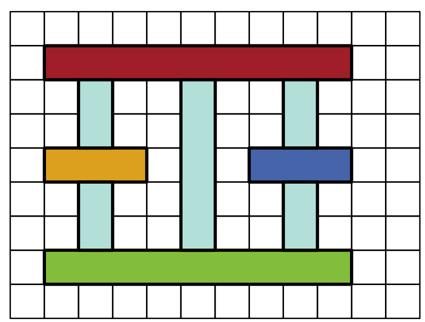
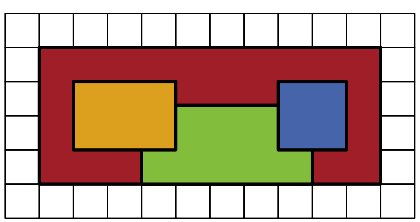
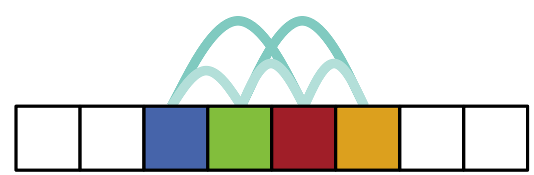
Grid-based Graph Drawing with ILP/SAT Modeling

joint work with T. Biedl, T. Bläsius, B. Niedermann, R. Prutkin, I. Rutter

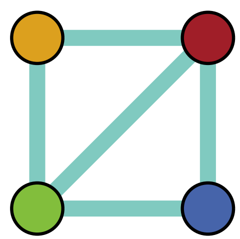


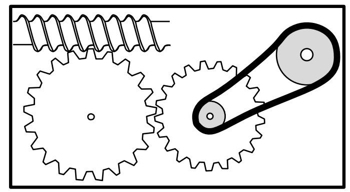




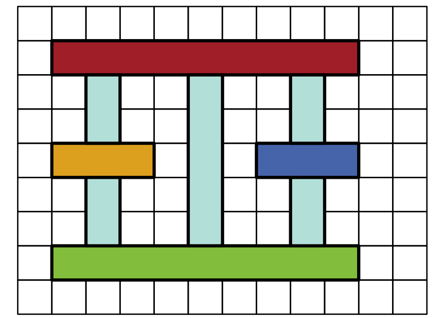
Motivation

Solve grid-based graph drawing problems in a unified framework:

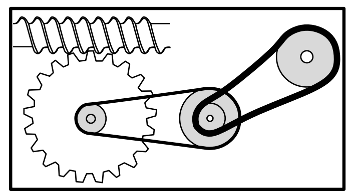




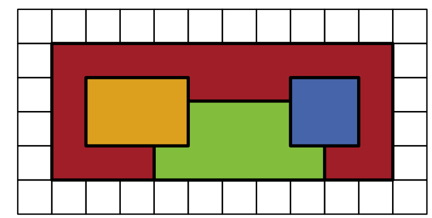
visibility representation



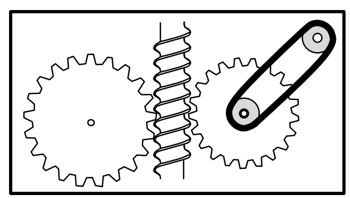
minimize width NP-hard



boxicity



recognition NP-hard



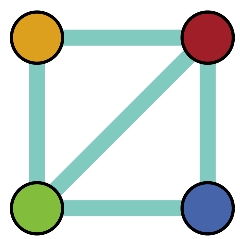
st-orientation

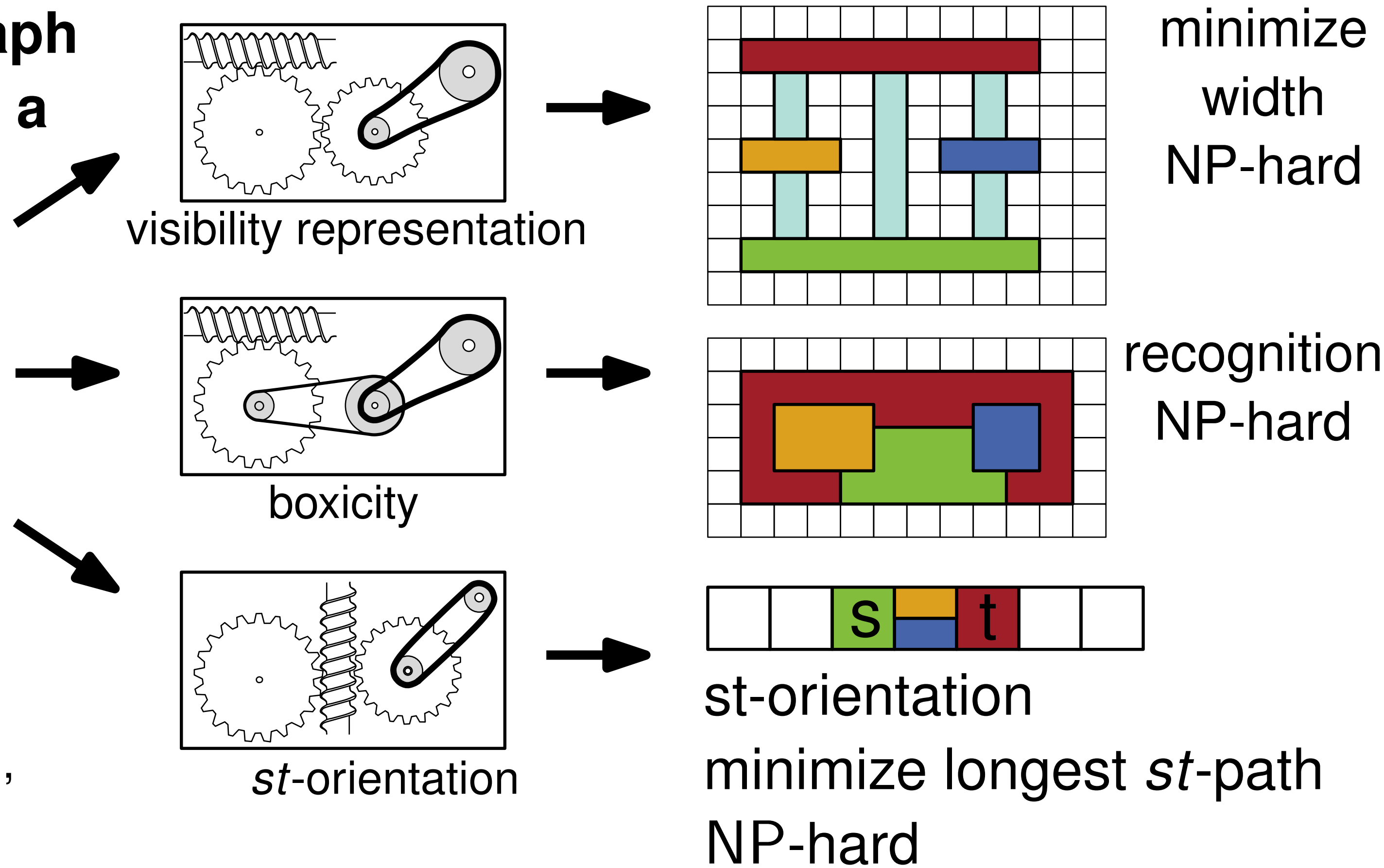
st-orientation
minimize longest *st*-path
NP-hard

pathwidth,
bandwidth,
...

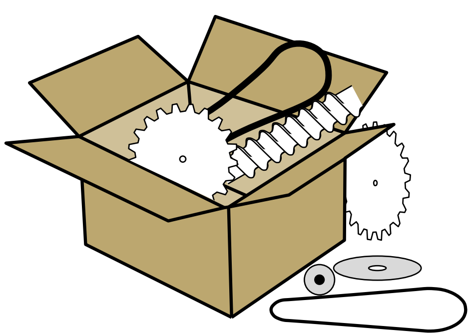
Motivation

Solve grid-based graph drawing problems in a unified framework:





pathwidth,
bandwidth,
. . .



Apply Integer Linear Programming (ILP) & SAT-solving:

- Collection of general constraints.
- Solving problem = assemble constraints.

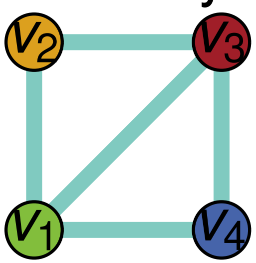
Approach

Given: Grid R, objects, constraints

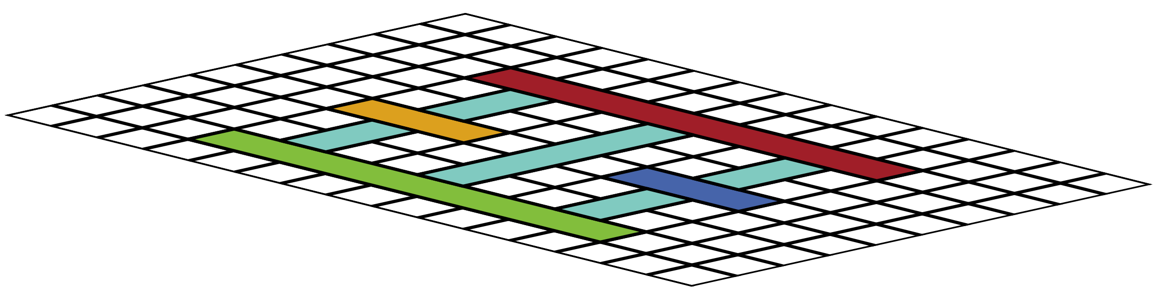
Find: Box on R for each object such that constraints are satisfied.

Example:

Visibility Representation



Vertices = horizontal boxes.
Edges = vertical boxes.



Approach

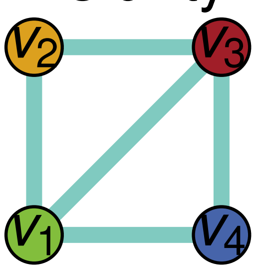
Given: Grid R, objects, constraints

Find: Box on R for each object such that constraints are satisfied.

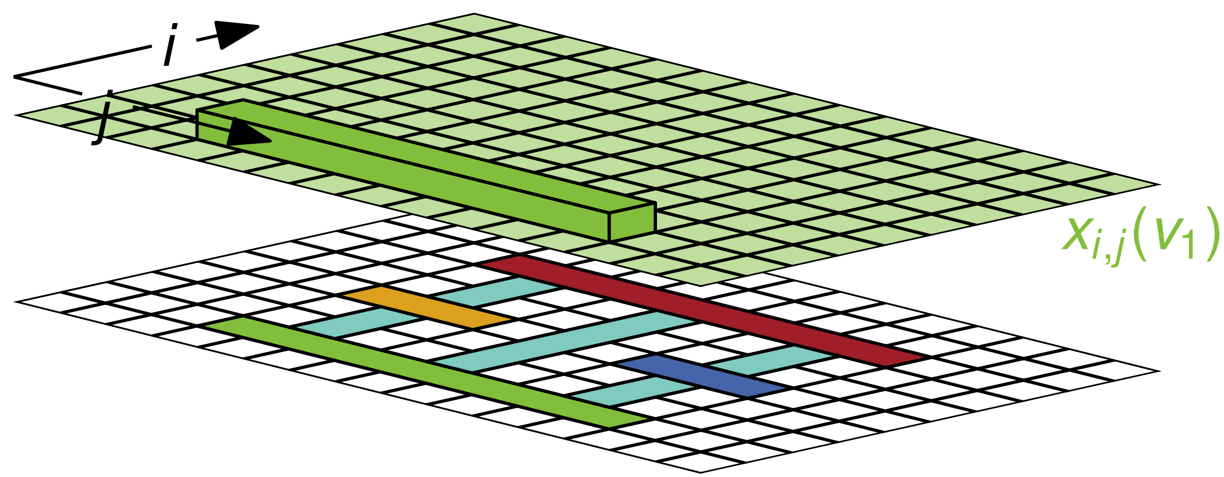
Representation of single box B:
- Grid of binary variables $x(B)_{i,j}$
- Meaning: $x(B)_{i,j} = 1$ iff grid point (i,j) belongs to B.

Example:

Visibility Representation



Vertices = horizontal boxes.
Edges = vertical boxes.



Approach

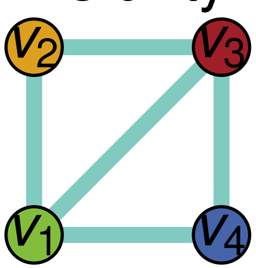
Given: Grid R, objects, constraints

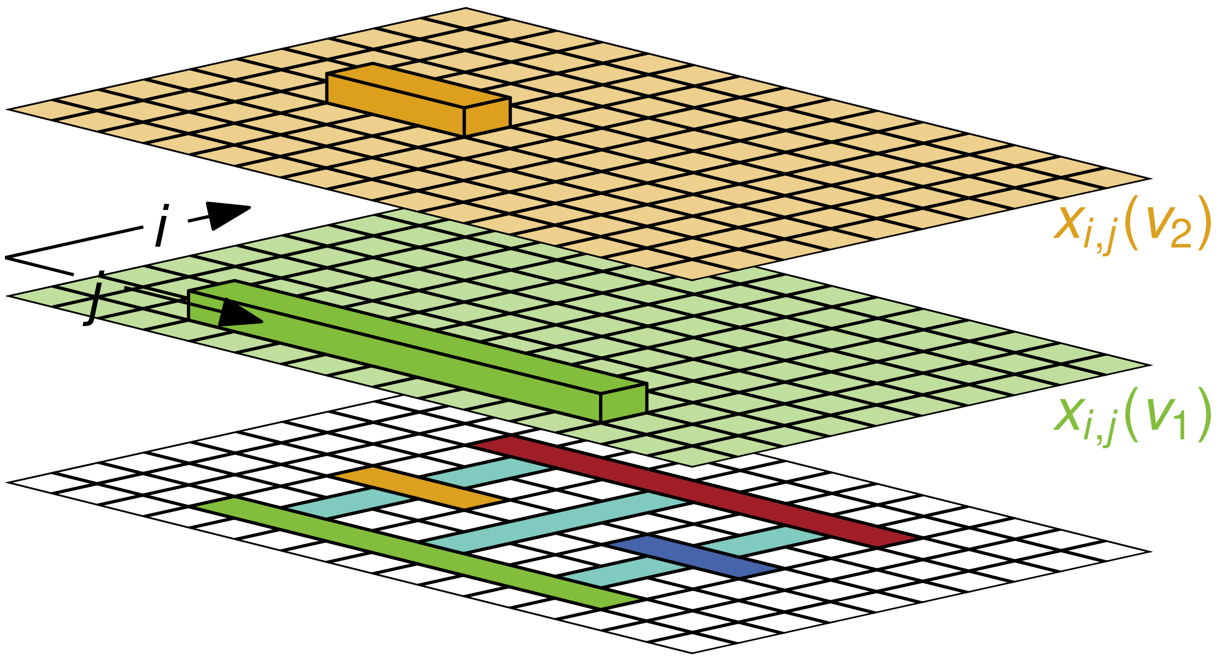
Find: Box on R for each object such that constraints are satisfied.

Representation of single box B:
- Grid of binary variables $x(B)_{i,j}$
- Meaning: $x(B)_{i,j} = 1$ iff grid point (i,j) belongs to B.

Example:

Visibility Representation





Vertices = horizontal boxes.
Edges = vertical boxes.

