Adjacency Labeling Schemes for Small Classes

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Example: Class $\mathcal{F}$ of forests has speed $|\mathcal{F}_n| \sim c^n \cdot n!$ for some $c > 0$.

labelling Schemes

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Given a class $\mathcal{C}$ find an algorithm $\mathcal{A}$ so that for every graph $G \in \mathcal{C}_n$ there is a vertex labelling $V(G) \mapsto \{0,1\}^*$ satisfying

$$\mathcal{A}(\ell(x), \ell(y)) = 1 \iff xy \in E(G), \quad \text{for every pair } x, y \in V(G).$$

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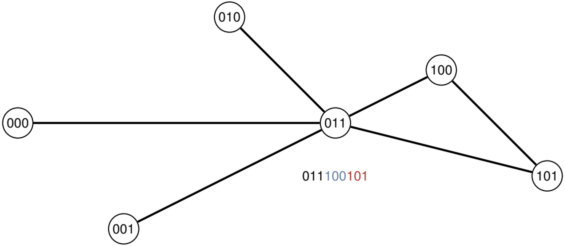
We say that $\mathcal{C}$ admits an $f(n)$-bit labelling scheme if the bit length of the longest label $\ell(v)$ of any vertex v of any graph $G \in \mathcal{C}_n$ is at most $f(n)$.

labelling Schemes - Bounded Degeneracy Example

G is *k*-degenerate if all subgraphs have a vertex with degree at most *k*.



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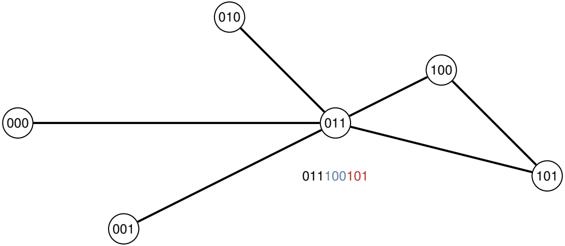

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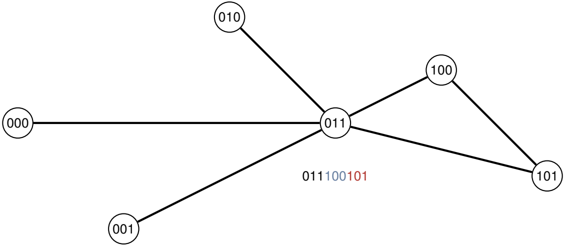

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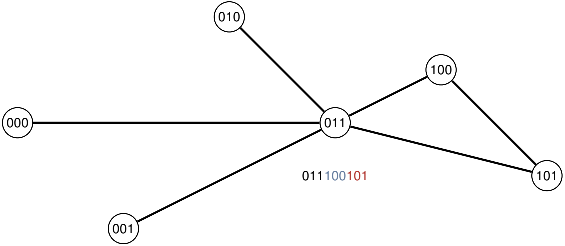

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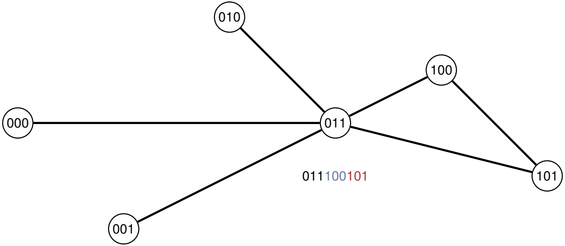
- 1st part of label: place in the order. 2nd part: the $\leq k$ neighbours after you.



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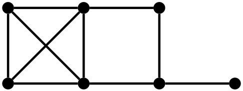

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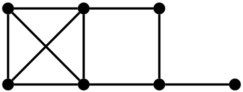

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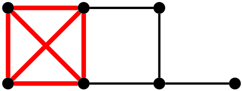

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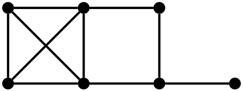

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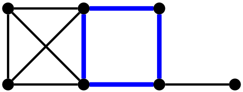

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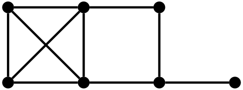

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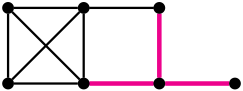

