Spectral Graph Theory and Algorithms, Summer 2017 June 20, 2017

Problem Set 1

Due Date: June 21, 2017

1. In class, we proved that for $A \in \Re^{n\times n}$, A symmetric, (real) eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$, and orthonormal eigenvectors $x_1, \ldots, x_n$, that

$$\lambda_{k+1} = \min_{x \perp span(x_1,\ldots,x_k)} \frac{x^T Ax}{x^T x}.$$

We asserted that it was also the case that

$$\begin{aligned}\lambda_{k+1} &= \min_{x \in span(x_{k+1},\ldots,x_n)} \frac{x^T Ax}{x^T x} \\ &= \max_{x \perp span(x_{k+2},\ldots,x_n)} \frac{x^T Ax}{x^T x} \\ &= \max_{x \in span(x_1,\ldots,x_{k+1})} \frac{x^T Ax}{x^T x}.\end{aligned}$$

Prove that the assertion is true.

2. (Optional) One problem with the characterization of eigenvalues of the previous problem is that it requires us to know the eigenvectors $x_1, \ldots, x_k$ (or $x_{k+2}, \ldots, x_n$) in order to compute λ_{k+1}. The *Courant-Fischer theorem* gives us a more general way of computing these eigenvalues. Prove that the following is true:

$$\begin{aligned}\lambda_{k+1} &= \min_{W \subseteq \Re^n : dim(W)=k+1} \ \max_{x \in W} \frac{x^T Ax}{x^T x} \\ &= \max_{W \subseteq \Re^n : dim(W)=n-k} \ \min_{x \in W} \frac{x^T Ax}{x^T x}.\end{aligned}$$