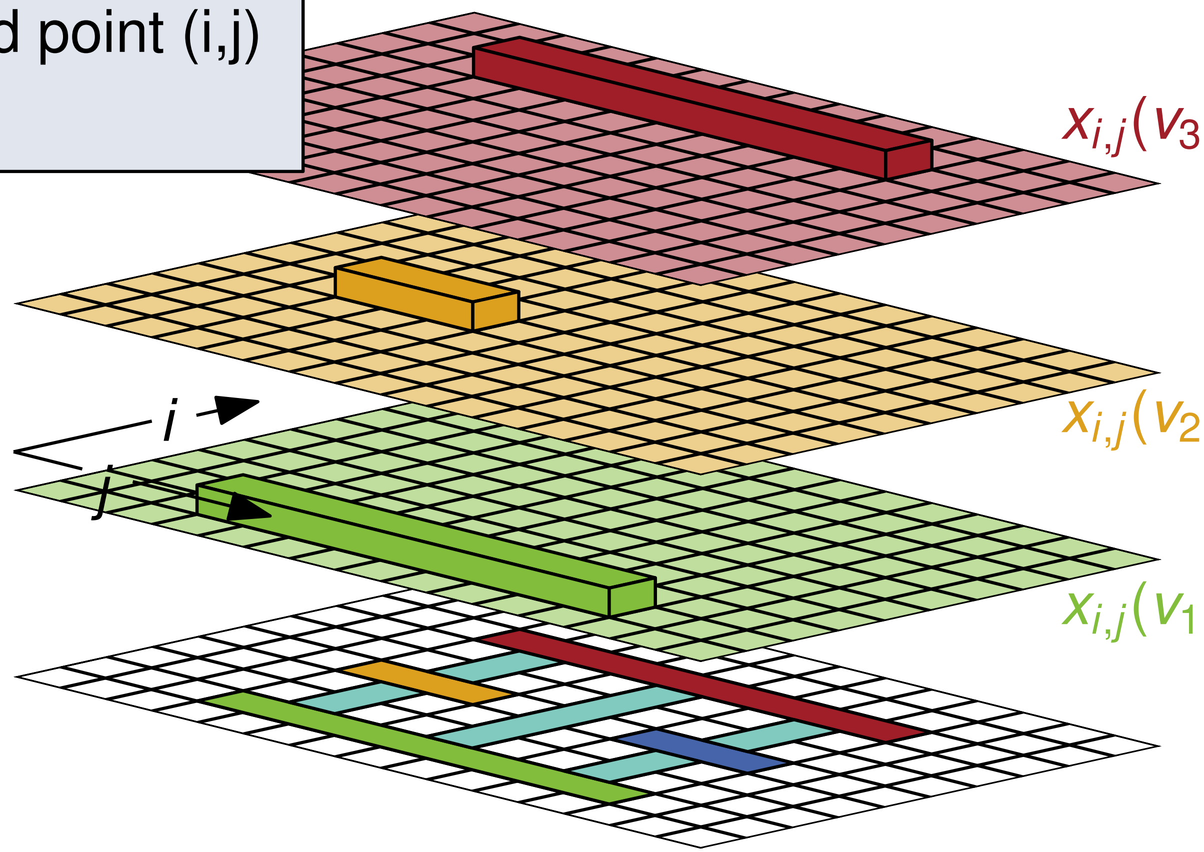
Approach

Given: Grid R, objects, constraints

Find: Box on R for each object such that constraints are satisfied.

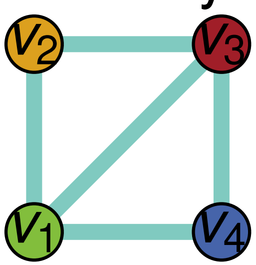
Representation of single box B:

- Grid of binary variables $x(B)_{i,j}$
- Meaning: $x(B)_{i,j} = 1$ iff grid point (i,j) belongs to B.



Example:

Visibility Representation



Vertices = horizontal boxes.
Edges = vertical boxes.

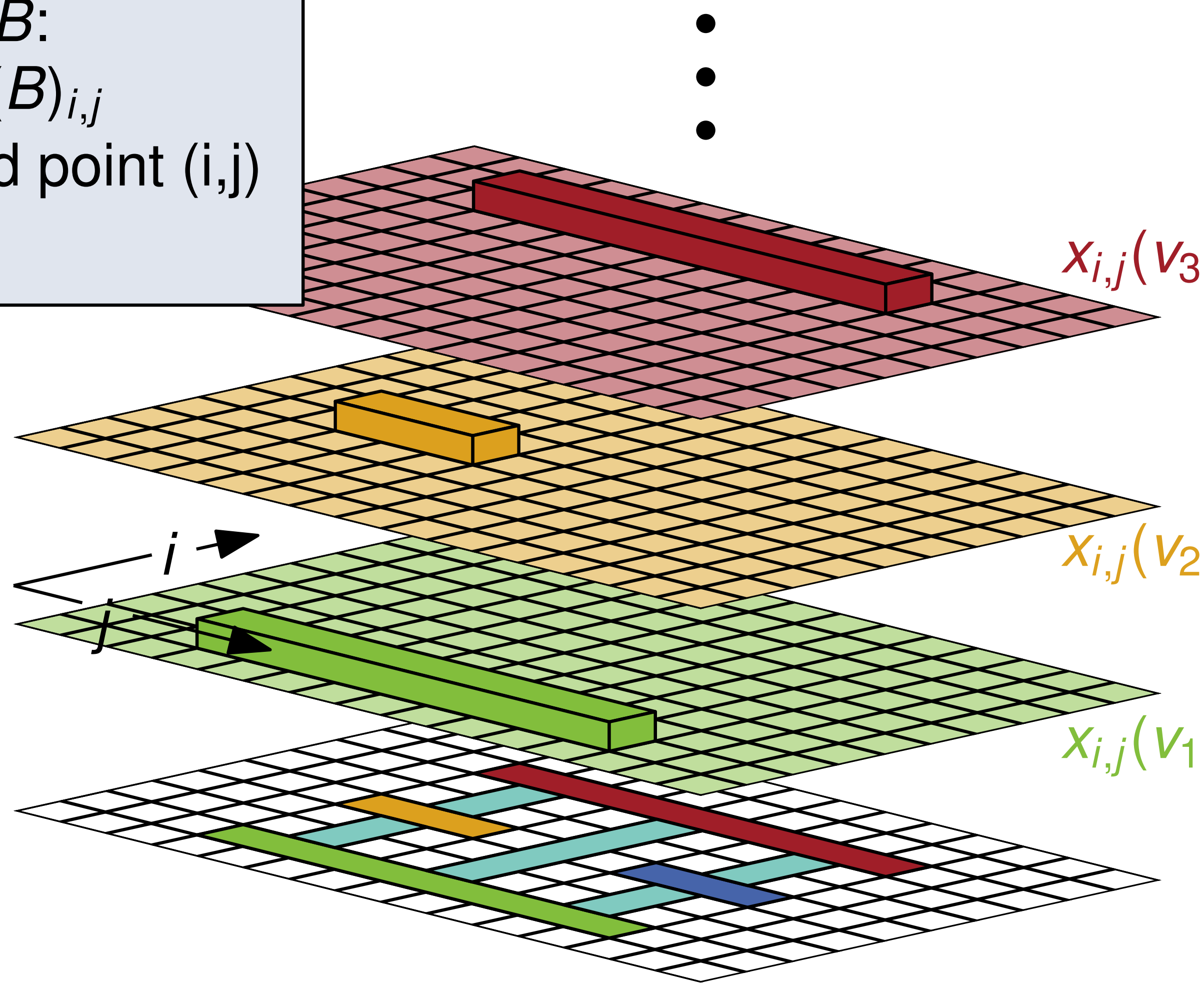
Approach

Given: Grid R, objects, constraints

Find: Box on R for each object such that constraints are satisfied.

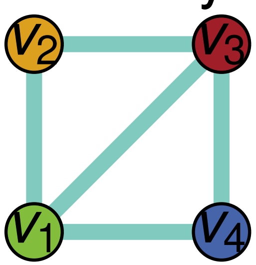
Representation of single box B:

- Grid of binary variables $x(B)_{i,j}$
- Meaning: $x(B)_{i,j} = 1$ iff grid point (i,j) belongs to B.



Example:

Visibility Representation



Vertices = horizontal boxes.
Edges = vertical boxes.

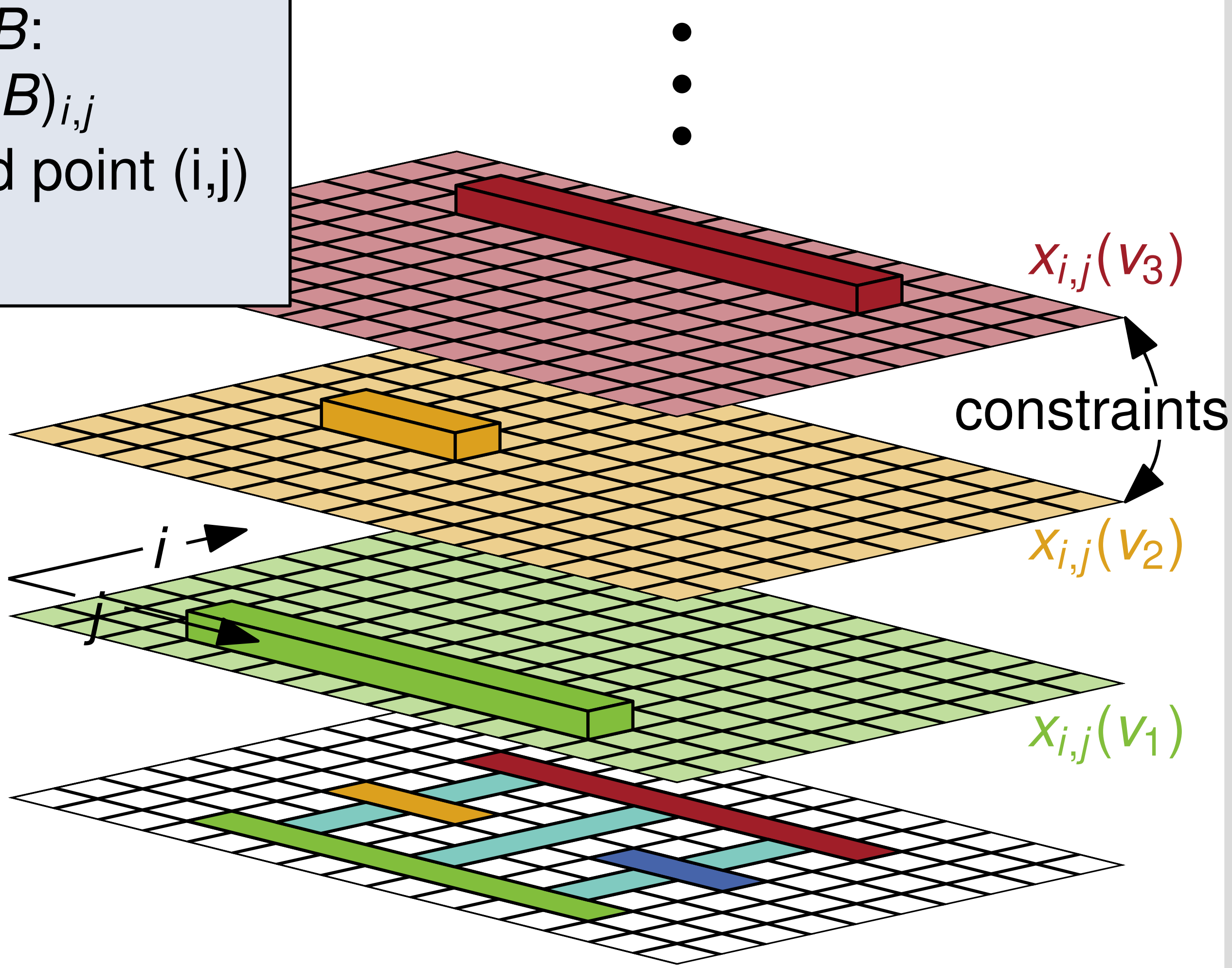
Approach

Given: Grid R, objects, constraints

Find: Box on R for each object such that constraints are satisfied.

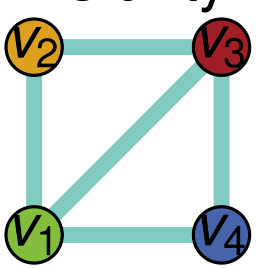
Representation of single box B:

- Grid of binary variables $x(B)_{i,j}$
- Meaning: $x(B)_{i,j} = 1$ iff grid point (i,j) belongs to B.



Example:

Visibility Representation

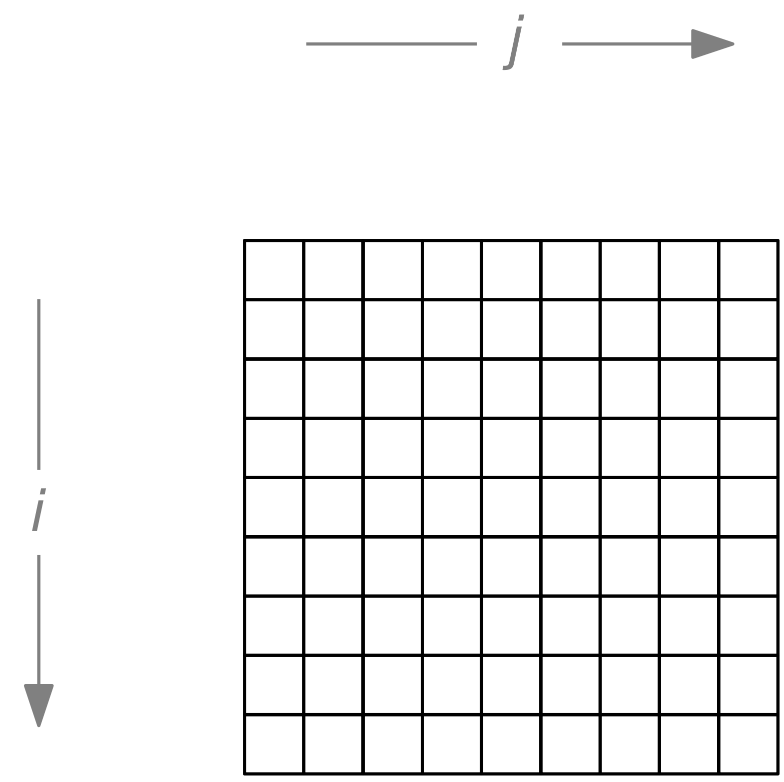


Vertices = horizontal boxes.
Edges = vertical boxes.

Representing Boxes

Represent box B by ILP constraints (here in 2D).

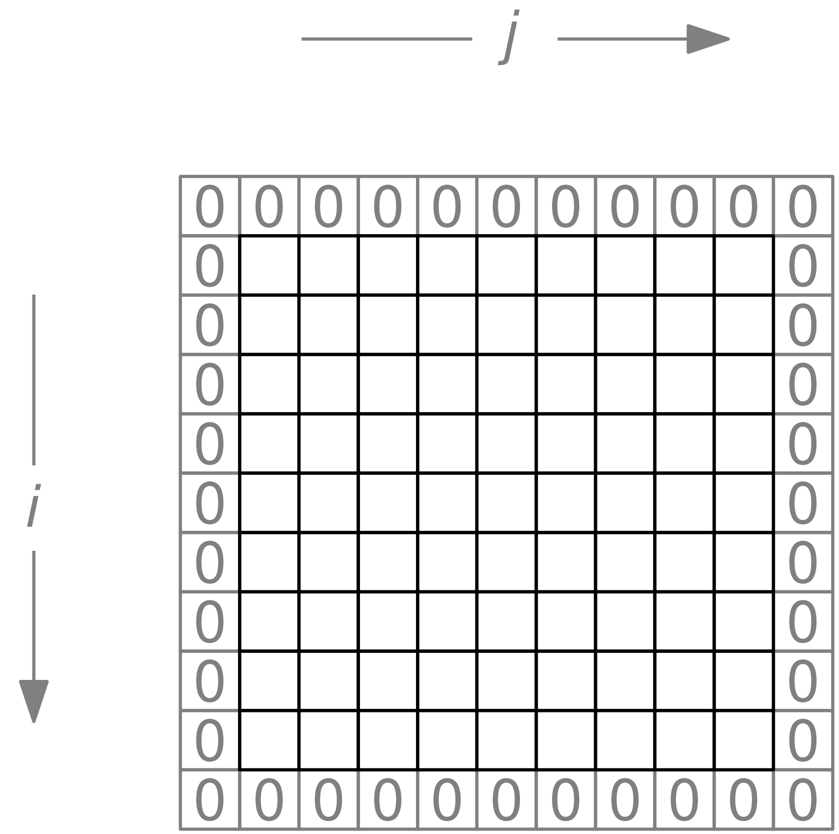
Grid of binary variables $x_{i,j}$



Representing Boxes

Represent box B by ILP constraints (here in 2D).

