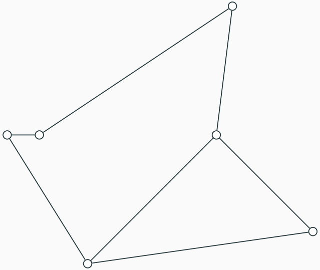
Extending Orthogonal Planar Graph Drawings is Fixed-Parameter Tractable

S. Bhore, R. Ganian, **L. Khazaliya**, F. Montecchiani, M. Nöllenburg

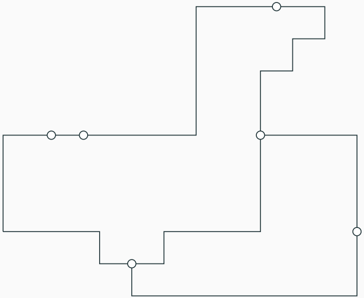
June 20, 2023

What we are interested in

Planar graph H

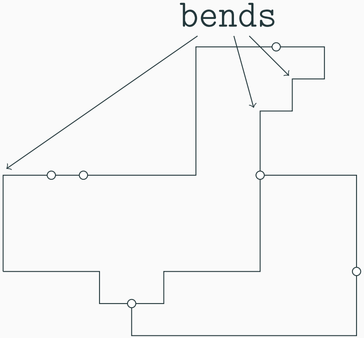


Orthogonal drawing $\Gamma(H)$ of H

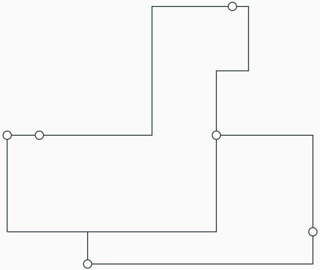


What we care about

Orthogonal drawing:



with 15 bends



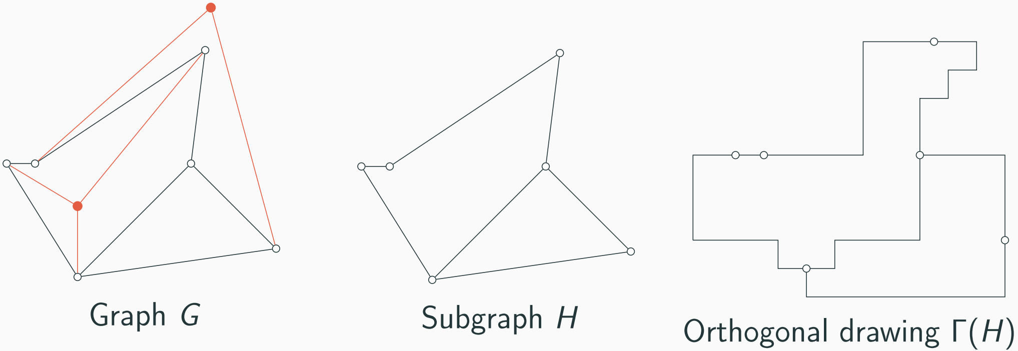
with 9 bends

Bend-minimal Orthogonal Extension Problem (BMOE)

INPUT: Graph G, an already fixed orthogonal drawing $\Gamma(H)$ for $H \subseteq G$, $\beta \in \mathbb{Z}$:

$$\langle G, H \subseteq G, \Gamma(H) \rangle, \beta \in \mathbb{Z}$$

TASK: Extend $\Gamma(H)$ to $\Gamma(G)$ using at most $\beta \geq 0$ additional bends



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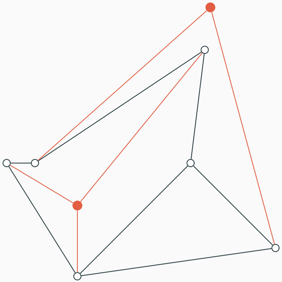
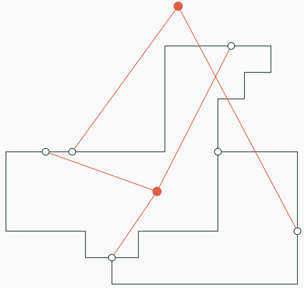