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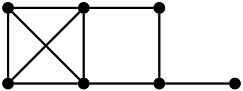

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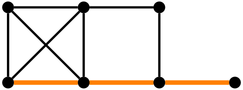

Link between labelling Schemes and Universal Graphs

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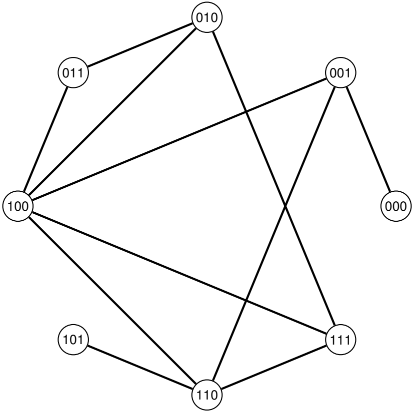

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Factorial: $|\mathcal{C}_n| = 2^{\Theta(n \log n)}$ (interval graphs, bounded degeneracy, unit disk...)

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Theorem [Hamed and Pooya Hatami, 2021]

For any $\delta > 0$, there exists a hereditary factorial class which does not admit an $n^{1/2-\delta}$-bit labelling scheme.

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First example of tight bounds for a class which are not "order optimal".

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This is a corollary of our more general result:

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- There are at least $2^{4m/5}$ subgraphs $T \subseteq F \subseteq H$ with $|\mathrm{Aut}(F)| \leqslant 2^{m/10}$.
- $$|\mathcal{C}_n| \geqslant \sum_F \frac{n!}{\mathrm{Aut}(F)} \geqslant 2^{4m/5} \cdot \frac{n!}{2^{m/10}} \geqslant n! \cdot 2^{7m/10} > n! \cdot c^n,$$

a contradiction.

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The Hatami brothers counterexample to the IGC is Weakly sparse!

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Theorem [Bonnet, Duron, S., Zamaraev 2025]

Every hereditary small class has neighbourhood complexity $\mathcal{O}(n \log n)$.

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Contiguity

G has contiguity k if one can find a linear order $v_1 \prec \cdots \prec v_n$ of $V(G)$ such that for any v_i, $N_G(v_i)$ is the disjoint union of k intervals.

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Used by Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, Toruńczyk for FO Model Checking on Monadically Stable Classes

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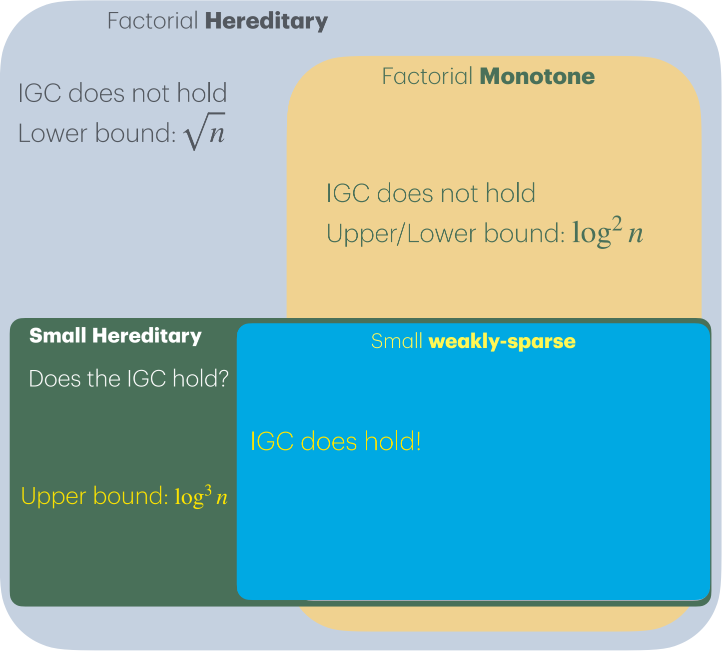
Factorial **Hereditary**
IGC does not hold
Lower bound: $\sqrt{n}$
Factorial **Monotone**
IGC does not hold
Upper/Lower bound: $\log^2 n$
Small Hereditary
Does the IGC hold?
Upper bound: $\log^3 n$
Small **weakly-sparse**
IGC does hold!

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