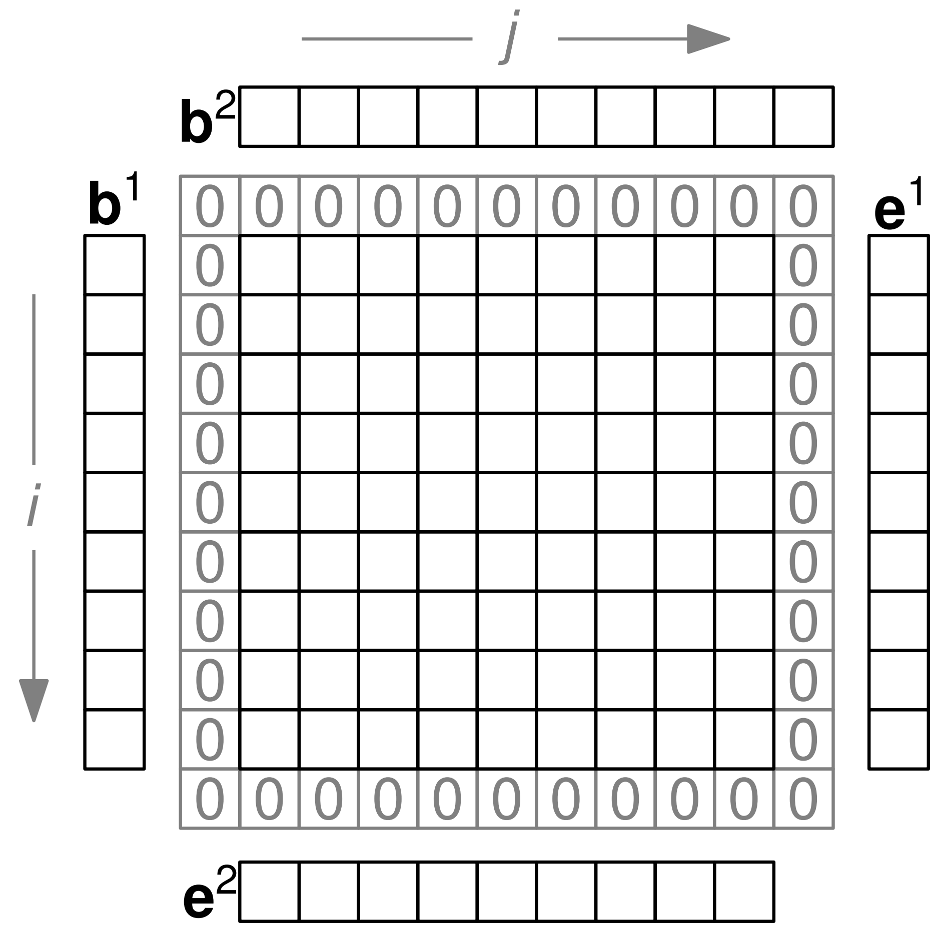
Grid of binary variables $x_{i,j}$



Representing Boxes

Represent box B by ILP constraints (here in 2D).

Grid of binary variables $x_{i,j}$



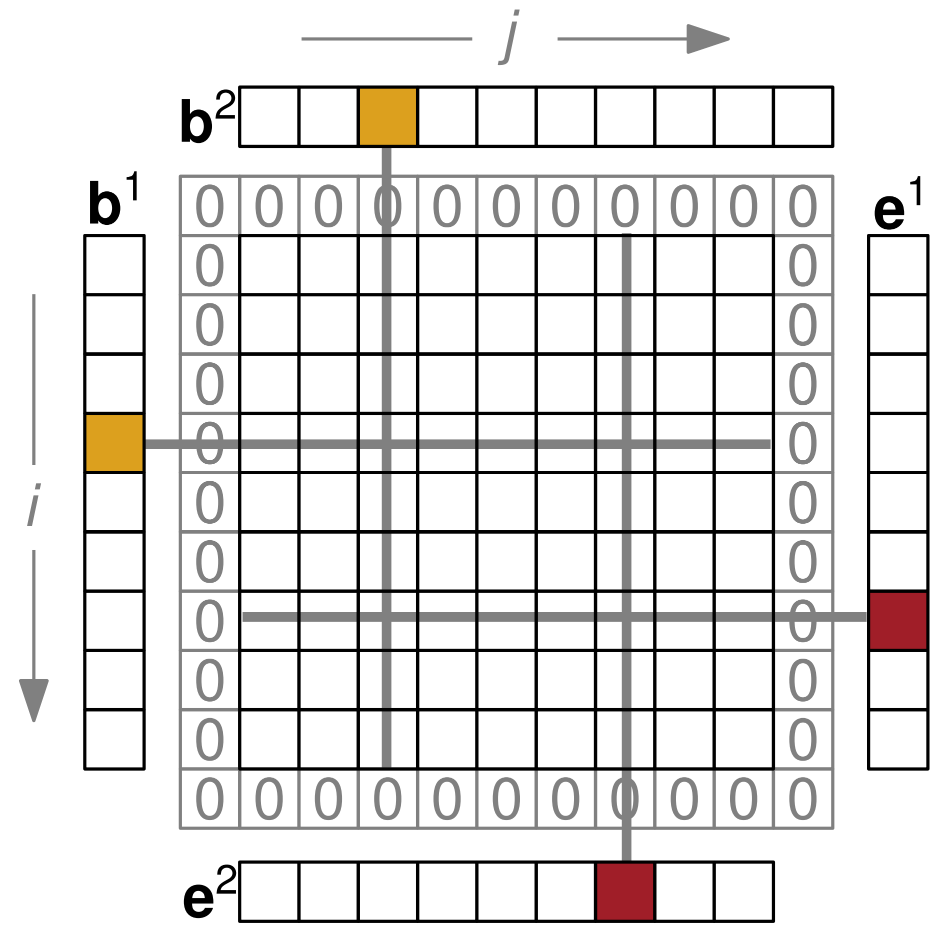
Representing Boxes

Represent box B by ILP constraints (here in 2D).

Grid of binary variables $x_{i,j}$

1.) There is a first/last row/column.

$\sum_i b_i^d = 1 \quad \sum_i e_i^d = 1$ for $d \in \{1, 2\}$



Representing Boxes

Represent box B by ILP constraints (here in 2D).

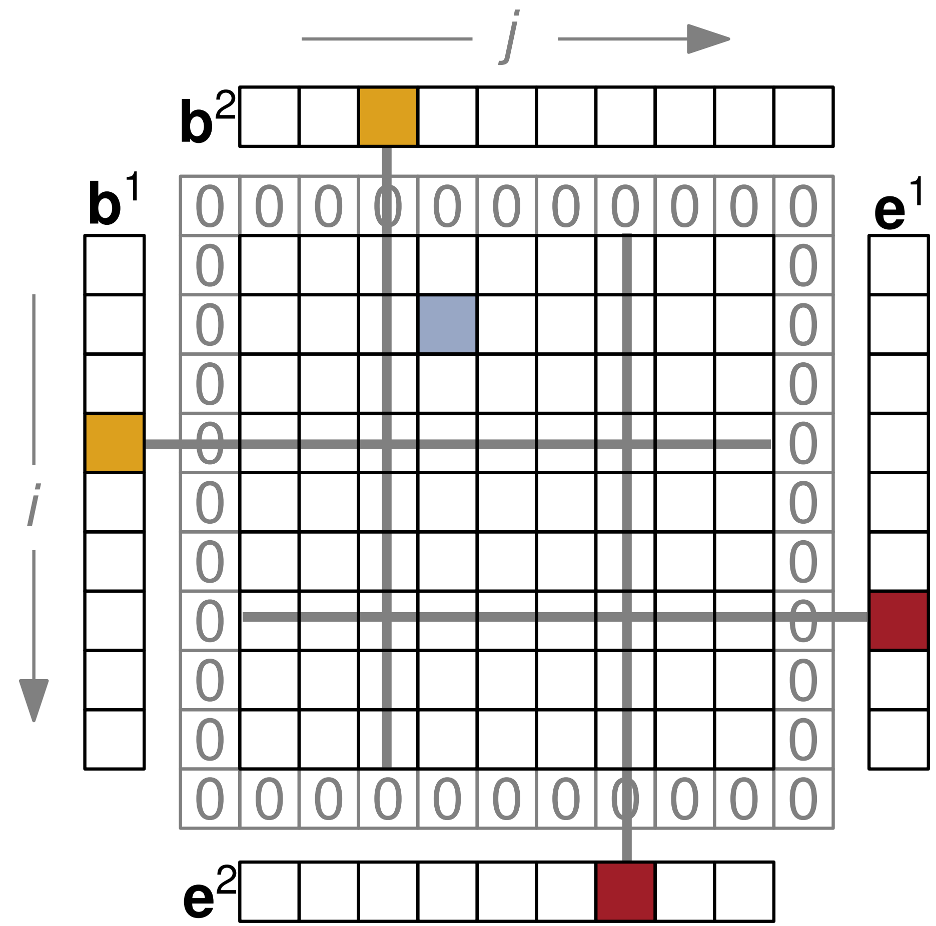
Grid of binary variables $x_{i,j}$

1.) There is a first/last row/column.

$\sum_i b_i^d = 1 \quad \sum_i e_i^d = 1$ for $d \in \{1, 2\}$

2.) Boxes are not empty.

$\sum_{i,j} x_{i,j} \geq 1$



Representing Boxes

Represent box B by ILP constraints (here in 2D).

Grid of binary variables $x_{i,j}$

1.) There is a first/last row/column.

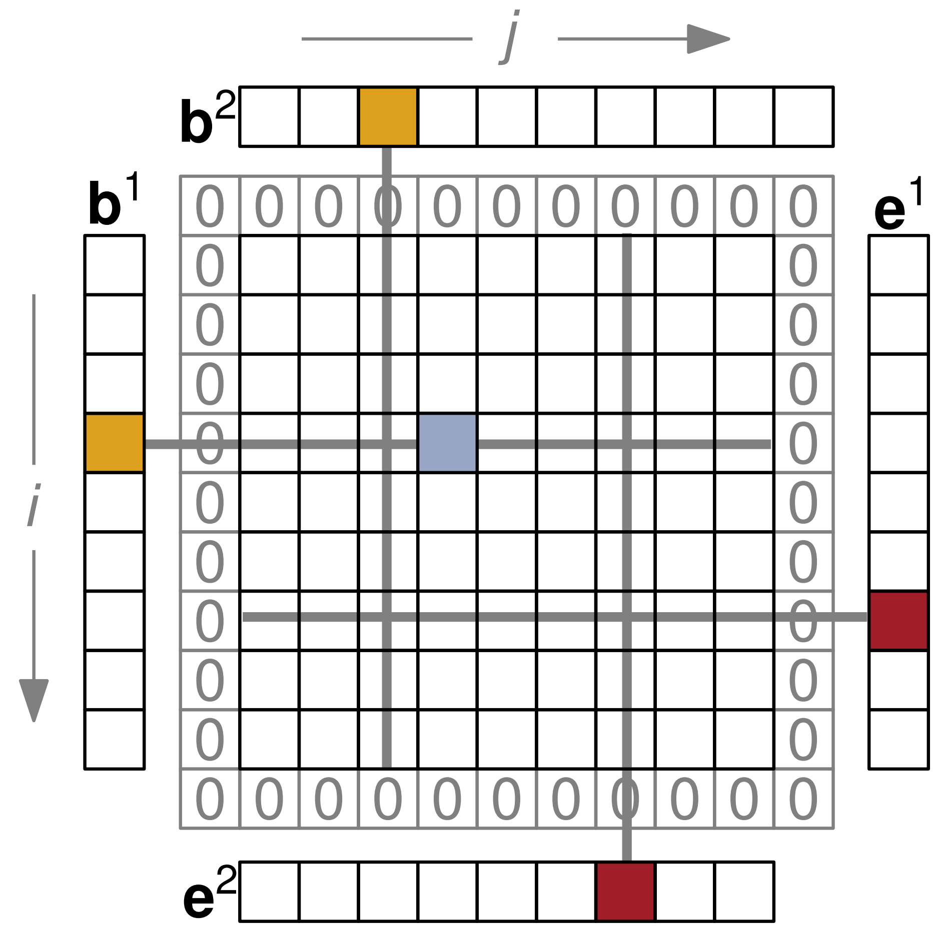
$\sum_i b_i^d = 1 \quad \sum_i e_i^d = 1$ for $d \in \{1, 2\}$

2.) Boxes are not empty.

$\sum_{i,j} x_{i,j} \geq 1$

3.) The first row of B is indicated by b_i^1.

$x_{i,j} \leq x_{i-1,j} + b_i^1$



Representing Boxes

Represent box B by ILP constraints (here in 2D).

Grid of binary variables $x_{i,j}$

1.) There is a first/last row/column.

$\sum_i b_i^d = 1 \quad \sum_i e_i^d = 1$ for $d \in \{1, 2\}$

2.) Boxes are not empty.

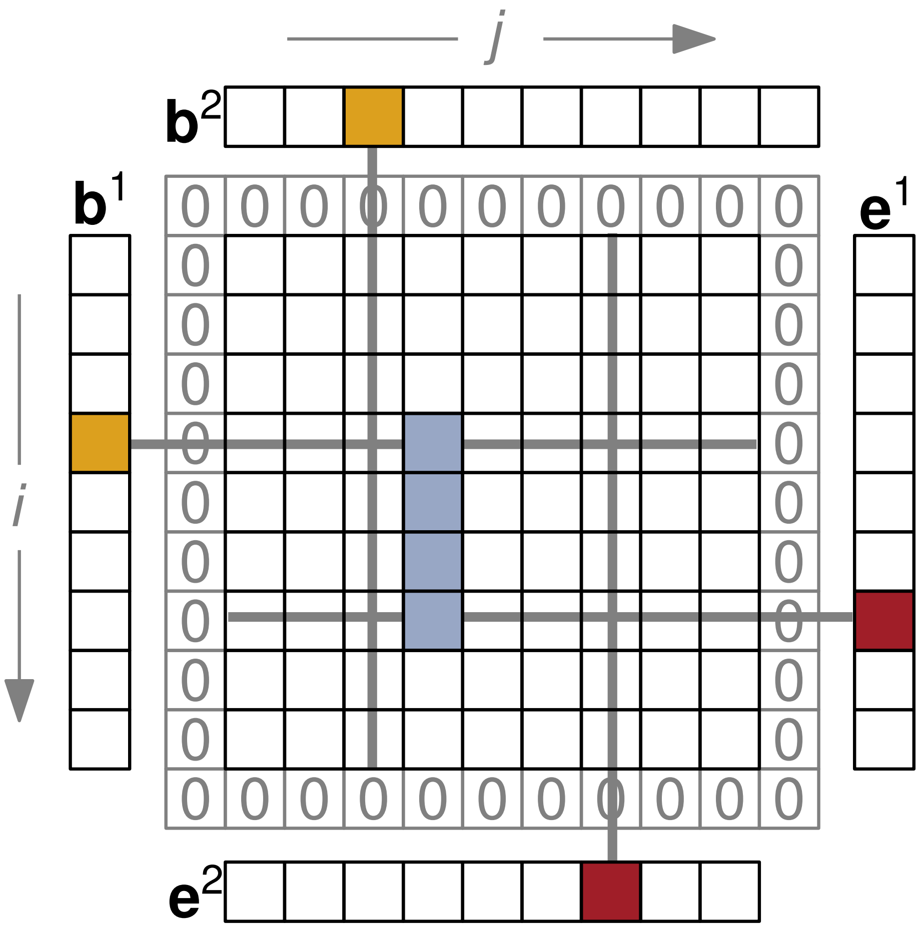
$\sum_{i,j} x_{i,j} \geq 1$

3.) The first row of B is indicated by b_i^1.