Graph G



Extension to G

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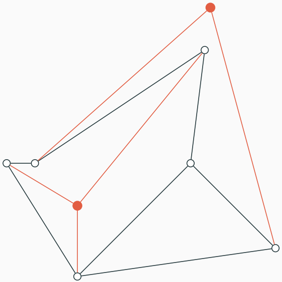

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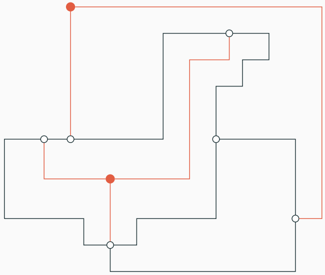


Bend-minimal orthogonal extension $\Gamma(G)$

Complexity results for an extension problems

(1) planar, linear-time algorithm

[Angelini et al., 2015]

(2) level planar, NP-hard

[Brückner and Rutter, 2017]

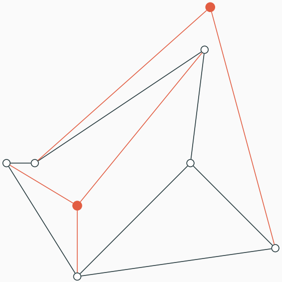
(3) upward planar, NP-hard

[Da Lozzo et al., 2020]

(4) bend-minimal orthogonal, NP-hard

[Angelini et al., 2021]

Bend-minimal Orthogonal Extension Problem (BMOE)



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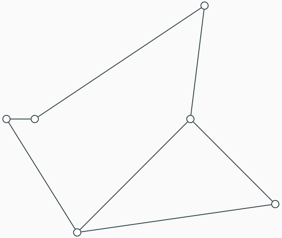
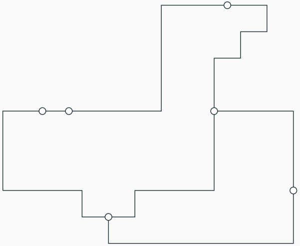


Subgraph H

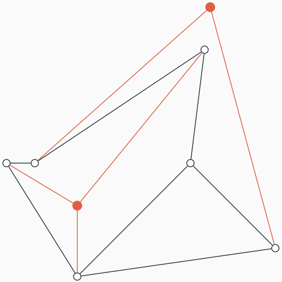


Orthogonal drawing $\Gamma(H)$

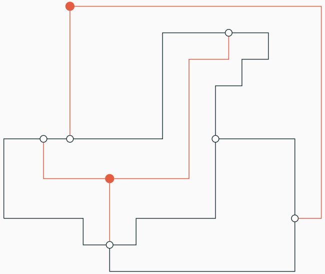
Let $\kappa = |V(G) \setminus V(H)| + |E(G) \setminus E(H)|$, i.e. the number of missing elements.

Contribution. If H is connected, the BMOE problem param. by κ is FPT.

Bend-minimal Orthogonal Extension Problem (BMOE)



Graph G



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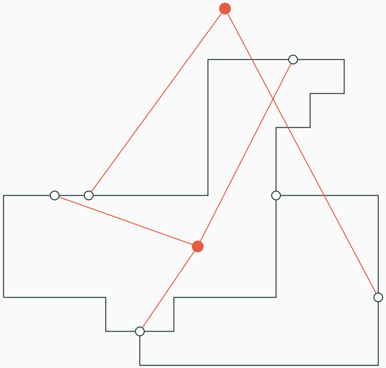
Overview [Key steps]

Bend-minimal Orthogonal Extension Problem (BMOE)



- Restrict BMOE to a single inner face (F-BMOE)
- Define sector graph based on bend-distances
- Tree-width of sector graph is bounded in $f(\kappa)$
- Universal point-set for each sector
- Dynamic Programming

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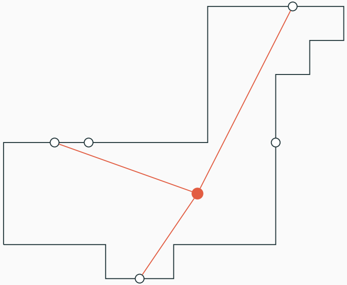

