Spectral Graph Theory and Algorithms, Summer 2017	21 June 2017
Lecture 3	
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Once again, we'll start by proving a general theorem about eigenvalues, and then show its application to some graph problems.

1 Eigenvalue Interlacing Theorem

The following theorem is known as the *eigenvalue interlacing theorem.*

Theorem 1 (Eigenvalue Interlacing Theorem) *Suppose $A \in \mathbb{R}^{n\times n}$ is symmetric. Let $B \in \mathbb{R}^{m\times m}$ with $m < n$ be a principal submatrix (obtained by deleting both i-th row and i-th column for some values of i). Suppose A has eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and B has eigenvalues $\beta_1 \leq \cdots \leq \beta_m$. Then*

$$\lambda_k \leq \beta_k \leq \lambda_{k+n-m} \quad \textit{for} \quad k = 1, \cdots, m$$

And if $m = n-1$,

$$\lambda_1 \leq \beta_1 \leq \lambda_2 \leq \beta_2 \leq \cdots \leq \beta_{n-1} \leq \lambda_n$$

Proof: WLOG, assume $A = \begin{bmatrix} B & X^T \\ X & Z \end{bmatrix}$. Let $\{x_1, \cdots, x_n\}$ be eigenvectors of A, $\{y_1, \cdots, y_m\}$ be eigenvectors of B. We define the following vector spaces:

$$V = span(x_k, \cdots, x_n), \qquad W = span(y_1 \cdots, y_k), \qquad \widetilde{W} = \left\{ \begin{pmatrix} w \\ 0 \end{pmatrix} \in \mathbb{R}^n, w \in W \right\}$$

Since $\dim(V) = n-k+1$ and $\dim(\widetilde{W}) = \dim(W) = k$, there exists $\widetilde{w} \in V \bigcap \widetilde{W}$ and $\widetilde{w} = \begin{pmatrix} w \\ 0 \end{pmatrix}$ for some $w \in W$. Then

$$\widetilde{w}^T A \widetilde{w} = \begin{bmatrix} w^T & 0 \end{bmatrix} \begin{bmatrix} B & X^T \\ X & Z \end{bmatrix} \begin{bmatrix} w \\ 0 \end{bmatrix} = w^T B w$$

Recall $\lambda_k = \min_{x \in V} \frac{x^T A x}{x^T x}$ and $\beta_k = \max_{x \in W} \frac{x^T B x}{x^T x}$. Then we see that

$$\lambda_k \leq \frac{\widetilde{w}^T A \widetilde{w}}{\widetilde{w}^T \widetilde{w}} = \frac{w^T B w}{w^T w} \leq \beta_k$$

[0]This lecture was drawn from some notes of Embree `http://www.caam.rice.edu/~caam440/chapter2.pdf`, Spielman's 2012 lecture notes, Lecture 3: `http://www.cs.yale.edu/homes/spielman/561/2012/lect03-12.pdf`, Lau's 2015 lecture notes, Lecture 2 (`https://cs.uwaterloo.ca/~lapchi/cs798/notes/L02.pdf`), and Cvetković, Rowlinson, and Simić, *An Introduction to the Theory of Graph Spectra*, Section 7.4. Scribe notes were written by Kun Dong and Rahmtin Rotabi.

The proof of the other inequality is similar. We now define the vector spaces

$$V = span(x_1, \cdots, x_{k+n-m}), \qquad W = span(y_k \cdots, y_m), \qquad \widetilde{W} = \left\{ \begin{pmatrix} w \\ 0 \end{pmatrix} \in \mathbb{R}^n, w \in W \right\}$$

Since $\dim(V) = k + n - m$, $\dim(\widetilde{W}) = \dim(W) = m - k + 1$, there exists $\widetilde{w} \in V \bigcap W$ and $\widetilde{w} = \begin{pmatrix} w \\ 0 \end{pmatrix}$ for some $w \in W$. We have $\widetilde{w}^T A \widetilde{w} = w^T B w$ as before. It follows that

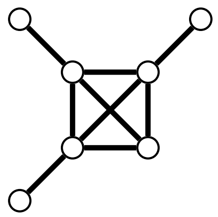
$$\lambda_{k+n-m} = \max_{x \in V} \frac{x^T A x}{x^T x} \geq \frac{\widetilde{w}^T A \widetilde{w}}{\widetilde{w}^T \widetilde{w}} = \frac{w^T B w}{w^T w} \geq \min_{x \in W} \frac{x^T B x}{x^T x} = \beta_k,$$

completing the proof. □

2 Clique and Chromatic Number

We now use the eigenvalue interlacing theorem to prove some statements about two particular graph quantities, the clique number and the chromatic number.

