# SAT – Propositional Satisfiability

SAT:
*Input:* a propositional formula $\varphi$ in CNF
*Question:* is $\varphi$ satisfiable?

Example (satisfiable):

$$(x_1 \vee x_2) \wedge (x_2) \wedge (\neg x_1 \vee x_3) \wedge (\neg x_1 \vee x_4) \wedge (x_4)$$

Example (unsatisfiable):

$$(x_1 \vee x_2) \wedge (\neg x_2) \wedge (\neg x_1) \wedge (\neg x_1 \vee x_4) \wedge (\neg x_4)$$

# SAT solvers & encodings

Best algorithms take $2^n$ time in the worst-case

but are very efficient in many cases (instances with millions of variables solved in seconds)

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**Strategy to solve your favorite NP-complete problem $Q$:**

1. encode instance of $Q$ into instance $\varphi$ of SAT (in poly-time)
2. call a SAT solver on $\varphi$
3. decode solution into a solution for $Q$

Example problem: does a graph $G$ have a clique of size $\geq m$?

# SAT encoding (in a picture)

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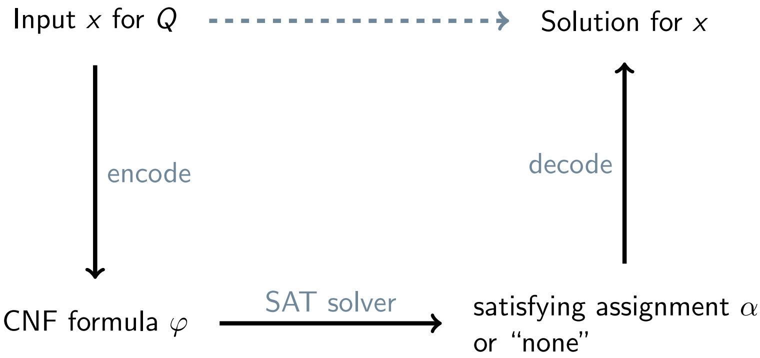

# Using SAT encodings

Given any graph $G$ and a natural number $k$, we build a formula $F(G, k)$ such that:

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**Idea**

The formula $F(G, k)$ is true if and only if $G$ has a parameter with value $k$.

Here $k$ can be any of the following graph parameters

- Treewidth (Samer & Veith 2009),
- Cliquewidth (Heule & Szeider 2013),
- Branchwidth (current work : Apply this approach for larger graphs using SAT for local improvement), ...

# Counting

- How to count something using *SAT* formula?
- Various techniques (Bjork 2009)
  - Unary encoding
  - Binary encoding
  - onehot encoding

# Cardinality Constraints using Unary counting

- Let $L(v)$ be *True* when vertex $v$ is being counted.
- Let $C(i)$ be true when counter has value $i$
- We can count 2 using following formula

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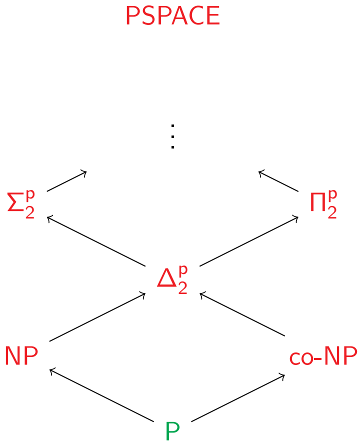
# Polynomial Hierarchy: Problems ‘Beyond NP’



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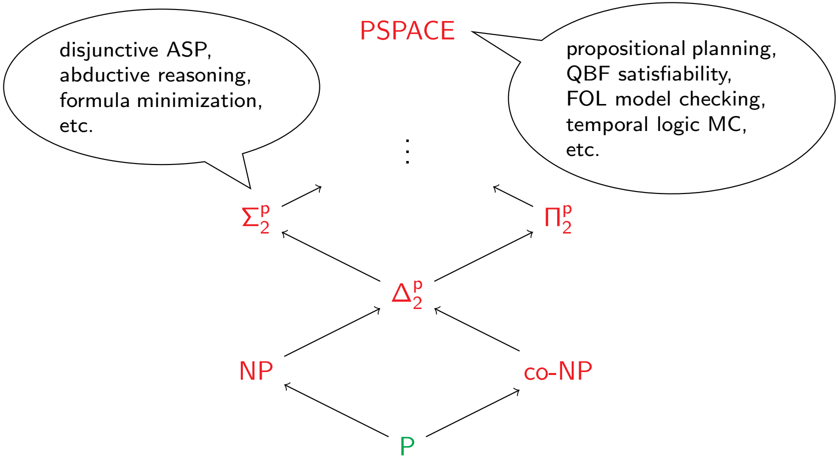

# Other kinds of reductions

Example problem: how big is the largest clique in a graph $G$?

Turing reductions: call a SAT solver multiple times

Example problem: find a subset of clauses that is *minimally unsatisfiable* (removing any clause makes it satisfiable) – MUS

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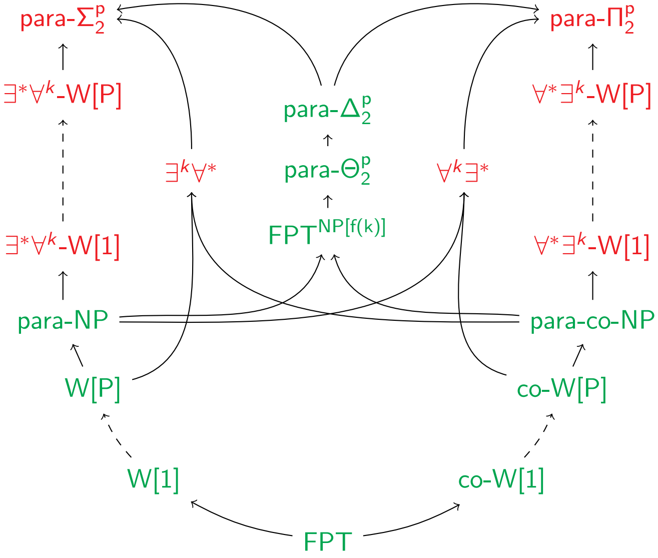
Fpt-reductions to SAT: allow fpt-time instead of poly-time

- Interesting for problems 'beyond NP'
- Example problem: find smallest logically equivalent CNF

# A new landscape



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