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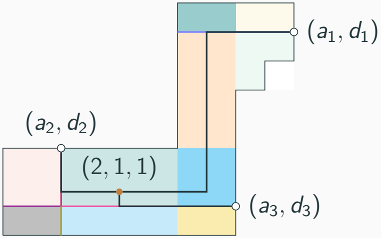

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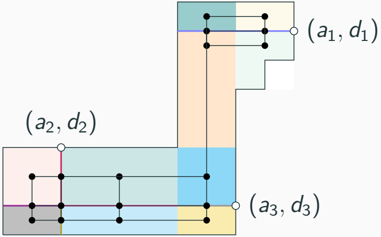

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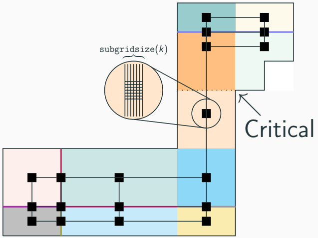

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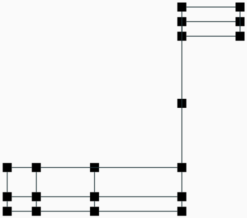

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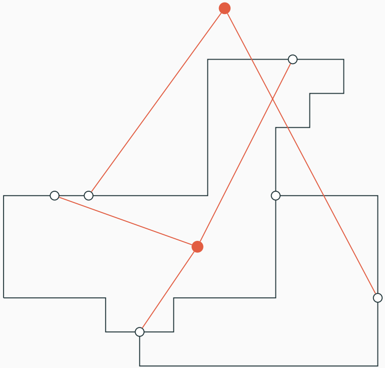
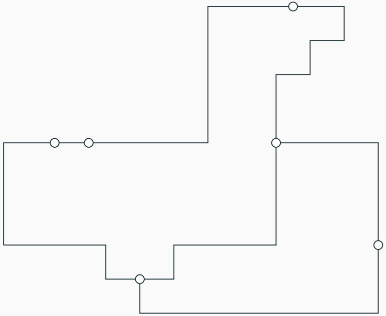
Basic definitions

Definitions



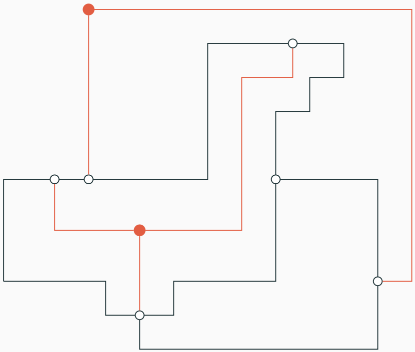
The complement $X = V(G) \setminus V(H)$ is the missing vertex set of G, and $E_X = E(G) \setminus E(H)$ the missing edge set.

Definitions



A planar orthogonal drawing $\Gamma(G)$ extends $\Gamma(H)$ if its restriction to the vertices and edges of H coincides with $\Gamma(H)$.

Definitions



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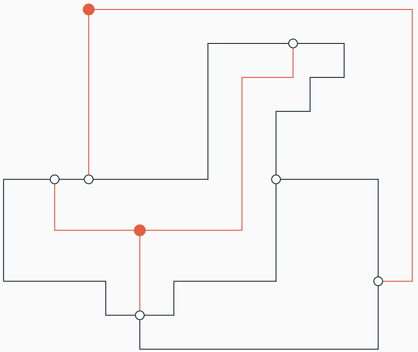

A feature point of an orthogonal drawing is a point representing either a vertex or a bend.

A vertex $a \in V(H)$ is called an anchor if it is incident to an edge in E_X.

