Spectral Graph Theory and Algorithms, Summer 2017	26 June 2017

Problem Set 5

Due Date: 27 June 2017

1. In this exercise, we'll look at a different way of bounding the largest eigenvalue and obtaining an approximation algorithm for the maximum cut problem; we'll do part of the problem today, and the other part in tomorrow's problem set. Let λ_n be the maximum eigenvalue of the normalized Laplacian $\mathcal{L}$, and let y be the corresponding eigenvector, so that

$$\lambda_n = \frac{y^T L_G y}{y^T D y} = \max_x \frac{x^T L_G x}{x^T D x} = \max_z \frac{z^T \mathcal{L} z}{z^T z}.$$

Assume that $\max_i |y(i)| \leq 1$. Let OPT denote the number of edges in a maximum cut, and let $S^* \subset V$ denote the set of vertices associated with that set, so that $|\delta(S^*)| = OPT$.

(a) Prove that if $OPT \geq (1-\epsilon)|E|$, then $\lambda_n \geq 2(1-\epsilon)$.

(b) Suppose we construct a solution $x \in \{-1, 0, +1\}^n$ as in Trevisan's algorithm (that is, pick $t \in (0,1]$ uniform, and let $x(i) = -1$ if $y(i) \leq -\sqrt{t}$, $x(i) = 1$ if $y(i) \geq \sqrt{t}$, and $x(i) = 0$ otherwise). Let sets $L = \{i \in V : x(i) = -1\}$, $R = \{i \in V : x(i) = 1\}$, $S = L \cup R$, and $V - S = \{i \in V : x(i) = 0\}$.
It can be shown that for all $0 \leq \beta \leq 1$,

$$E\left[|\delta(L,R)| + \beta|\delta(S)|\right] \geq \beta(1-\beta) \sum_{(i,j)\in E} (y(i) - y(j))^2.$$

(If you want some extra work, you can prove this yourself; it might help to know Bergström's inequality, which states that for $a, b \geq 0$ and $0 \leq \beta \leq 1$, $\beta(1-\beta)(a+b)^2 \leq (1-\beta)a^2 + \beta b^2$).
Prove that if $\lambda_n \geq 2(1-\epsilon)$, then

$$E\left[|\delta(L,R)| + \beta|\delta(S)|\right] \geq 2(1-\epsilon)\beta(1-\beta)E\left[2|E(S)| + |\delta(S)|\right].$$