Widths, Games and Logic

Robert Ganian

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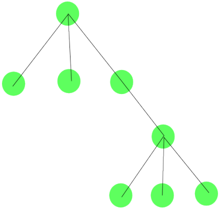
ALGORITHMS AND COMPLEXITY GROUP

Trees

Trees



Nice and simple – most problems are easy on trees.

Treewidth

Treewidth measures how "tree-like" a graph is.
Why?

Treewidth

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Why?

- "tree-like" structure can be exploited for algorithms
- Many problems FPT parameterized by treewidth
- Applications in all kinds of fields, many real-world systems have low treewidth
- $\sim$9k hits on google scholar, $\sim$100k hits on google

Undisputed king of structural parameters

Does my graph have bounded treewidth?

Several equivalent definitions of treewidth

This talk: Definition via Cops and Robber game
(Easy to use) (Also: most fun)

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- k cops vs. 1 robber
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- Each cop can land on one vertex. Vertex is blocked and cannot be used by robber until cop lifts off
  - Before cop actually lands, the robber can still move.

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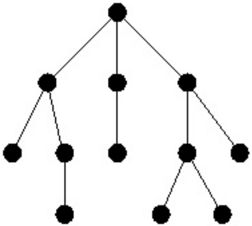
How many cops do we need to catch the robber? Subtract 1 and you get treewidth.

- There are many algorithms for computing treewidth: FPT, approximation, heuristics...

Cops and Robber examples



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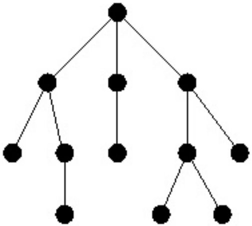

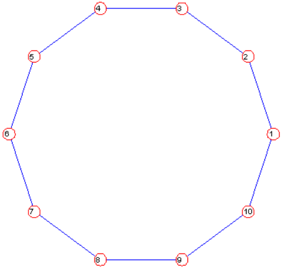

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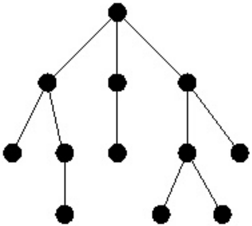

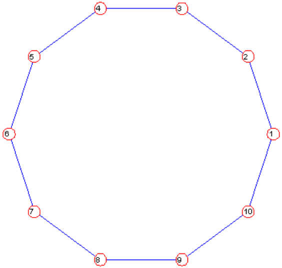

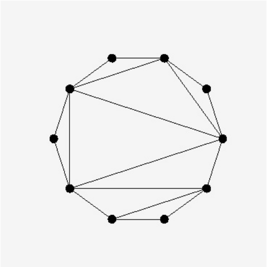

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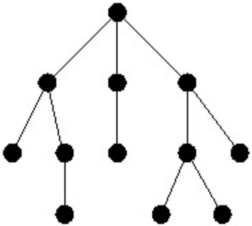

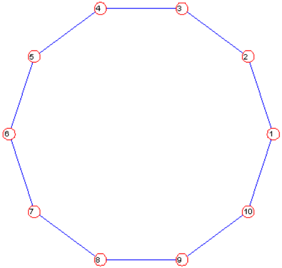

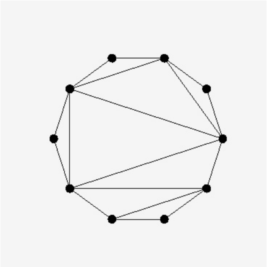

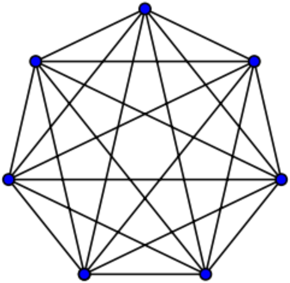

Using treewidth

Any problem which can be expressed by a small MSO formula is FPT par. by treewidth

Example of **Algorithmic Meta-Theorem**... we also develop them

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MSO formulas

- Quantifiers over individual vertices/edges, vertex/edge sets
- Can check $\neq$, $\in$, vertex-edge incidence "inc" (and vertex-vertex and edge-edge adjacency "adj")
- standard logical connectives

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There exists an independent set of vertices

$$\exists C \forall a, b : (a \in C \land b \in C) \implies \neg \mathrm{adj}(a, b)$$

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Can also optimize over size of sets

- Find smallest/largest set A such that ...
- Vertex Cover, Independent Set, Dominating Set...

Clique-width

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Solution: Clique-width

- Clique-width measures how close a graph is to... not just cliques
- More general than treewidth
  - Whenever treewidth is bounded, then so is clique-width
  - Can also be small on graphs with high treewidth (cliques)
- Allows solution of many problems (but less than treewidth)

Does my graph have small clique-width?

Building game (like Lego)

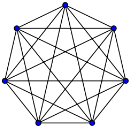
You can use labels $1 \ldots k$ on vertices, and have these operations:

- Create a new graph with one vertex with label i
- Make disjoint union of graphs
- add all edges between labels i and j
- change all labels i to j

Clique-width = minimum number of labels you need to build the graph

Examples

- Complete graph



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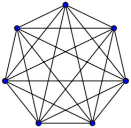

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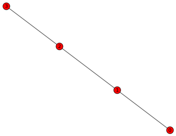

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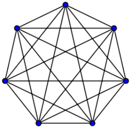

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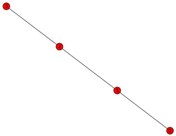

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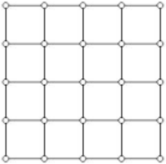

- Grid
- Tree? (Think when bored)

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- Cannot speak about edges

Can we express 3-Colorability?

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rank-width $\sim$ clique-width 2.0

- Computing rank-width $\times$ Computing clique-width

Relation to our research:

- Use treewidth, clique-width, rank-width in our results
- Study other measures of structure (parameter ecology)
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Thank you for your attention.