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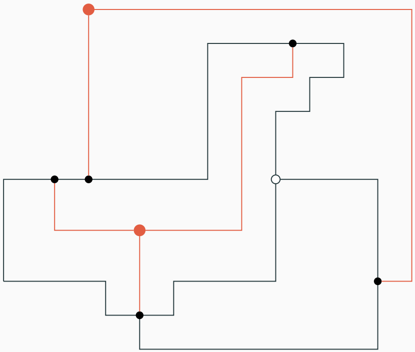

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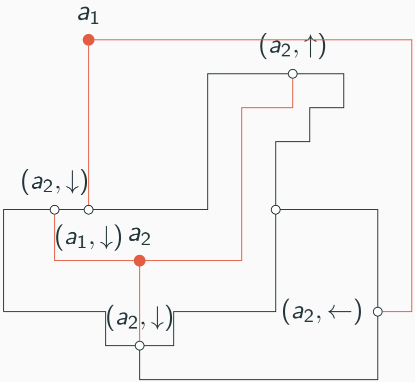

A port candidate is a pair (a, d), i.e. for $ax \in E_X$ $a \in V(H)$, $d \in \{\downarrow, \uparrow, \leftarrow, \rightarrow\}$.

A port-function $\mathcal{P}$ is an ordered set of port candidates which contains precisely one port candidate for each missing edge $ax \in E_X, a \in V(H)$.

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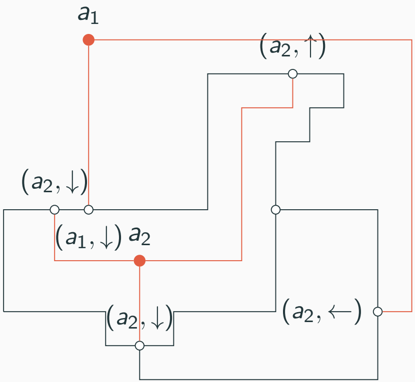

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Branching and F-BMOE

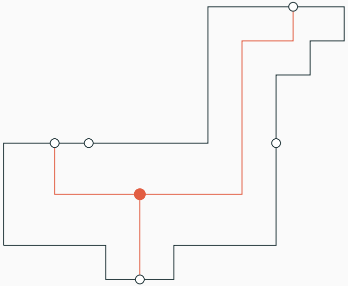
Turing reduction

Lemma. There is an algorithm that solves an instance of BMOE in time $2^{\mathcal{O}(\kappa)} \cdot T(|\mathcal{I}|, k)$, where $T(|\mathcal{I}|, k)$ is the time required to solve an instance $\mathcal{I}$ of $\mathrm{F\text{-}BMOE}$ with instance size $|\mathcal{I}|$ and parameter value k.

BMOE on just one Face (F-BMOE)

INPUT: Graph G_f (just one face), fixed orthogonal drawing $\Gamma(H_f)$ for $H_f \subseteq G_f$, set of missing vertices X_f, a port function for X_f:

$$\langle G_f, H_f \subseteq G_f, \Gamma(H_f)\rangle,$$
$$X_f = V(G_f) \setminus V(H_f);\ \text{port-function } \mathcal{P}.$$



TASK: Compute the minimum number of bends needed to extend $\Gamma(H_f)$ to $\Gamma(G_f)$ and

(1) missing edges and vertices are only drawn in f;

(2) each edge $ax \in E_X$ connects to $a \in V(H_f)$ via its port candidate in $\mathcal{P}$; or

(3) determine that no such extension exists.

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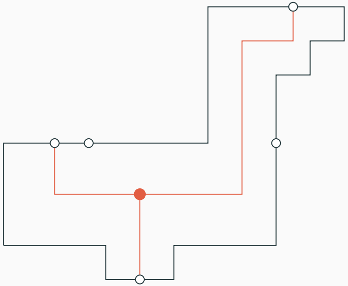

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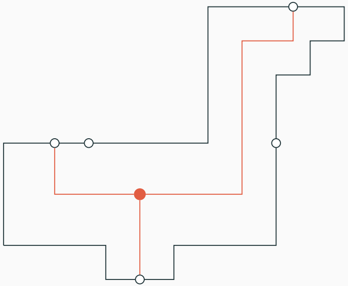

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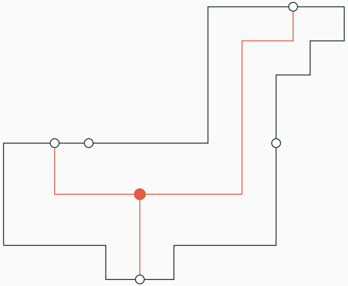

