Column Generation

A short Introduction

Martin Riedler
AC Retreat

ALGORITHMS AND
COMPLEXITY GROUP

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Basic Idea

- We already heard about Cutting Planes:
  - use an exponential number of constraints
  - use clever separation methods to only add violated constraints to the model until a feasible model is obtained

Basic Idea

- We already heard about Cutting Planes:
  - use an exponential number of constraints
  - use clever separation methods to only add violated constraints to the model until a feasible model is obtained
- Column Generation
  - this is the dual principle
  - use an exponential number of variables
  - only consider variables that have the potential to increase the objective value
  - can also be applied during Branch&Bound $\Rightarrow$ Branch&Price
  - or even in combination with Cutting Planes

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A small example

The Cutting Stock Problem

Definition

Consider a paper company that has a supply of large rolls of paper, of width $W \in \mathbb{N}$.

- Customers have demands for smaller widths of papers.
- In particular b_i rolls of width w_i, $i = 1, \ldots, m$, need to be produced (assume $b_i \in \mathbb{N}$, $w_i \in \mathbb{N}$, $w_i \leq W$, for each i)

The goal is to minimize the number of large rolls used while satisfying customer demands.

A small example (cont.)

The Cutting Stock Problem

- Smaller rolls can be obtained by slicing a large roll in a certain way, called a pattern
- Example: A large roll of with $W = 70$ can, e.g., be cut into
  - 3 rolls of width $w_1 = 17$ and
  - 1 roll of width $w_2 = 15$
  - with a waste of 4.

A small example (cont.)

The Cutting Stock Problem

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- Example: A large roll of with $W = 70$ can, e.g., be cut into
  - 3 rolls of width $w_1 = 17$ and
  - 1 roll of width $w_2 = 15$
  - with a waste of 4.
- *Use one variable $x_j \in \mathbb{Z}^m$ for each feasible pattern j*
  $\widehat{=}$ number of large rolls that are cut according to this specific pattern
- Pattern j represented as vector (column) $\mathbf{A}_j$, ith entry a_{ij} denotes the number of rolls of width w_i.
- Above example: $\mathbf{A}_j = (3, 1, 0, \ldots, 0)'$.

A small example: ILP

The Cutting Stock Problem

- Let $\mathbf{A} = (a_{ij})$ be the matrix consisting of all feasible cutting patterns $j = 1, \ldots, n$
- Each column $\mathbf{A}_j$ is a feasible pattern if

$$\sum_{i=1}^{m} w_i a_{ij} \leq W$$

$$a_{ij} \in \mathbb{N}_0, \; i = 1, \ldots, m$$

A small example: ILP

The Cutting Stock Problem

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Compact ILP (set partitioning formulation):

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} x_j \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \in \mathbb{Z}_+^m \end{aligned}$$

A small example: LP

The Cutting Stock Problem

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} x_j & \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i & i = 1, \ldots, m \\
& x_j \geq 0 & j = 1, \ldots, n
\end{aligned}
$$

Initialization of restricted master problem (RMP)

- Consider, e.g., m trivial patterns where pattern j consists of only one roll of with w_j.

A small example: Pricing Subproblem

The Cutting Stock Problem

Intuitively

- Find a variable corresponding to a feasible pattern with minimum reduced costs

Formally

- Obtained via the dual of the LP model
- In our case this turns out to be an *integer knapsack problem*

$\Rightarrow$ This problem can be solved fairly efficient (e.g., using dynamic programming or branch-and-bound)

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When to consider Column Generation?

- Sometimes, such formulations are somehow natural
  - Vehicle Routing (variables correspond to feasible tours)
  - Network Design (variables correspond to feasible paths or connections)
  - Cutting and Packing (variables corresponding to packing assignments / cutting patterns)
- Column generation may yield state-of-the-art results if the problem is rather "restricted"
- Sometimes we can get stronger LP-bounds by reformulation ($\Rightarrow$ *Dantzig-Wolfe Decomposition*)
- Column generation provides a decomposition of the problem into master and subproblems
  - This decomposition may have a natural interpretation in the contextual setting allowing for the incorporation of additional important constraints.

Metaheuristics and Column Generation

Two basic possibilities to boost column generation

- Apply (meta-)heuristics to the pricing subproblems.
- Apply metaheuristic to IMP and extract connections of good solutions, adding them as columns.
  - Sequential (batch) approach:
    The metaheuristic is performed first to create initial columns.
  - Parallel approach:
    The metaheuristic repeatedly delivers columns to the column generation algorithm.

Thank you for your Attention!

Questions?