Definition 1 *The clique number of G, $\omega(G)$, is the size of the largest $S \subseteq V$ such that for all $i, j \in S$, $(i, j) \in E$.*
Example:



$\omega(G) = 4$

Definition 2 *The chromatic number $\chi(G)$ is the fewest number of colors needed such that we can assign one color to each vertoex and for all $(i, j) \in E$, i, j are assigned different colors.*

Observation 1 *$\chi(G) \geq \omega(G)$.*
The observation follows since every vertex in the maximum clique needs to be assigned a different color: if two vertices in the clique are assigned the same color, then since there is an edge between them, the two endpoints of that edge are not assigned different colors.

Consider the complete graph on n nodes $G \equiv K_n$; that is, there is an edge between every pair of vertices. Then $\omega(G) = n = \chi(G)$. The adjacency matrix of G is $A = J - I$ where J is the matrix of all ones. Let

$$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Then

$$Ae = (J - I)e = ne - e = (n-1)e$$

Therefore e is an eigenvector for eigenvalue $n-1$.

For any vector v such that $\langle e, v\rangle = 0$, $Av = (J-I)v = 0 - v = -v$. This means any v such that $\langle e, v\rangle = 0$ is an eigenvector of eigenvalue -1. Thus the spectrum of A is $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$.

Now consider an arbitrary graph G, and let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues to its adjacency matrix.

Claim 2 *$\lambda_1 \geq \omega(G) - 1$.*

Proof: For the largest clique S in G, let B be the principal submatrix with columns and rows corresponding to S. Let $m = |S| = \omega(G)$, then $B = J_m - I_m$. If β_1 is the largest eigenvalue of B, then $\beta_1 = \omega(G) - 1$. By the Interlacing Theorem, $\lambda_1 \geq \beta_1 = \omega(G) - 1$. $\square$

We can in fact prove something slightly stronger. The following theorem strengthens that bound of the claim since $\omega(G) \leq \chi(G)$.

Theorem 3 (Wilf 1967) *$\chi(G) \leq \lfloor \lambda_1 \rfloor + 1$*

Before we can prove this we need a lemma. Let $d(i)$ be the degree of node i, $\Delta = \max_i d(i)$ and $d_{ave} = \frac{\sum_{i\in V} d(i)}{n}$.

Lemma 4 *$d_{ave} \leq \lambda_1 \leq \Delta$ for G connected.*

Proof:

$$\lambda_1 = \max_x \frac{x^T A x}{x^T x} \geq \frac{e^T A e}{e^T e} = \frac{\sum_{i,j} a_{ij}}{n} = \frac{\sum_{i\in V} d(i)}{n} = d_{ave}$$

Let x_1 be the associated eigenvector. By Perron-Frobenius Theorem, we can assume $x_1 > 0$. Let $i^* = \arg\max_i x(i)$, then

$$(Ax_1)(i^*) = \lambda_1 x_1(i^*)$$

$$\implies \lambda_1 = \frac{(Ax_1)(i^*)}{x_1(i^*)} = \frac{\sum_{j:(j,i^*)\in E} x_1(j)}{x_1(i^*)} = \sum_{j:(j,i^*)\in E} \frac{x_1(j)}{x_1(i^*)} \leq \sum_{j:(j,i^*)\in E} 1 = d(i^*) \leq \Delta$$

$\square$

Notice that we needed connectivity in the proof above to invoke the Perron-Frobenius theorem for the inequality $\lambda_1 \leq \Delta$; we did not need it for the lower bound $d_{ave} \leq \lambda_1$.

We also observe the following.

Observation 2 *$\chi(G) \leq \Delta + 1$.*

This is true because if we color the graph greedily, we will never get stuck: if we color a vertex, it has at most Δ neighbors that have already been colored, and so we can color it with the $(\Delta+1)$st color.

Proof of Theorem 3:

The proof is by induction on n.