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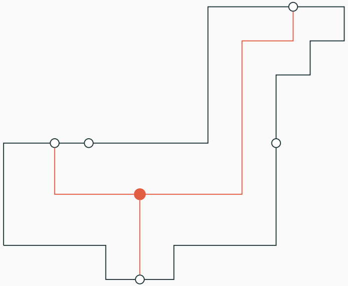

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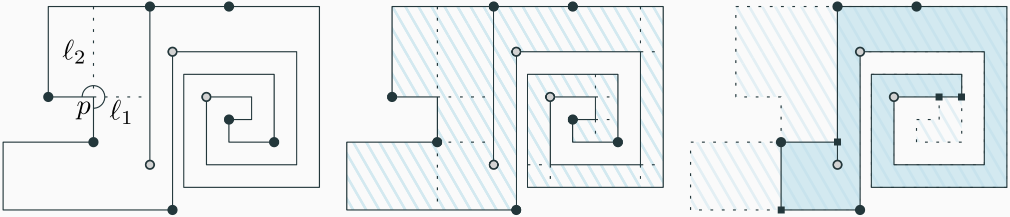
Preprocessing

Pruning

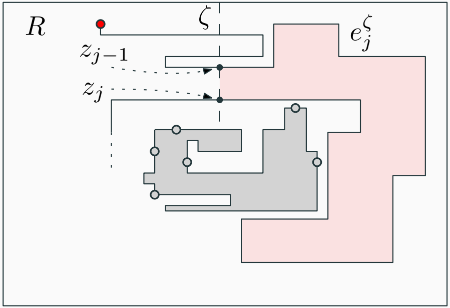


Left: A reflex corner p and its projections ℓ_1 and ℓ_2.

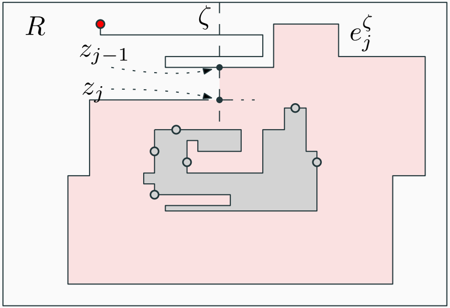
Middle: A face (striped) with all its non-essential reflex corners and projections.

Right: The corresponding clean instance.

Outer face



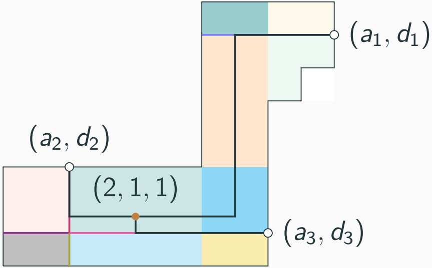
A ζ-handle



A ζ-spiral

Discretizing the Instances

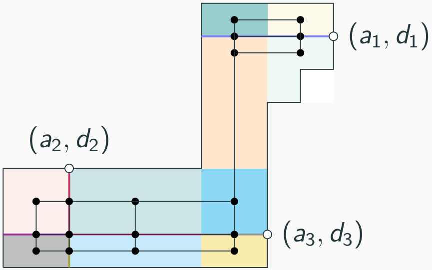
Sectors and the Sector Graph



For a point $p \in f$, the bend distance $\text{bd}(p, (a, d))$ to a port candidate (a, d) is the min $q \in \mathbb{Z}$ such that there exists an orthogonal polyline with q bends connecting p and a in the interior of f which arrives to a from direction d.

For point $p \in f$ and for a port-function $\mathcal{P} = ((a_1, d_1), \ldots, (a_q, d_q))$, we define a bend-vector of p as the tuple $\text{vect}(p) = (\text{bd}(p, (a_1, d_1)), \ldots, \text{bd}(p, (a_q, d_q)))$.

