Spectral Graph Theory and Algorithms, Summer 2017 23 June 2017

Problem Set 4

Due Date: 26 June 2017

1. This problem concerns generalized Laplacians and the Colin de Verdiére invariant as discussed in lecture; we will show that if G is outerplanar and 2-vertex-connected, then $\mu(G) \leq 2$. Recall that an outerplanar graph is one such that there is a planar embedding in which all the vertices of the graph are on the external face. It is known that a graph is outerplanar iff it doesn't contain K_4 or $K_{2,3}$ as a minor. In what follows assume G is outerplanar and 2-vertex connected, and that we have a plane embedding of G in which all the vertices are on the external face F. Since G is 2-vertex-connected and outerplanar, we can assume that F is a cycle.

   (a) Let M be a generalized Laplacian of G with one negative eigenvalue. Assume that $\mu(G) > 2$. Argue that there is $x \in \ker(M)$ such that there are two adjacent vertices u and v with $x(u) = x(v) = 0$.

   (b) Assume that x has minimal support, and that $x \in \ker(M)$ has two adjacent vertices u and v with $x(u) = x(v) = 0$. We will prove that F cannot contain vertices from both $\text{supp}^+(x)$ and $\text{supp}^-(x)$. To do this, suppose otherwise, and pick some $p \in \text{supp}^+(x)$ and $q \in \text{supp}^-(x)$, both in F. ithout loss of generality, assume that the vertices are ordered u, v, q, p going in clockwise order around F. Let v' be the first vertex not in $\text{supp}(x)$ going counterclockwise around F from q, and let u' be the first vertex not in $\text{supp}(x)$ going clockwise around F from p. The vertices u', v', q, and p must all be distinct and appear in that order around F. Prove that there must be a path P from u' to p in which every vertex in the path except u' is in $\text{supp}^+(x)$; similarly, there must be a path N from v' to q in which every vertex in the path except v' is in $\text{supp}^-(x)$. Use the existence of these two paths to argue that the graph cannot be planar, and derive a contradiction.

   (c) Conclude the proof and show that if G is outerplanar, then it must be the case that $\mu(G) \leq 2$.