$x_{i,j} \leq x_{i-1,j} + b_i^1$

4.) The last row of B is indicated by e_i^1.

$x_{i,j} \leq x_{i+1,j} + e_i^1$



Representing Boxes

Represent box B by ILP constraints (here in 2D).

Grid of binary variables $x_{i,j}$

1.) There is a first/last row/column.

$\sum_i b_i^d = 1 \quad \sum_i e_i^d = 1$ for $d \in \{1, 2\}$

2.) Boxes are not empty.

$\sum_{i,j} x_{i,j} \geq 1$

3.) The first row of B is indicated by b_i^1.

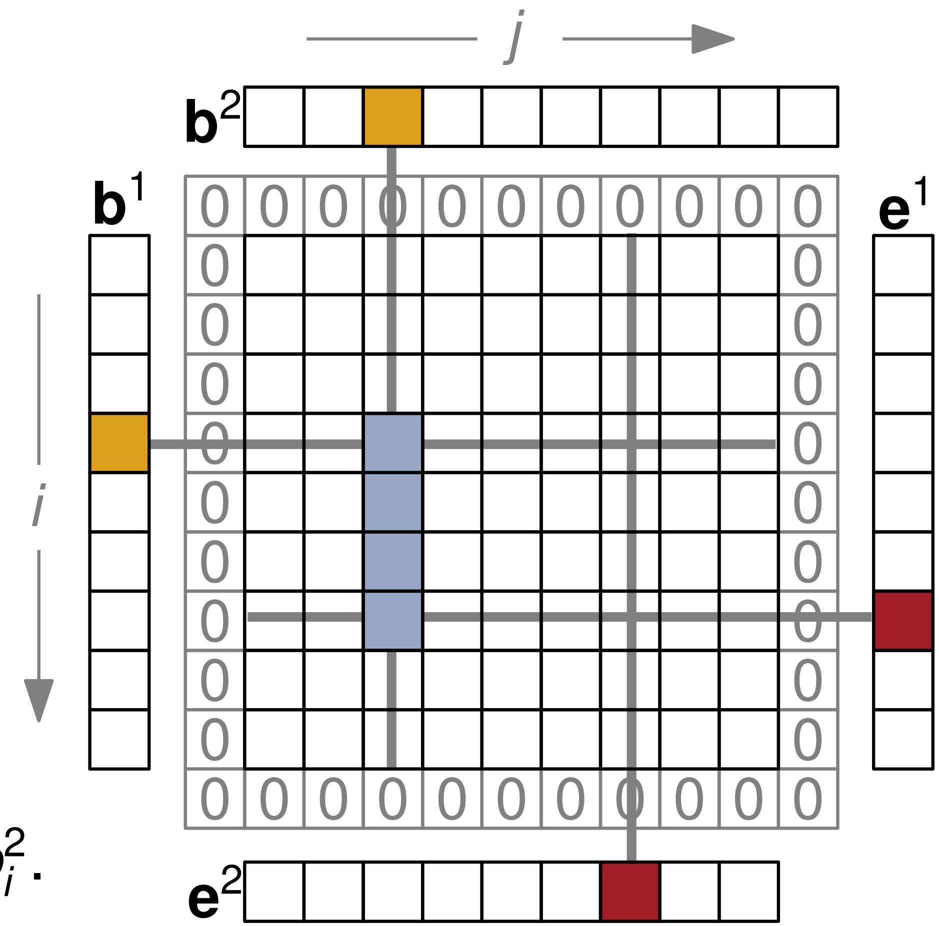
$x_{i,j} \leq x_{i-1,j} + b_i^1$

4.) The last row of B is indicated by e_i^1.

$x_{i,j} \leq x_{i+1,j} + e_i^1$

5.) The first column of B is indicated by b_i^2.

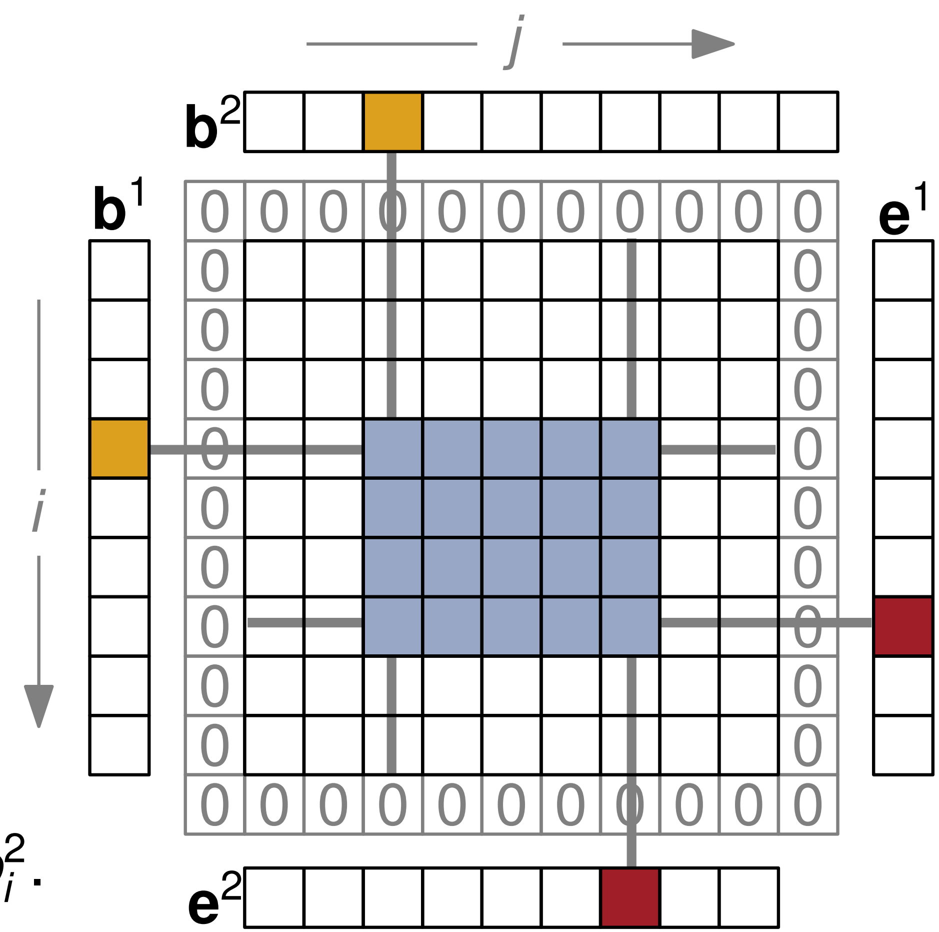
$x_{i,j} \leq x_{i,j-1} + b_j^2$



Representing Boxes

Represent box B by ILP constraints (here in 2D).

Grid of binary variables $x_{i,j}$



1.) There is a first/last row/column.

$\sum_i b_i^d = 1 \quad \sum_i e_i^d = 1$ for $d \in \{1,2\}$

2.) Boxes are not empty.

$\sum_{i,j} x_{i,j} \geq 1$

3.) The first row of B is indicated by b_i^1.

$x_{i,j} \leq x_{i-1,j} + b_i^1$

4.) The last row of B is indicated by e_i^1.

$x_{i,j} \leq x_{i+1,j} + e_i^1$

5.) The first column of B is indicated by b_i^2.

$x_{i,j} \leq x_{i,j-1} + b_j^2$

6.) The last column of B is indicated by e_i^2.

$x_{i,j} \leq x_{i,j+1} + e_j^2$

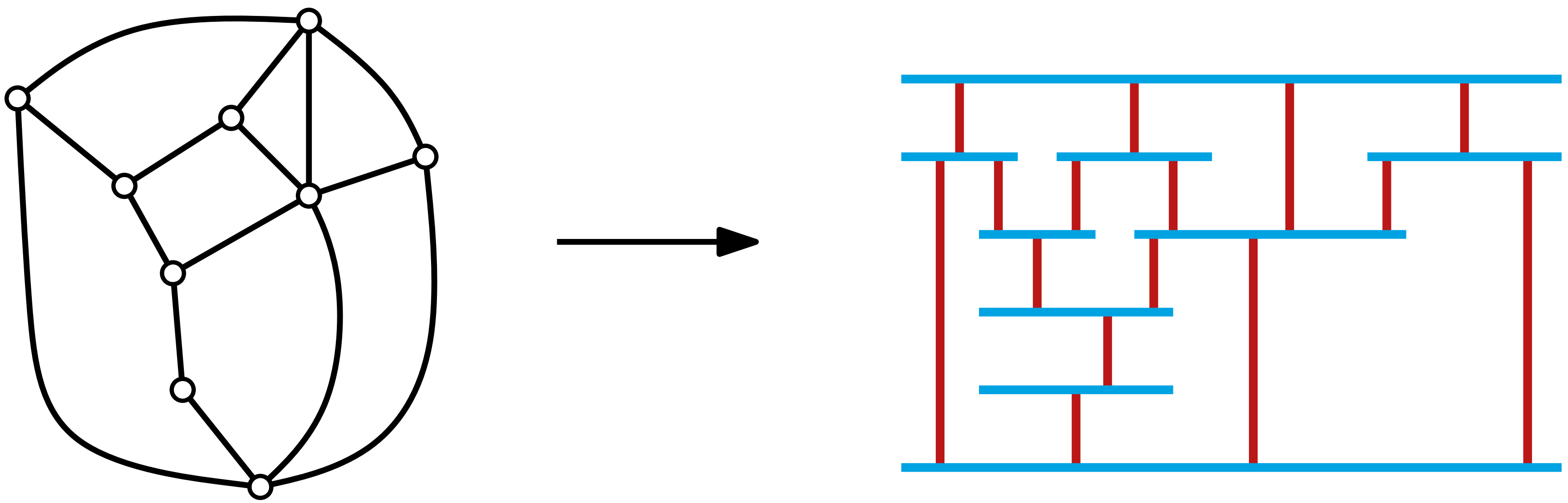
Visibility Representations

Def: A **visibility representation** of a graph $G = (V, E)$ draws

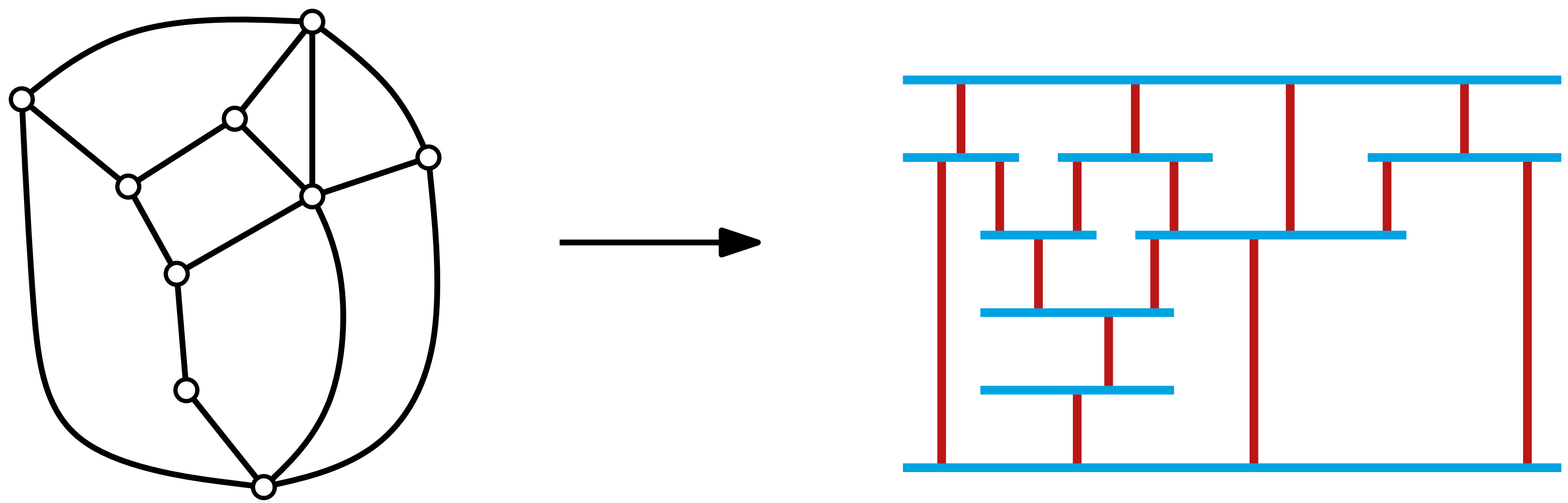
- each vertex v as a horizontal line segment $\Gamma(v)$
- each edge $e = (u, v)$ as a vertical line segment $\Gamma(e)$

such that

- no two vertex segments intersect
- no two edge segments intersect
- each edge segment $\Gamma(u, v)$ has its endpoints on $\Gamma(u)$ and $\Gamma(v)$ and intersects no other vertex segment



Properties of Visibility Representations



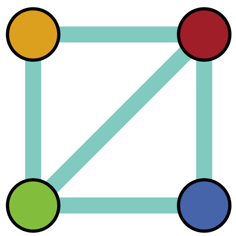
- graph *G* must be planar (obviously)
- every planar graph has a visibility representation

[Wismath ’85], [Tamassia, Tollis ’86]

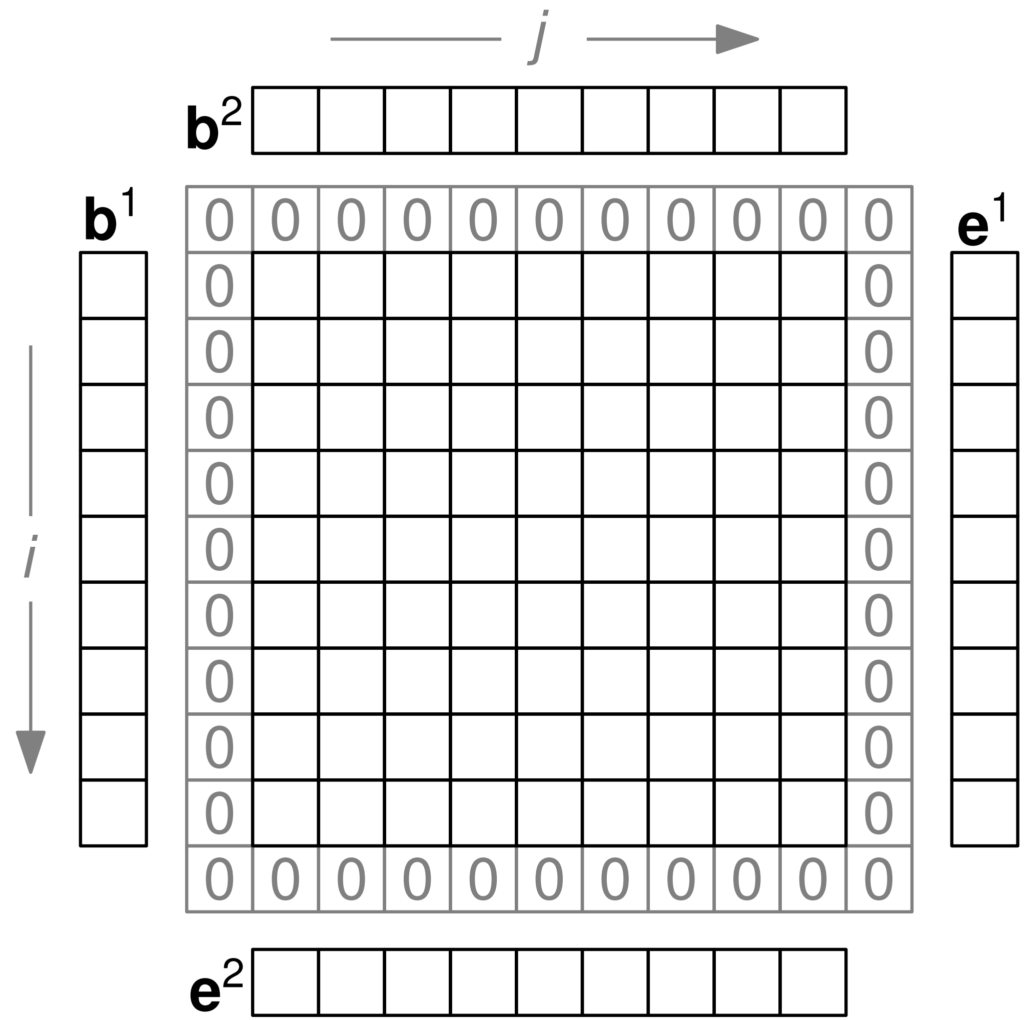
- minimizing the area of a visibility representation is NP-hard

