Spectral Graph Theory and Algorithms, Summer 2017 27 June 2017

Problem Set 6

Due Date: 28 June 2017

1. Today's problem set is a continuation of yesterday's; we continue to look at a different way of bounding the largest eigenvalue and obtaining an approximation algorithm for the maximum cut problem. Let λ_n be the maximum eigenvalue of the normalized Laplacian $\mathcal{L}$, and let y be the corresponding eigenvector, so that

$$\lambda_n = \frac{y^T L_G y}{y^T D y} = \max_x \frac{x^T L_G x}{x^T D x} = \max_z \frac{z^T \mathcal{L} z}{z^T z}.$$

Assume that $\max_i |y(i)| \leq 1$. Let OPT denote the number of edges in a maximum cut, and let $S^* \subset V$ denote the set of vertices associated with that set, so that $|\delta(S^*)| = OPT$.

As with yesterday, suppose we construct a solution $x \in \{-1, 0, +1\}^n$ as in Trevisan's algorithm (that is, pick $t \in (0, 1]$ uniform, and let $x(i) = -1$ if $y(i) \leq -\sqrt{t}$, $x(i) = 1$ if $y(i) \geq \sqrt{t}$, and $x(i) = 0$ otherwise). Let sets $L = \{i \in V : x(i) = -1\}$, $R = \{i \in V : x(i) = 1\}$, $S = L \cup R$, and $V - S = \{i \in V : x(i) = 0\}$. You can assume all the inequalities are true that were stated on yesterday's problem set.

(a) Consider $\rho(G) = \max_{S \subset V} \rho(S)$, where

$$\rho(S) = \max_{\text{partition } S \text{ into } L,R} \frac{|\delta(L,R)| + \frac{1}{2}|\delta(S)|}{|E(S)| + |\delta(S)|}.$$

Set $A = 2(1-\epsilon)\beta(1-\beta)$, and restrict $\frac{1}{2} \leq A + \beta < 1$. Prove that we can use the algorithm to find an S, L, and R such that

$$\rho(G) \geq \frac{|\delta(L,R)| + \frac{1}{2}|\delta(S)|}{|E(S)| + |\delta(S)|} \geq \frac{1-2\beta}{2(1-A-\beta)}.$$

It may be helpful to observe that $|\delta(L,R)| \leq |E(S)|$ and $1 - 2A - 2\beta \leq 0$ so that

$$(1-2A-2\beta)|\delta(L,R)| \geq (1-2A-2\beta)|E(S)|.$$

(b) Use the above to find an α-approximation algorithm for the maximum cut problem for as large an α as you can, preferably $\alpha > .5$. Getting $\alpha \geq .529$ should be doable (hint: try $\beta = 1/4$ to start). If you can get $\alpha > .614$, you have a publishable paper.