix). Because G is connected, multiples of e are the only eigenvectors of L_G

with eigenvalue of 0. Thus every column of $\text{adj}(L_G)$ must be some multiple of e. But we know that for the ith column of $\text{adj}(L_G)$, its ith entry is $\tau(G)$, so the column itself must be $\tau(G)e$, and the lemma statement follows. $\square$

We conclude with one more theorem.

Theorem 7 *Let $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ be the eigenvalues of L_G. Then*

$$\tau(G) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i.$$

Proof: The theorem is true if G is not connected, since then $\lambda_2 = 0$ and $\tau(G) = 0$.

Otherwise, we will look at linear term of the characteristic polynomial in two different ways. In the first way, the characteristic polynomial is

$$(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) = \lambda(\lambda - \lambda_2)(\lambda - \lambda_3)...(\lambda - \lambda_n),$$

so the linear term is

$$(-1)^{n-1}\prod_{i=2}^{n}\lambda_i.$$

For the second way, we want the linear term of $\det(\lambda I - L_G)$; the matrix looks like the following:

$$\begin{pmatrix} {\scriptstyle \lambda - d(1)} & & & \\ & \ddots & -L_G & \\ & -L_G & \ddots & \\ & & & {\scriptstyle \lambda - d(n)} \end{pmatrix}$$

If we think about the determinant as the sum over all permutations of the products of the entries corresponding to the permutation, then we get a linear term in λ whenever an (i,i) term is part of the permutation, but no other diagonal entries are part of the permutation; also, if the (i,i) term is part of the permutation then no other entry from row and column i is part of the permutation. Finally, since all the other entries are negations of their entry in L_G, we get that if we have a linear term in λ because we include the (i,i) term of the matrix as part of the permutation, the linear term is $(-1)^{n-1}\det(L_G[i])$. Summing over all (i,i) entries, the linear term of λ in $\det(\lambda I - L_G)$ is

$$(-1)^{n-1}\sum_{i=1}^{n}\det(L_G[i]) = (-1)^{n-1}\cdot n \cdot \tau(G).$$

Thus we have that $\tau(G) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i$.

$\square$