[Lin, Eades ’03]

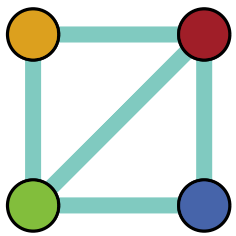
Modeling Visibility Representation



$G = (V, E)$

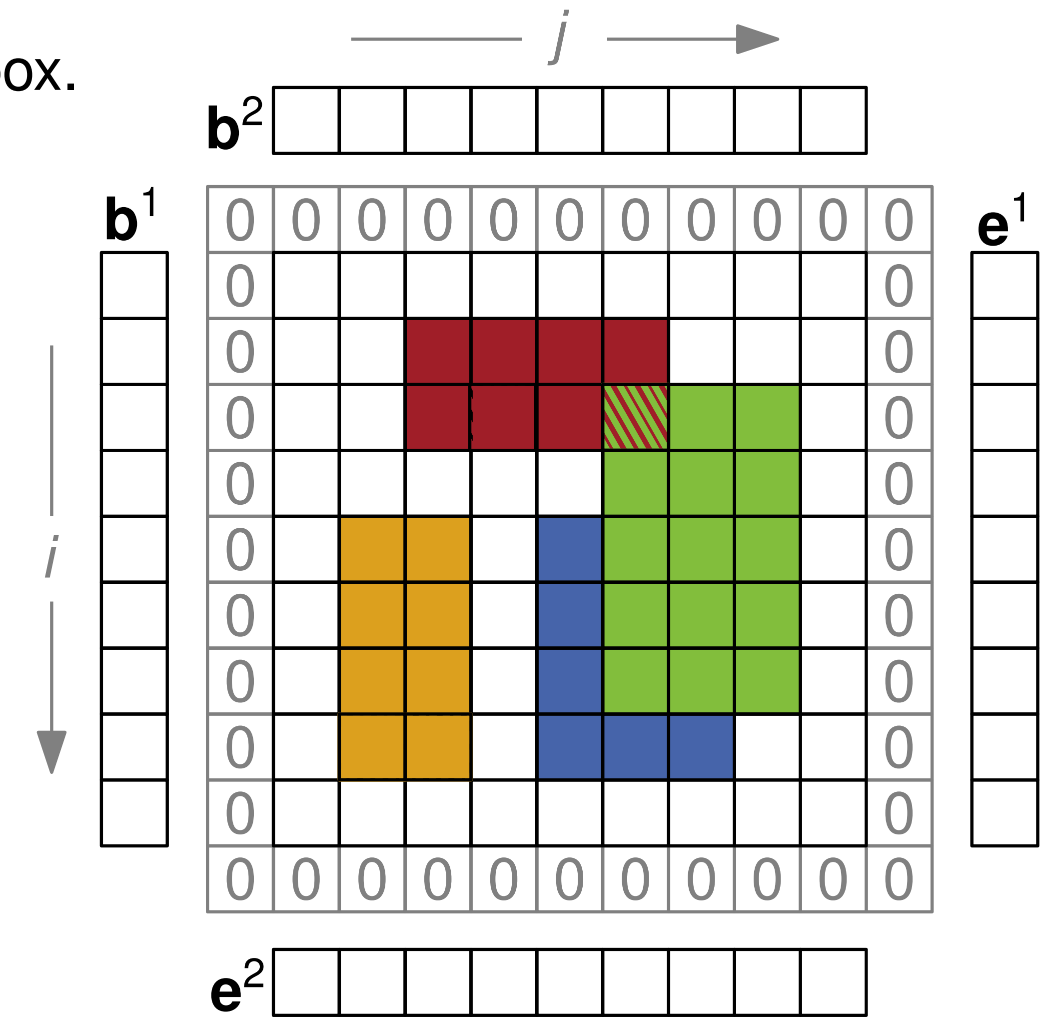


Modeling Visibility Representation

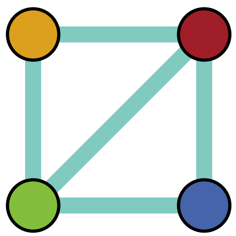


$G = (V, E)$

1.) $\forall v \in V$ introduce box.



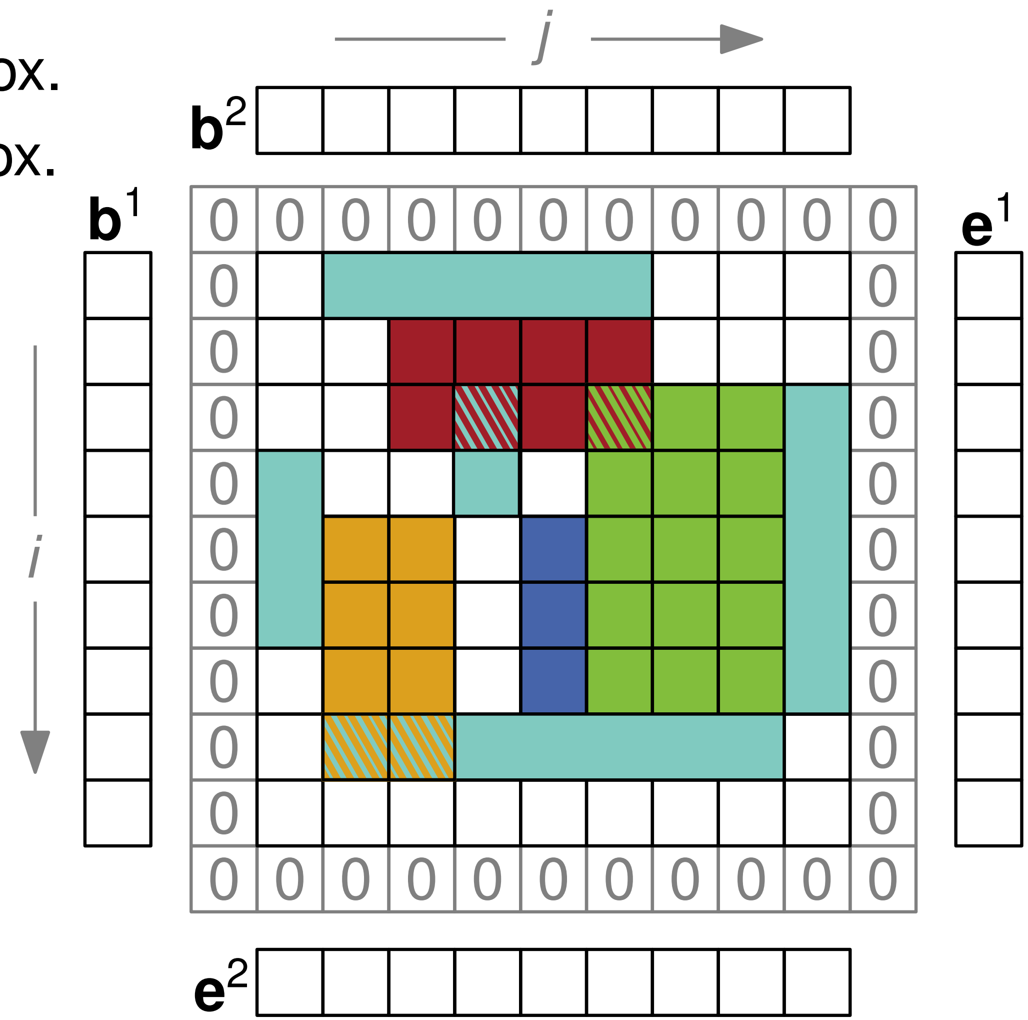
Modeling Visibility Representation



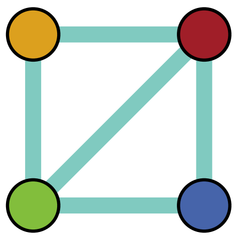
$G = (V, E)$

1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.



Modeling Visibility Representation

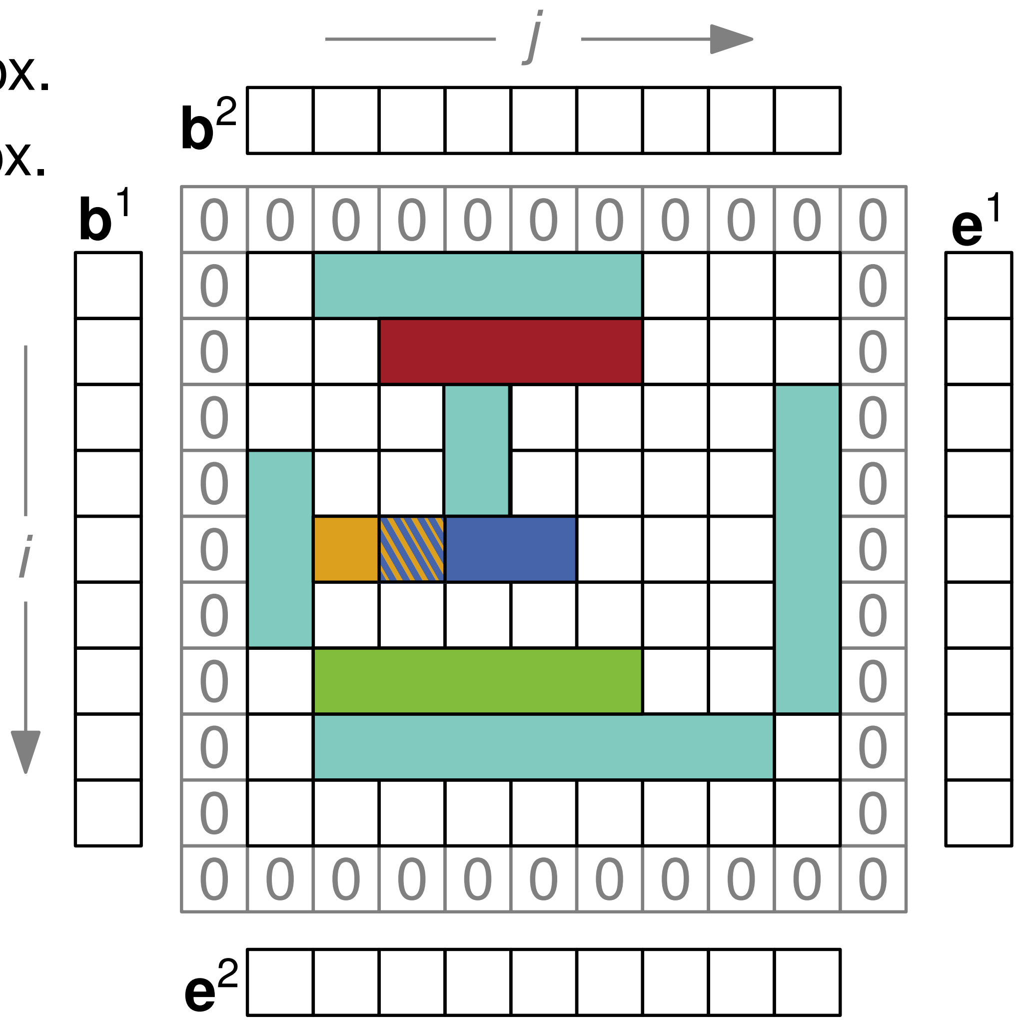


$G = (V, E)$

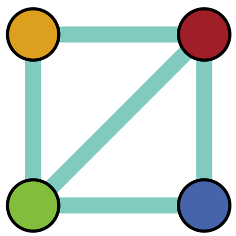
1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.

3.) Vertices are horizontal segments:



Modeling Visibility Representation



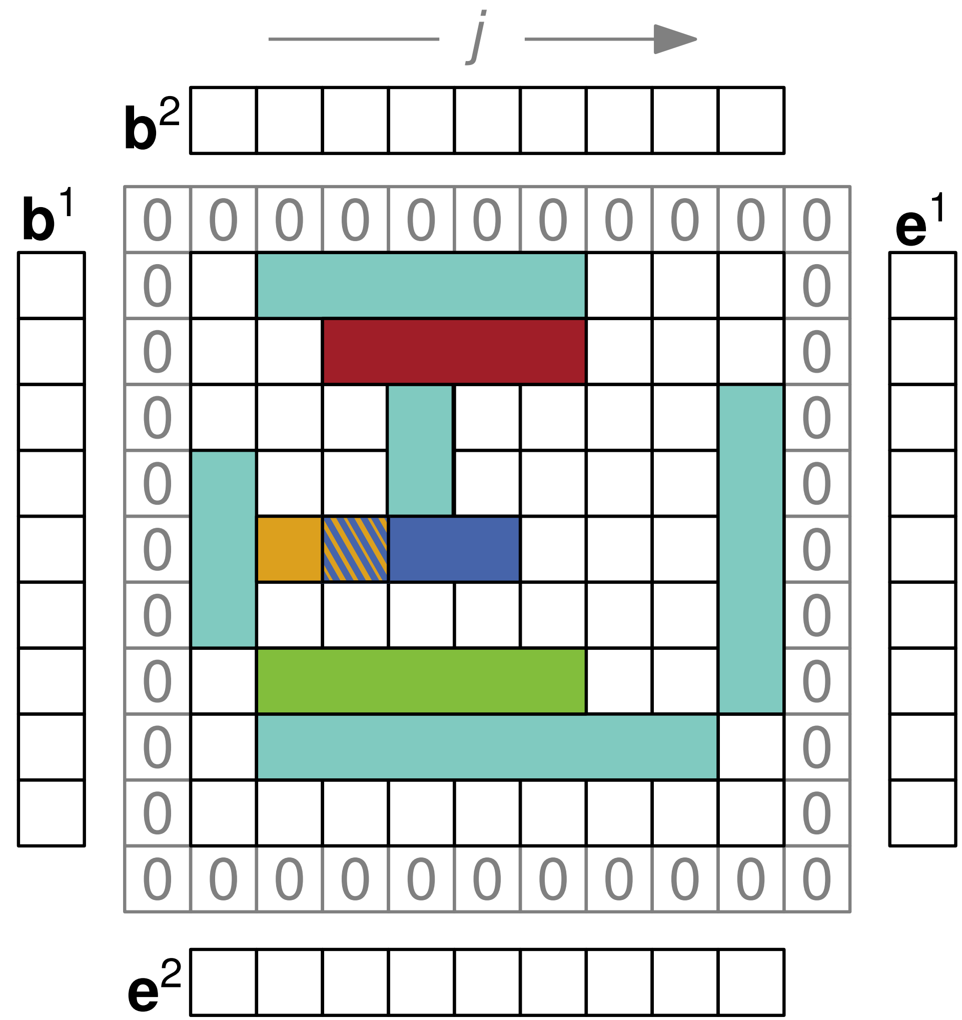
$G = (V, E)$

1.) $\forall v \in V$ introduce box.