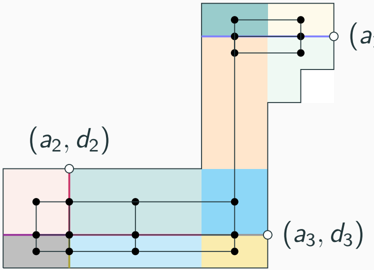
The Sector Graph



Sectors A and B are adjacent if there exists a point p in A and a direction $d \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}$ such that the first point outside of A hit by the ray starting from p in direction d is in B.

The number of vertices in $\mathcal{G}$ is upper-bounded by $9x^2$, where x is the number of feature points in $\Gamma(H_F)$.

The Sector Graph



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Exploiting the Treewidth

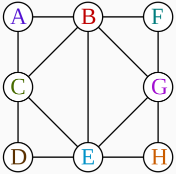
Tree decomposition

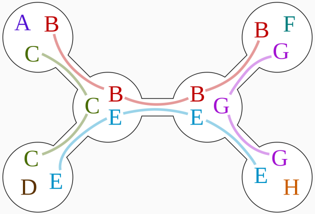
Def. A tree decomposition of a graph G is a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a bag, with three following conditions:

$\mathcal{T}1$. $\bigcup_{t \in V(T)} X_t = V(G)$;

$\mathcal{T}2$. For every $vw \in E(G)$, there exists a node t of T such that bag X_t contains both v and w;

$\mathcal{T}3$. For every $v \in V(G)$, the set $T_v = \{t \in V(T) | v \in X_t\}$ induces a connected subtree of T.





Def. The width of $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is $\max_{t \in V(T)} |X_t| - 1$.

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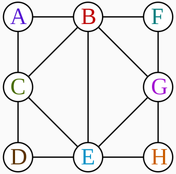

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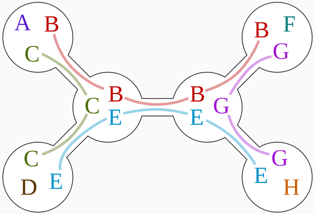
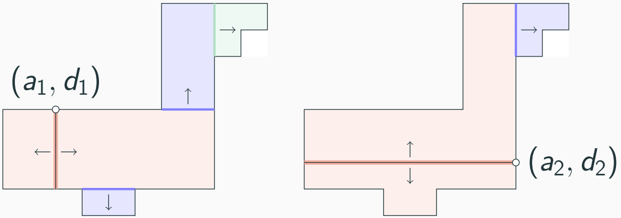


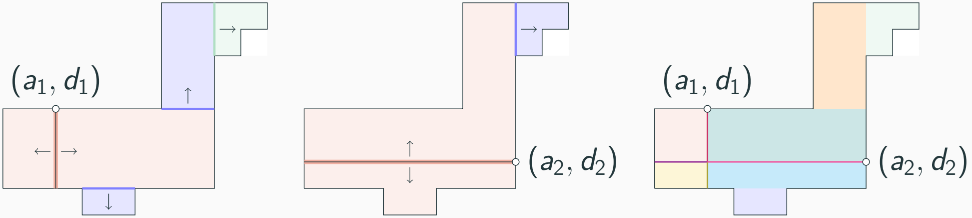
<u>Def.</u> The width of $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is $\max_{t \in V(T)} |X_t| - 1$.

Sector Graphs Are Tree-Like

Lemma. The sector graph $\mathcal{G}_1$ is a tree.

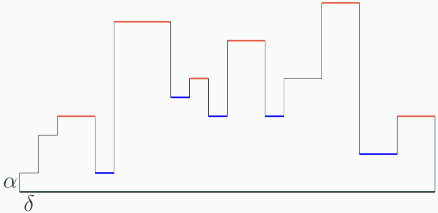


Sector Graphs Are Tree-Like



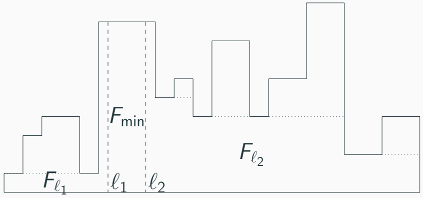
Baseline and Histogram

Each sector admits at least one baseline.