Base case $n = 2$,

 $\lambda_1 = 1, \quad \chi_1(G) = 2$

 $\lambda_1 = 0, \quad \chi_1(G) = 1$

Inductive step: Suppose the theorem holds on all graphs with at most $n-1$ vertices. By the Lemma, G has a vertex of degree less than $\lfloor \lambda_1 \rfloor$. Remove this vertex v and call the resulting graph G'. Let B be its adjacency matrix and β_1 be its largest eigenvalue. By the Interlacing Theorem, $\beta_1 \le \lambda_1$. By induction, we can color G' with $\lfloor \beta_1 \rfloor + 1$ colors, which is less than $\lfloor \lambda_1 \rfloor + 1$ colors. We can then finish coloring G by coloring v with one of the $\lfloor \lambda_1 \rfloor + 1$ colors since degree of v is less than $\lfloor \lambda_1 \rfloor$. □

3 Graph Laplacians

So far we've used the adjacency matrix of a graph to represent a graph as a matrix, and have looked at the spectra of such matrices. We will now introduce another way of representing a graph as a matrix.

Let's let $e_i \in \{0,1\}^n$ be the standard basis vectors (1 in the i-th coordinate, 0's else where).

A *Laplacian* of an undirected graph $G = (V, E)$,

$$L_G = \sum_{(i,j) \in E} (e_i - e_j)(e_i - e_j)^T.$$

Each term $(e_i - e_j)(e_i - e_j)^T$ is an $|V| \times |V|$ matrix that has $+1$ in the (i,i) and (j,j) coordinate, -1 in the (i,j) and (j,i) coordinate and the rest of the entries are all zero. Now, we define the following notation:

- $d(i)$: degree of i in G.
- D: $diag(d(i))$ is the $|V| \times |V|$ diagonal matrix where $D(i,i) = d(i)$.
- A: Adjacency matrix of graph A.

With this notation we can write $L_G = D - A$.

If G has weights $w(i,j), \forall (i,j) \in E$, then the *weighted Laplacian*,

$$L_G = \sum_{(i,j) \in E} w(i,j)(e_i - e_j)(e_i - e_j)^T.$$

Define $W = (w(i,j)) \in \Re^{n \times n}$ where $w(i,j) = 0$ if $(i,j) \notin E$ and $D = diag(d(i))$, where $d(i) = \sum_{(i,j) \in} w(i,j)$. Then $L_G = D - W$. We will sometimes denote this matrix by $L_{G,w}$.

An interesting and useful fact is that the Laplacian L_G is positive semidefinite. Let's briefly remember what this means, as well as some useful facts about such matrices.

Definition 3 *A matrix $A \in \Re^{n \times n}$ is positive semidefinite, if $x^T A x \ge 0$ for all $x \in \Re^n$. If A is positive semidefinite we write $A \succeq 0$.*

Given what we know about matrices, the following fact is easy to prove, but we will skip its proof.

Fact 1 *For a symmetric matrix A the following are equivalent:*

(i) $A \succeq 0$.

(ii) $A = VV^T$ *for some matrix* V.

(iii) A has all non-negative eigenvalues.

We can now show that L_G is positive semidefinite, which we will do in two different ways.

Claim 5 $L_G \succeq 0$.

Proof:
First proof:
Note L_G is symmetric.
We observe that if $A \succeq 0$ and $B \succeq 0$ then $A + B \succeq 0$, since

$$x^T(A+B)x = x^T Ax + x^T Bx \geq 0$$

for all $x \in \Re^n$. Note that by (ii), $(e_i - e_j)(e_i - e_j)^T \succeq 0$. So, by summing up all these terms we will get L_G and based on the observation above we can say $L_G \succeq 0$. $\square$
Second proof:
Also we know that for any $x \in \Re^n$,

$$\begin{aligned} x^T L_G x &= x^T \left(\sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T \right) x \\ &= \sum_{(i,j)\in E} x^T (e_i - e_j)(e_i - e_j)^T x \\ &= \sum_{(i,j)\in E} (x(i) - x(j))(x(i) - x(j)) \\ &= \sum_{(i,j)\in E} (x(i) - x(j))^2 \qquad \geq 0. \end{aligned}$$

$\square$

We will usually write the eigenvalues of L_G, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and since we know that L_G is positive semi-definite we can write $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

What is the spectrum of L_G? We observe that e (all 1s vector) is an eigenvector of eigenvalue 0 for L_G, since:

$$L_G e = \sum_{(i,j)\in E} (e_i - e_j)(e_i - e_j)^T e = \sum_{(i,j)\in E} (e_i - e_j) \cdot 0 = 0 \cdot e.$$

Thus $\lambda_1 = 0$.

4 Graph Laplacians and Connectivity

Now we switch our focus to λ_2, which is much more interesting. We will see a very close connection between λ_2 and various notions of the connectivity of the graph.