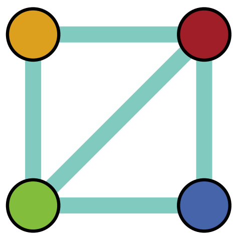
2.) $\forall e \in E$ introduce box.

3.) Vertices are horizontal segments:

$$b_i^1(v) = e_i^1(v)$$



Modeling Visibility Representation



$G = (V, E)$

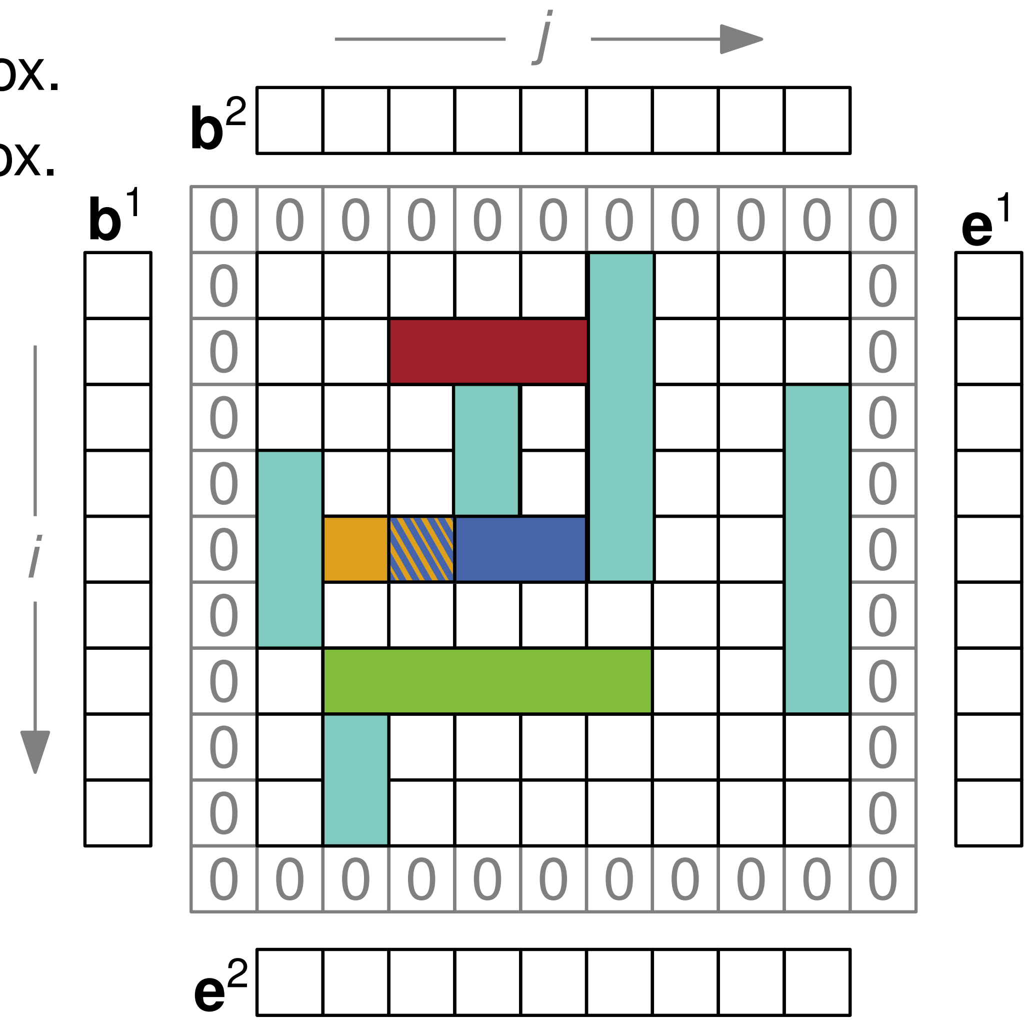
1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.

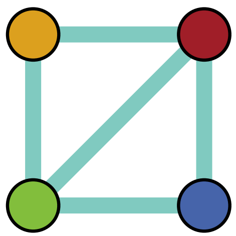
3.) Vertices are horizontal segments:

$$b_i^1(v) = e_i^1(v)$$

4.) Edges are vertical segments:



Modeling Visibility Representation



$G = (V, E)$

1.) $\forall v \in V$ introduce box.

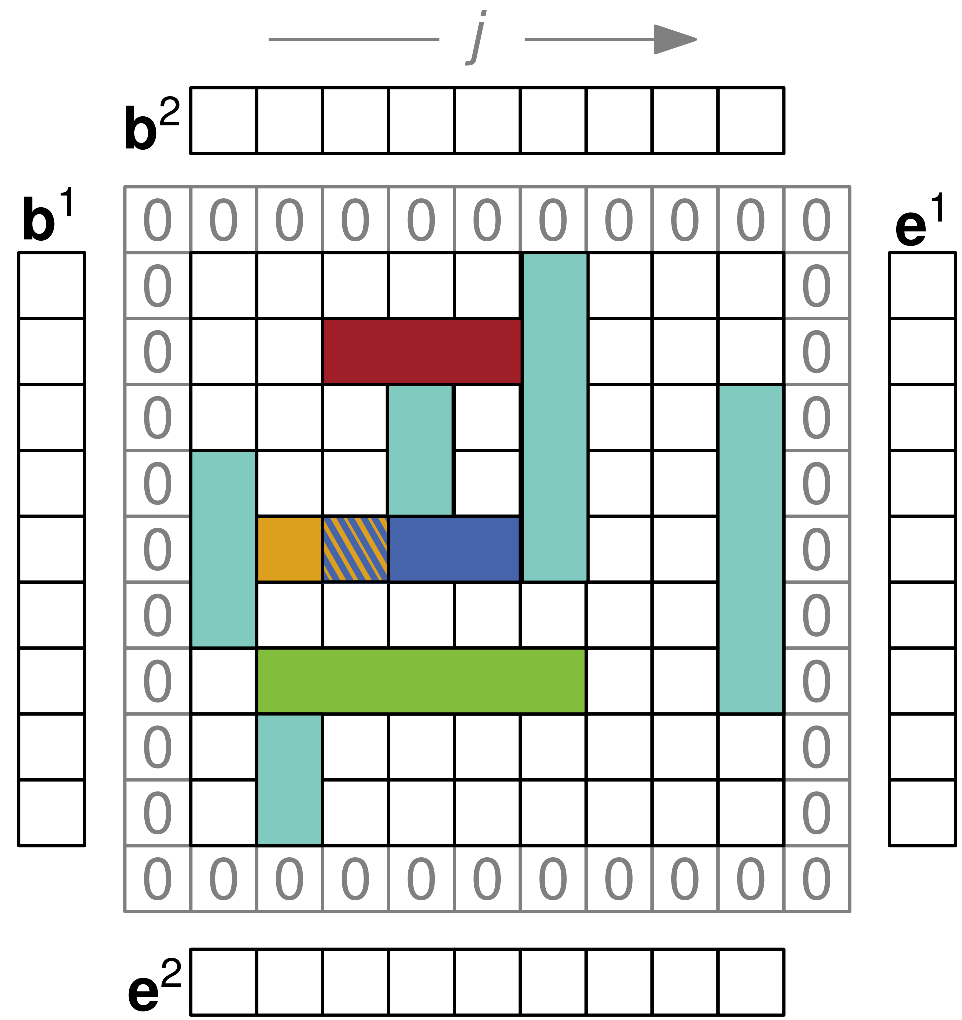
2.) $\forall e \in E$ introduce box.

3.) Vertices are horizontal segments:

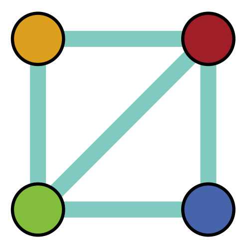
$$b_i^1(v) = e_i^1(v)$$

4.) Edges are vertical segments:

$$b_i^2(e) = e_i^2(e)$$



Modeling Visibility Representation



$G = (V, E)$

1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.

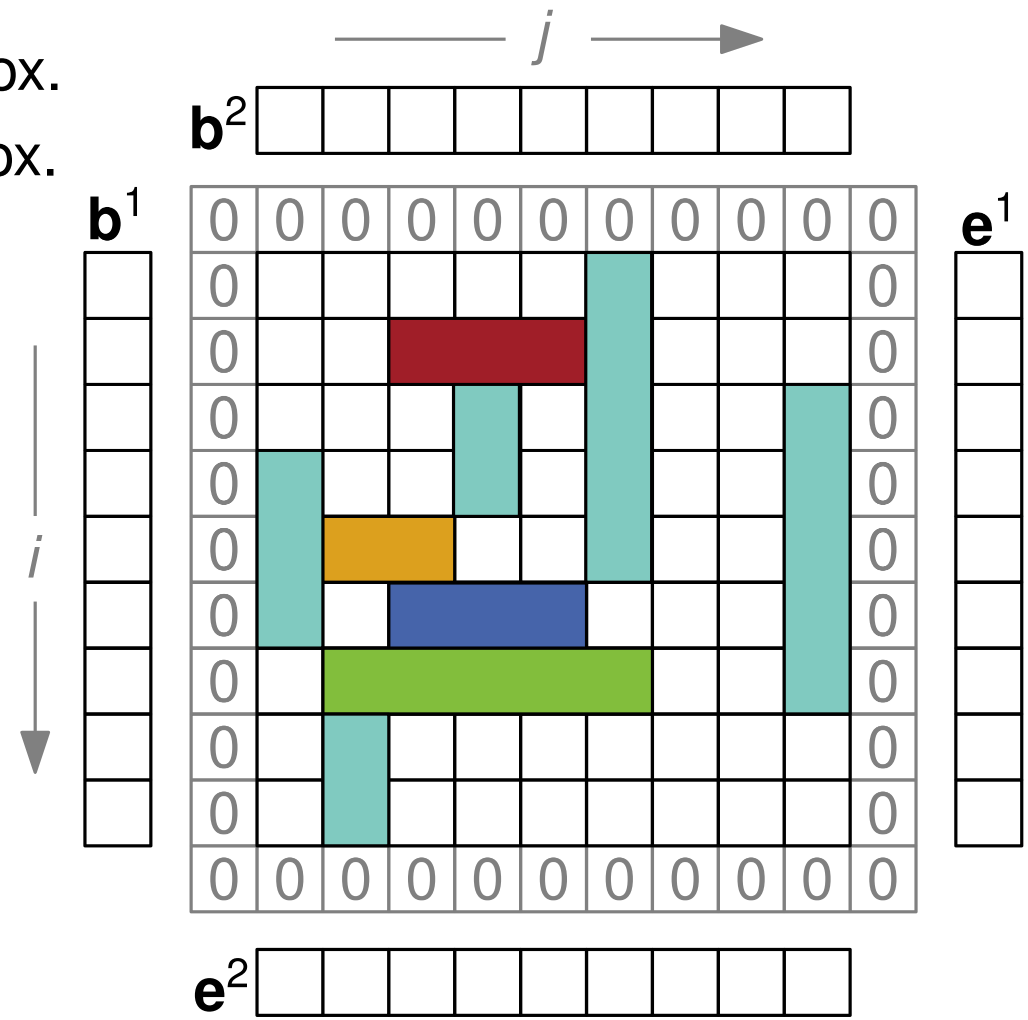
3.) Vertices are horizontal segments:

$$b_i^1(v) = e_i^1(v)$$

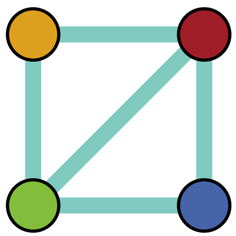
4.) Edges are vertical segments:

$$b_i^2(e) = e_i^2(e)$$

5.) Vertices do not overlap:



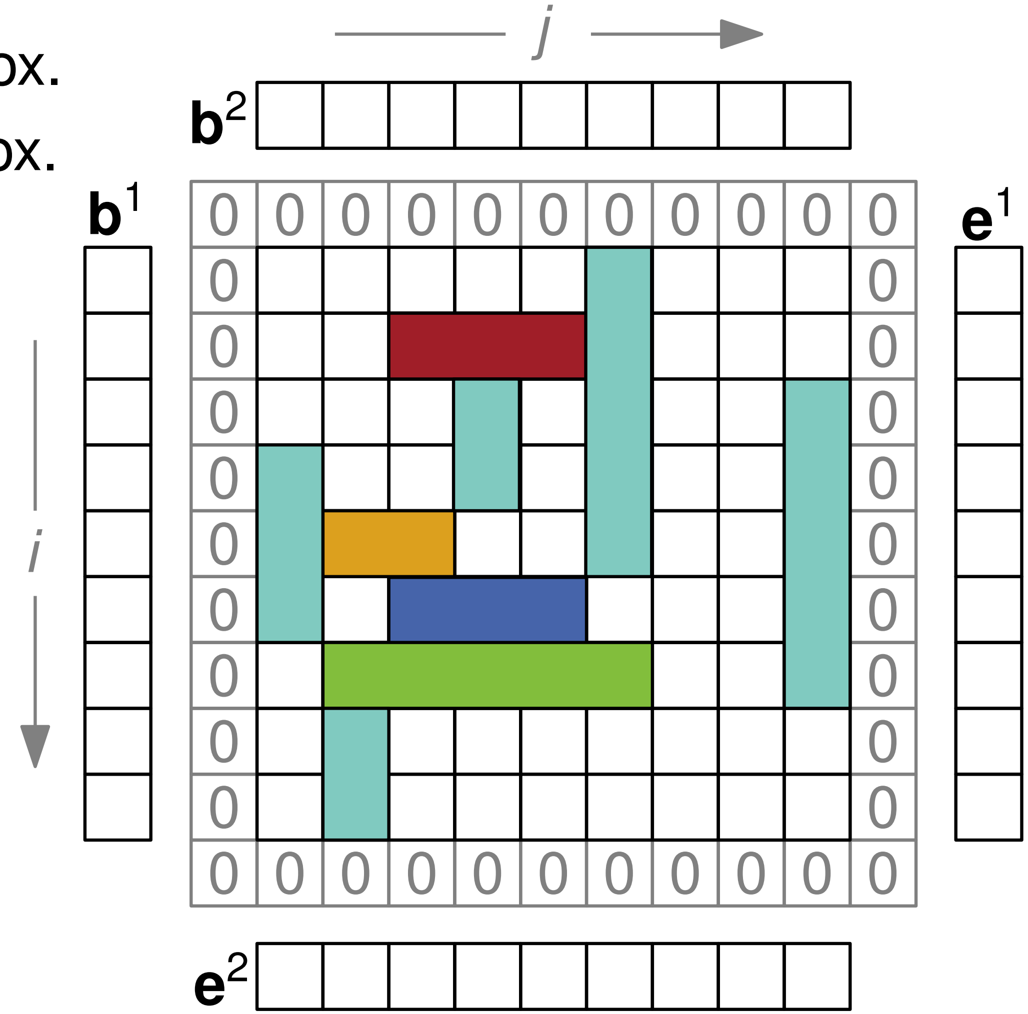
Modeling Visibility Representation



$G = (V, E)$

1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.