The segments colored red (blue) are local maxima (minima).

Baseline and Histogram



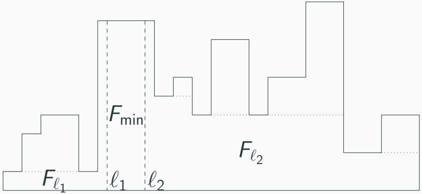
Cases of relative location of the $F_{\min}$ sector in F relative to the F-baseline

Each sector is subdivided into a linear in the number of local maxima number of new sectors.

Let $\mathcal{G}$ be a sector graph of a face f of the drawing $\Gamma(G)$:

$$\mathrm{tw}(\mathcal{G}) \leq (4+4k)^{4k}.$$

Baseline and Histogram



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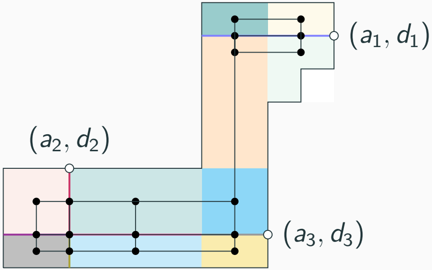
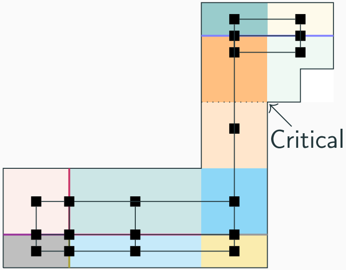
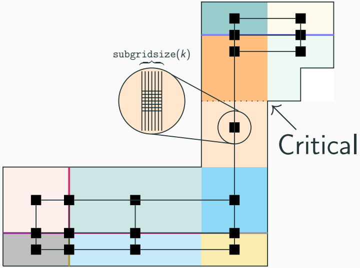
From Sector Graph to The Skeleton

For each sector and for each direction, there are at most $4k$ critical reflex corners.





From Sector Graph to The Skeleton



- Replace each vertex of the skeleton with a tiny square grid of size $\mathcal{O}(k^3)$.
- Grid points in each subsector is enough to host all vertices and bends of one column/row of the skeleton

The Final Step: DP

The Final Step

An instance $\mathcal{I} = \langle G_f, H_f, \Gamma(H_f), \mathcal{P} \rangle$ with $k = |V(G_f) \setminus V(H_f)|$ of $\mathrm{F\text{-}BMOE}$

- admits a sector graph $\mathcal{G}$ of treewidth at most $(4 + 4k)^{4k}$;
- a bend-minimal extension of $\Gamma(H_f)$ to an orthogonal planar drawing of G_f can be assumed to only contain feature points on the sector-grid points [at most $\texttt{gridsize}(k)$ many per sector].

Lemma. $\mathrm{F\text{-}BMOE}$ can be solved in time $2^{k^{\mathcal{O}(1)}} \cdot |V(G_f)|$.

The Final Step

Lemma. $\mathrm{F\text{-}BMOE}$ can be solved in time $2^{k^{\mathcal{O}(1)}} \cdot |V(G_f)|$.

- There is an algorithm that solves an instance $\mathcal{I}$ of BMOE in time $2^{\mathcal{O}(\kappa)} \cdot T(|\mathcal{I}|, k)$,

where $T(a, b)$ is the time required to solve an instance of $\mathrm{F\text{-}BMOE}$ with instance size a and parameter value b.

Corollary. BMOE can be solved in time $2^{\kappa^{\mathcal{O}(1)}} \cdot n$,
where n is the number of feature points of $\Gamma(H)$.

Further directions

Further

We have proved that the Bend-Minimal Orthogonal Extension Problem if FPT in the number of missing elements.

- What if H is not connected?
- The approach can be adjusted to minimize the number σ of bends per edge.
- Can we extend the result to planar drawings using a fixed number of slopes?

Thanks for attention!

Further directions

- What if H is not connected?
- Minimize the number of bends per edge.
- Add a fixed number of slopes.

Contents