Theorem 6 $\lambda_2 = 0$ *iff G is disconnected.*

Proof: If G is disconnected then, we can partition it into G_1 and G_2 such that there are no edges between G_1 and G_2. Furthermore, we can re-index the nodes so that

$$L_G = \begin{bmatrix} L_{G_1} & 0 \\ 0 & L_{G_2} \end{bmatrix}.$$

Then both vectors

$$\begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 1 \\ 1 \end{bmatrix}.$$

(where first $|V_{G_1}|$ entries of the first vector is 1 and the rest are zero and the opposite for the second vector) will be eigenvectors of L_G and orthogonal to each other. Since the eigenvalues associated with both vectors are 0, this implies that $\lambda_2 = 0$.

To see the other direction, let x_2 be an eigenvector of eigenvalue λ_2. We can assume $\langle x_2, e\rangle = 0$. If $\lambda_2 = 0$, then $x_2^T G x_2 = x_2^T(\lambda_2 x_2) = 0$. So then,

$$x_2^T L_G x_2 = \sum_{(i,j)\in E} (x_2(i) - x_2(j))^2 = 0.$$

The summation of squared real values is 0, therefore each of them is equal to zero. Therefore, $x_2(i) = x_2(j)$ for all $(i,j) \in E$. Consider $V_1 = \{i \in V : x_2(i) \geq 0\}$ and $V_2 = \{i \in V : x_2(i) < 0\}$. It's clear there are no edges between V_1 and V_2. Since $\langle x_2, e\rangle = 0$, there should be both positive and negative entries in x_2 which proves that $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, and hence G has at least two components. □

For this reason λ_2 is called the *algebraic connectivity* of G. The proof above easily extends to prove the followng.

Claim 7 $\lambda_k = 0$ *iff* G *has at least* k *components.*

We now show another connection between λ_2 and the connectivity of the graph G.

Definition 4 $\kappa(G)$ *is the* vertex connectivity *of* G*; it is the smallest nonnegative integer such that we can remove up to* $\kappa(G) - 1$ *vertices and associated edges from* G *and* G *is still connected.*

We will show the following shortly. Let $G - S$ be the graph that results from removing the vertices in S from the graph, as well as all edges incident on the vertices in S.

Lemma 8 $\lambda_2(L_G) \leq \lambda_2(L_{G-S}) + |S|$, *for all* $S \subseteq V$.

Note that we easily get the following corollary.

Corollary 9 $\lambda_2(G) \leq \kappa(G)$.

Proof: Let S be a set of vertices of size $\kappa(G)$ that disconnects G. Then

$$\lambda_2(G-S)=0 \Rightarrow \lambda_2(G) \leq 0+\kappa(G).$$

$\square$

Proof of Lemma 8: Let x_2 be the eigenvector of L_{G-S} corresponding to $\lambda_2(L_{G-S})$, with $x_2^T x = 1$, $\langle x_2, e\rangle = 0$.

Then we know

$$x_2^T L_G x_2 = \sum_{(i,j)\in E} (x_2(i)-x_2(j))^2 = \lambda_2(L_{G-S})$$

for $G-S=(V-S,E')$. Note that $x_2 \in \Re^{|V|-|S|}$. We want a vector $x \in \Re^{|V|}$, so we let

$$x(i) = \begin{cases} x_2(i), & \text{if } i \in S \\ 0, & \text{otherwise} \end{cases}.$$

With this definition x is a unit vector since, $x^T x = x_2^T x_2 = 1$ and $\langle x, e\rangle = \langle x_2, e\rangle = 0$. Then we have that

$$\begin{aligned}
\lambda_2(L_G) = \min_{z\in\Re^n:\langle z,e\rangle=0} \frac{z^T L_G z}{z^T z} &\leq \frac{x^T L_G x}{x^T x} \\
&= x^T L_G x \\
&= \sum_{(i,j)\in E} (x(i)-x(j))^2 \\
&= \sum_{(i,j)\in E'} (x(i)-x(j))^2 + \sum_{i\in S}\sum_{j:(i,j)\in E} (x(i)-x(j))^2 \\
&= \sum_{(i,j)\in E'} (x_2(i)-x_2(j))^2 + \sum_{i\in S}\sum_{j:(i,j)\in E} (x_2(j))^2 \\
&\leq \sum_{(i,j)\in E'} (x_2(i)-x_2(j))^2 + \sum_{i\in S} 1 \text{ } (x_2 \text{ has unit norm}) \\
&= \lambda_2(L_{G-S}) + |S|.
\end{aligned}$$

$\square$