3.) Vertices are horizontal segments:

$$b_i^1(v) = e_i^1(v)$$

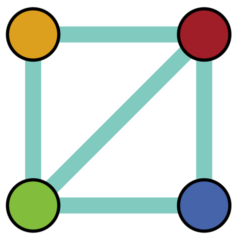
4.) Edges are vertical segments:

$$b_i^2(e) = e_i^2(e)$$

5.) Vertices do not overlap:

$$\sum_{v \in V} x_{i,j}(v) \leq 1$$

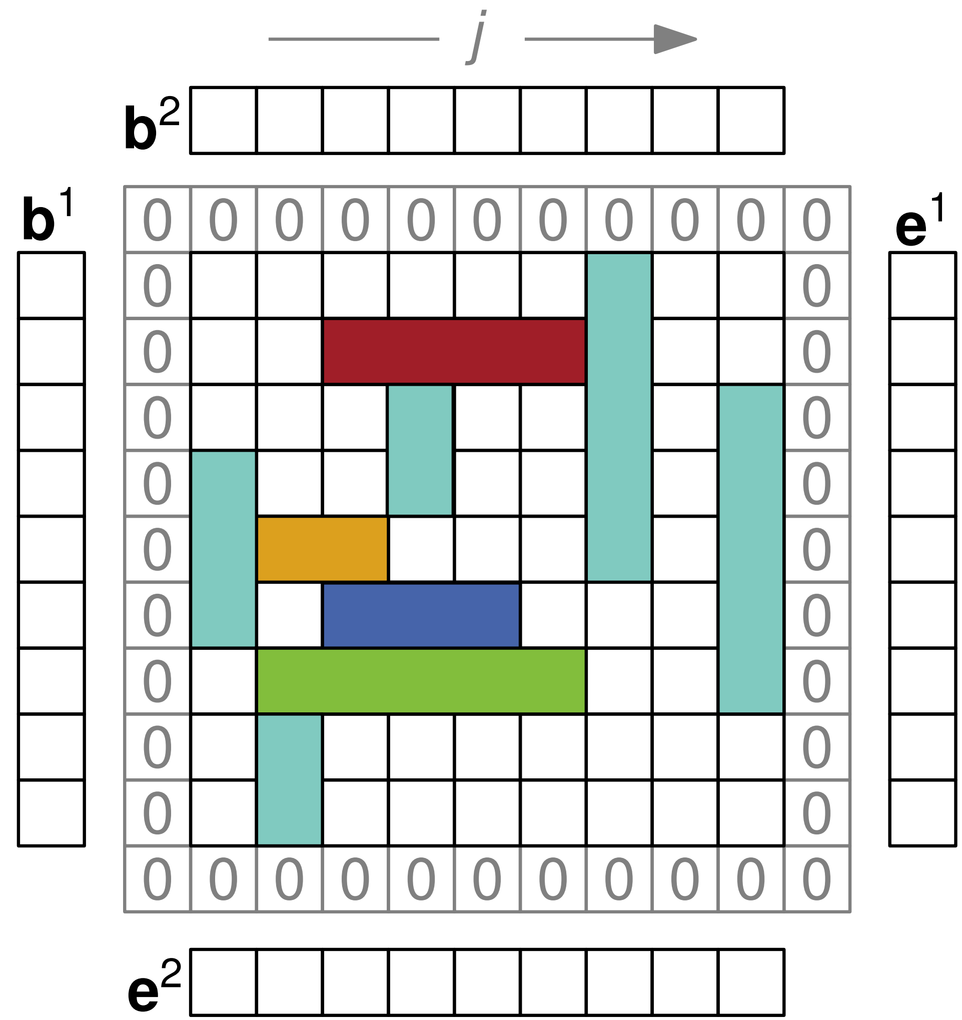
Modeling Visibility Representation



$G = (V, E)$

1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.



3.) Vertices are horizontal segments:

$$b_i^1(v) = e_i^1(v)$$

4.) Edges are vertical segments:

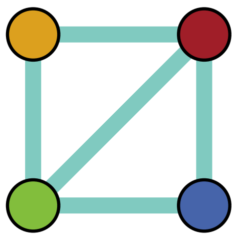
$$b_i^2(e) = e_i^2(e)$$

5.) Vertices do not overlap:

$$\sum_{v \in V} x_{i,j}(v) \leq 1$$

6.) Edges do not overlap non-incident vertices:

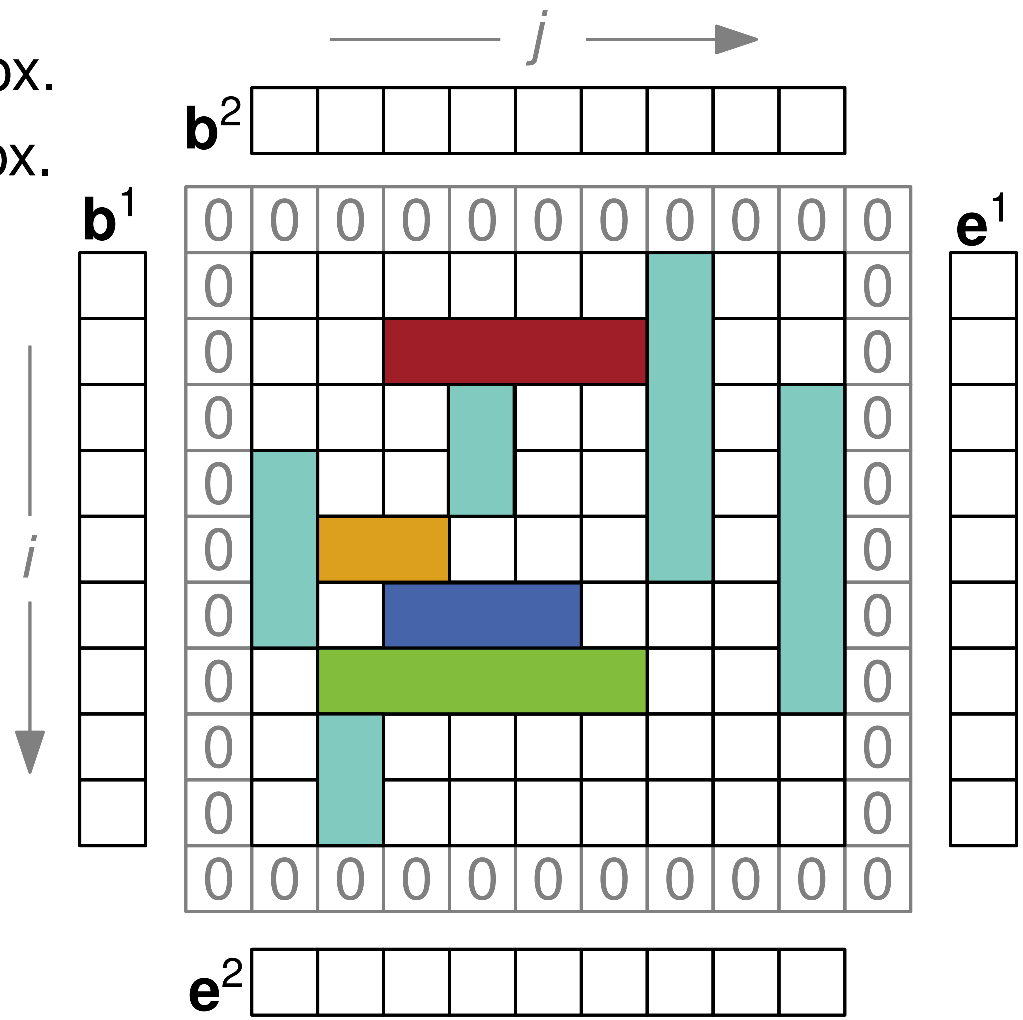
Modeling Visibility Representation



$G = (V, E)$

1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.



3.) Vertices are horizontal segments:

$$b_i^1(v) = e_i^1(v)$$

4.) Edges are vertical segments:

$$b_i^2(e) = e_i^2(e)$$

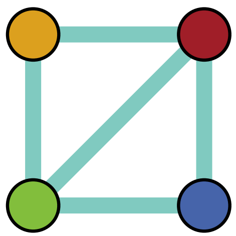
5.) Vertices do not overlap:

$$\sum_{v \in V} x_{i,j}(v) \leq 1$$

6.) Edges do not overlap non-incident vertices:

$$\sum_{v \in V \setminus e} x_{i,j}(v) \leq (1 - x_{i,j}(e)) \text{ for all } e \in E$$

Modeling Visibility Representation

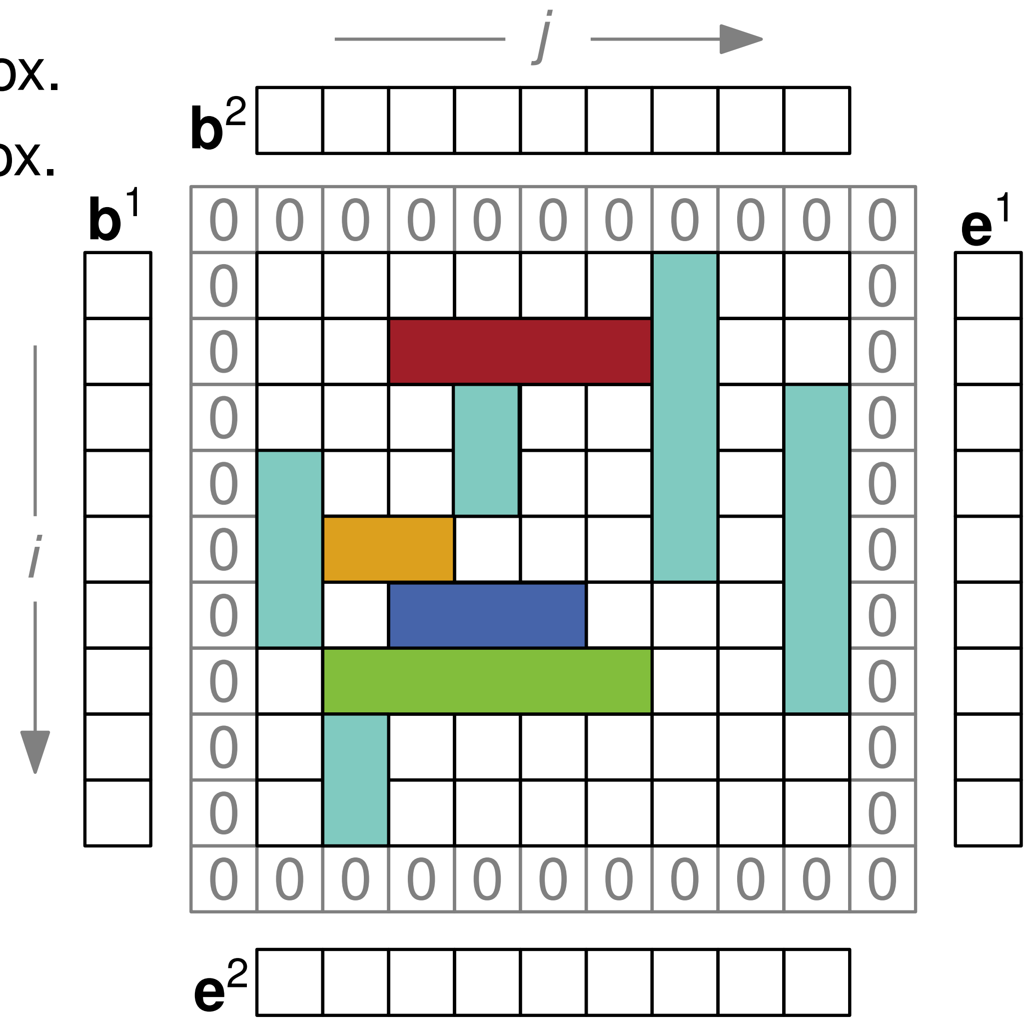


$G = (V, E)$

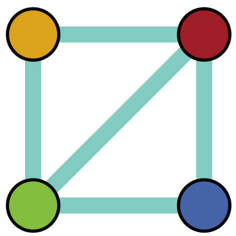
1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.

7.) Edges intersect incident vertices:



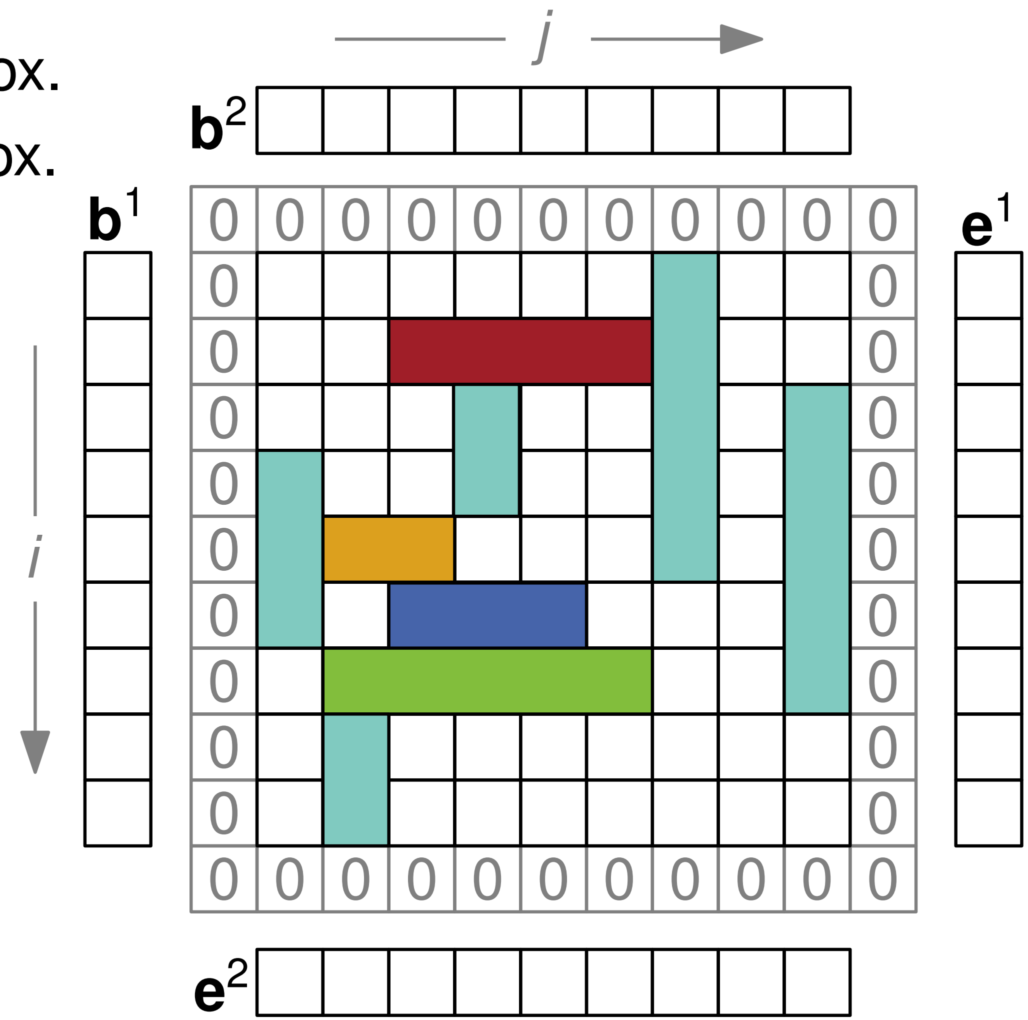
Modeling Visibility Representation



$G = (V, E)$

1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.



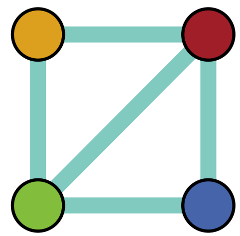
7.) Edges intersect incident vertices:

For all $e = \{u, v\} \in E$:

a.) Grid of binary variables

$x_{i,j}(e, u)$ $\quad x_{i,j}(e, v)$

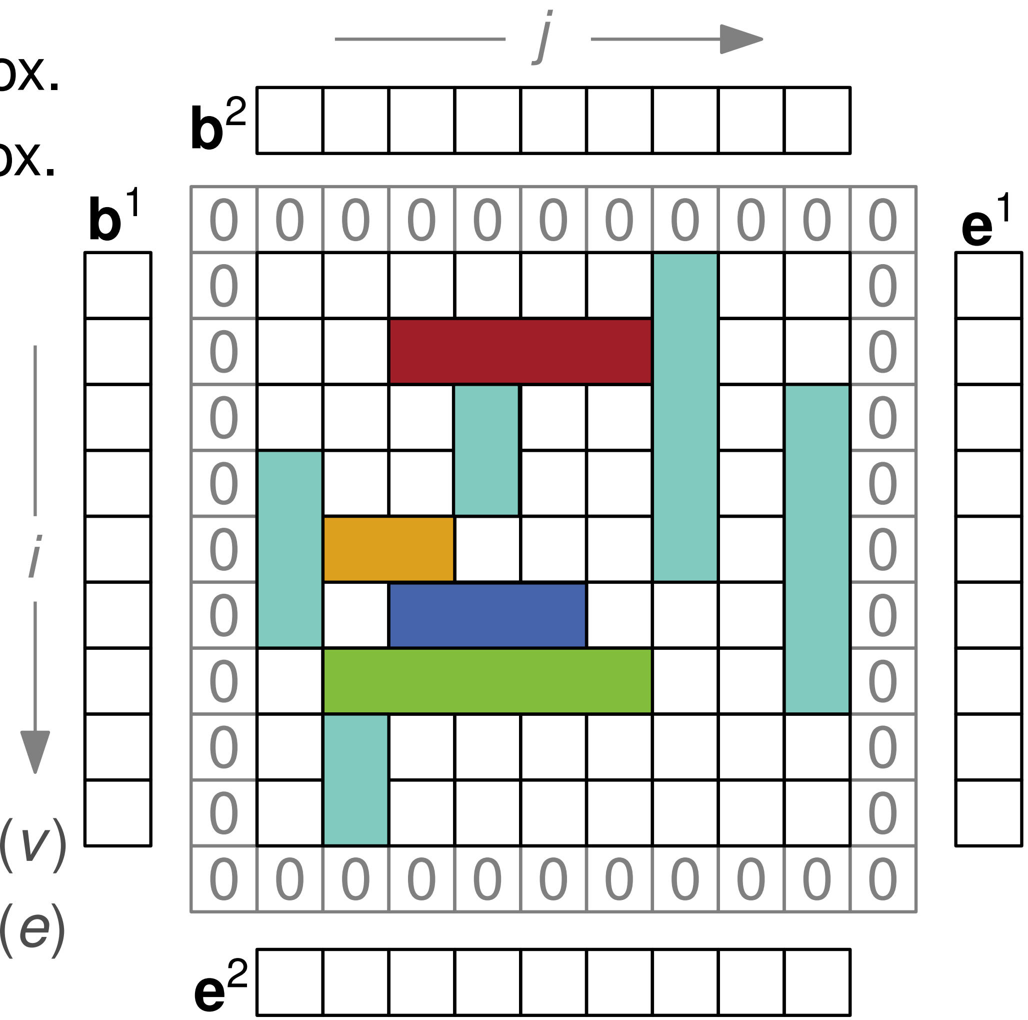
Modeling Visibility Representation



$G = (V, E)$

1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.



7.) Edges intersect incident vertices:

For all $e = \{u, v\} \in E$:

a.) Grid of binary variables

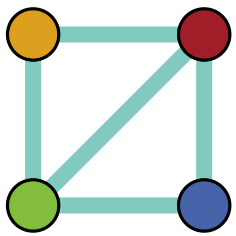
$x_{i,j}(e, u) \quad x_{i,j}(e, v)$

b.) Common point

$x_{i,j}(e, u) \leq x_{i,j}(u) \quad x_{i,j}(e, v) \leq x_{i,j}(v)$

$x_{i,j}(e, u) \leq x_{i,j}(e) \quad x_{i,j}(e, v) \leq x_{i,j}(e)$

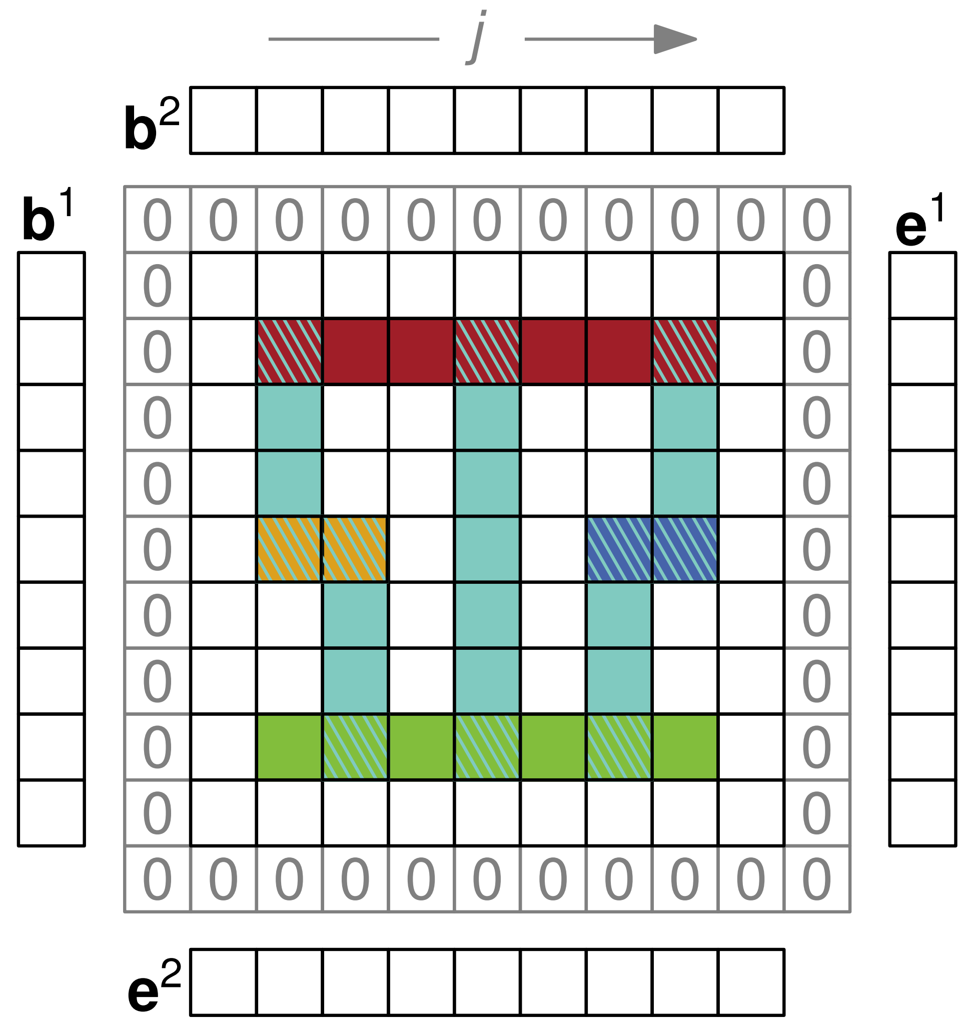
Modeling Visibility Representation



$G = (V, E)$

1.) $\forall v \in V$ introduce box.

2.) $\forall e \in E$ introduce box.



7.) Edges intersect incident vertices:

For all $e = \{u, v\} \in E$:

a.) Grid of binary variables

$x_{i,j}(e, u)$ $\quad x_{i,j}(e, v)$

b.) Common point

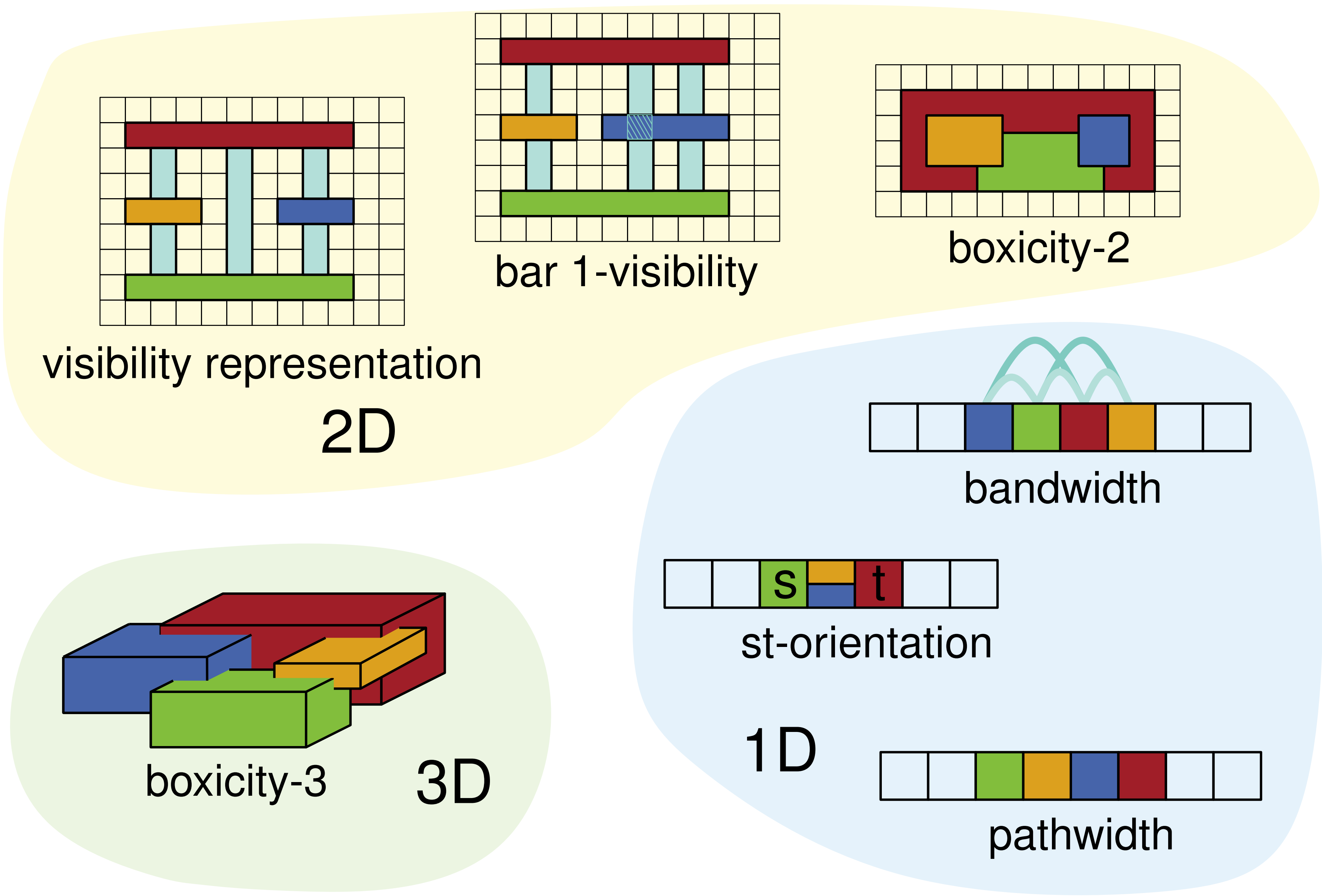
$x_{i,j}(e, u) \leq x_{i,j}(u)$ $\quad x_{i,j}(e, v) \leq x_{i,j}(v)$

$x_{i,j}(e, u) \leq x_{i,j}(e)$ $\quad x_{i,j}(e, v) \leq x_{i,j}(e)$

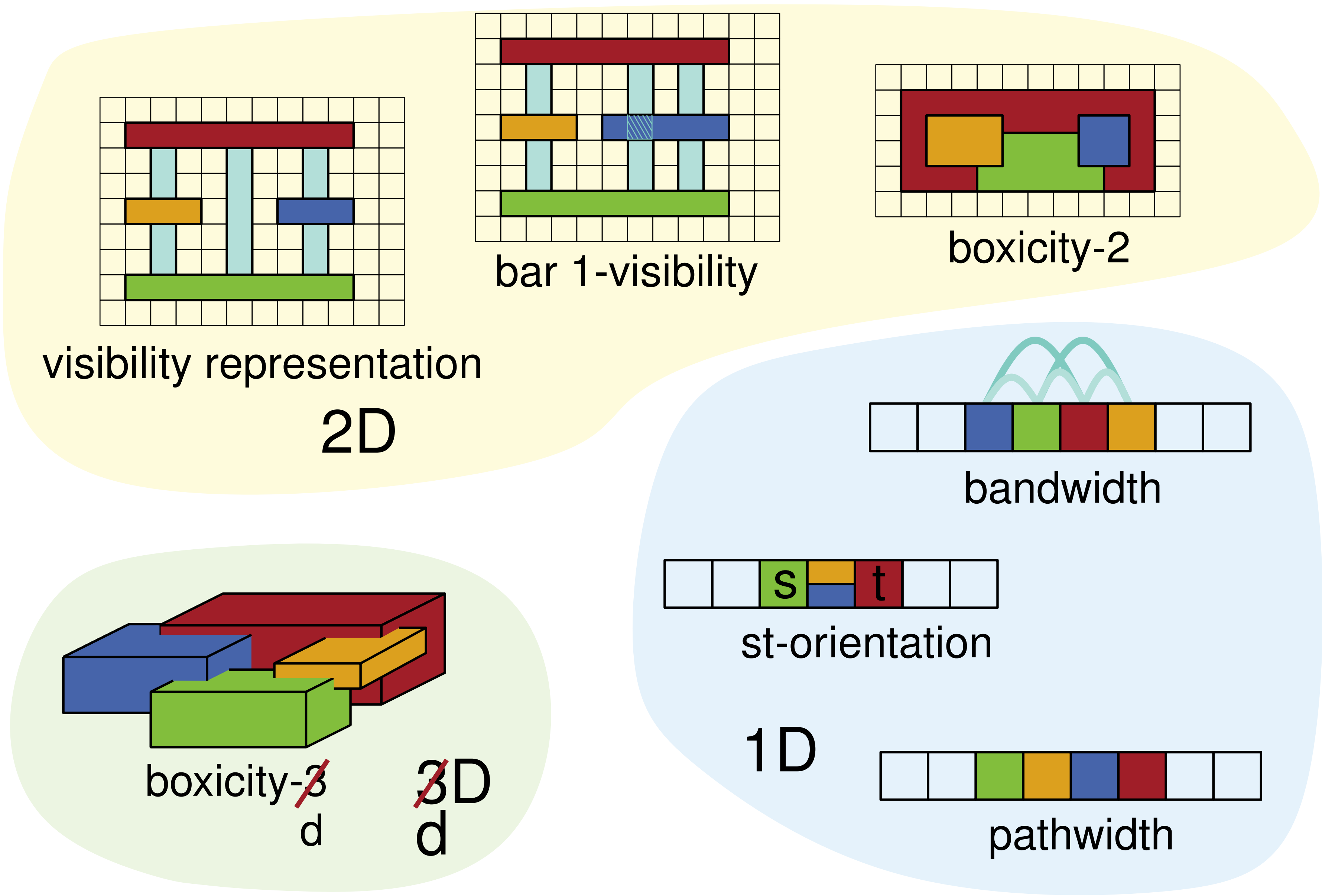
c.) Existence of common point.

$\sum_{i,j} x_{i,j}(e, u) \geq 1$ $\quad \sum_{i,j} x_{i,j}(e, v) \geq 1$

Overview – Considered Problems



Overview – Considered Problems



Summary

- flexible framework for modeling many grid-based graph layout problems
- suitable for NP-hard problems
- less efficient than tailored algorithms
- gets faster if ILP constraints are transformed into SAT
- easy-to-use tool PIGRA for working with the framework