Crossing Number is NP-hard for Constant Path-width (and Tree-width)

Petr Hliněný
MUNI Brno, Czech Republic

Liana Khazaliya
TU Wien, Vienna, Austria

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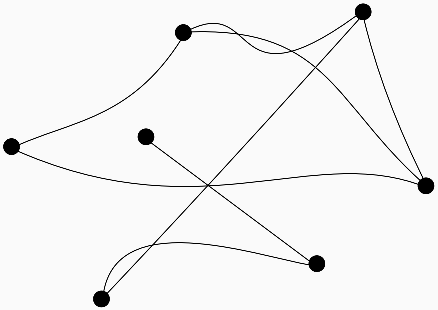
Part 1.

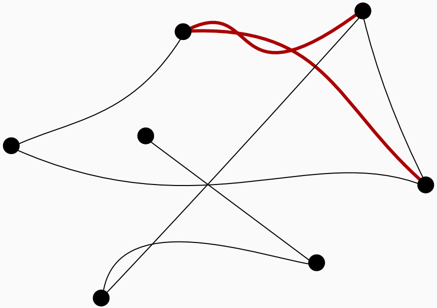
Crossing Number: Overview

Crossings and Crossing Number



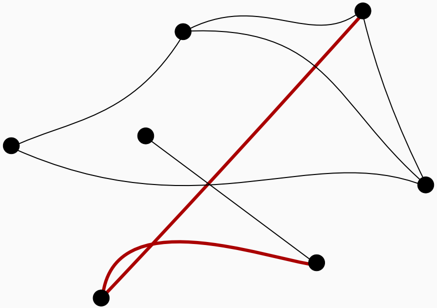
Vertices of G are distinct points in the plane; every edge uv is a simple arc joining u to v.

Crossings and Crossing Number



- any pair of edges crosses at most once;

Crossings and Crossing Number



- any pair of edges crosses at most once;
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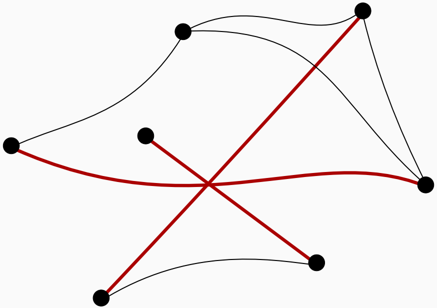

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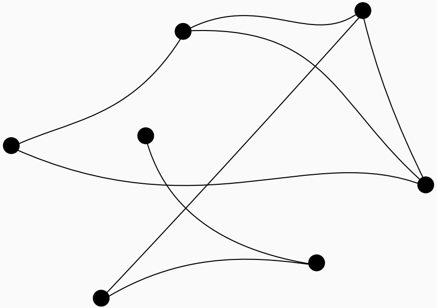
Crossings and Crossing Number



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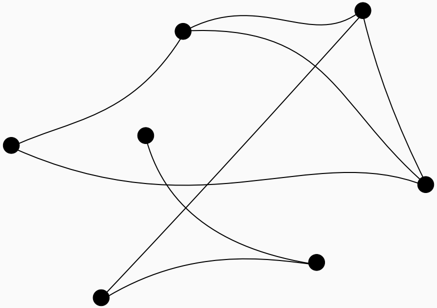
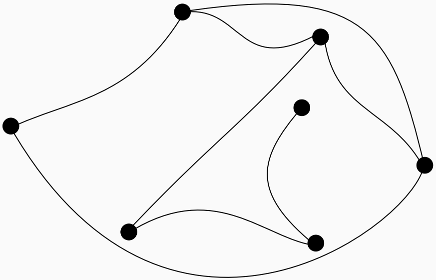


4 crossings



0 crossings

The minimum such k for a given G is the crossing number $\mathrm{cr}(G)$.

Crossing Number

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Crossing Number
Input: A graph G and $k \in \mathbb{Z}_{\geq 0}$
Question: Does there exist a drawing $\mathcal{G}$ of G with $\leq k$ edge crossings

Some examples

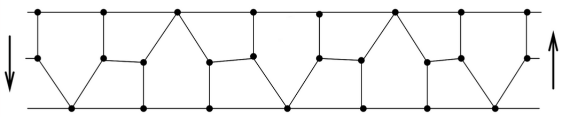
- $\mathrm{cr}(K_5) = 1$, $\mathrm{cr}(K_6) = 3$, ..., $\mathrm{cr}(K_{12}) = 150$

  but $\mathrm{cr}(K_{13})$ is still unknown

  Conjecture. $\mathrm{cr}(K_n) = \frac{1}{4} \cdot \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor \cdot \lfloor \frac{n-2}{2} \rfloor \cdot \lfloor \frac{n-3}{2} \rfloor$

- The two *minimal* graphs of the crossing number ≥ 1 are K_5 and $K_{3,3}$.

- There exists an infinite family of simple 3-connected graphs that are minimal to having the crossing number ≥ 2: [Kochol, 1987]



Brief complexity status

NP-hardness

- The general case [Garey and Johnson, 1983]
- The degree-3 and minor-monotone cases [Hliněný, 2004]
- And for almost-planar (planar graphs plus one edge) [Cabello and Mohar, 2010]

Approximations

- No constant factor approximation for some $c > 1$ [Cabello, 2013]
- Randomized subpolynomial-approximation when bounded degree [Chuzhoy and Tan, 2022]

$$\left(2^{\mathcal{O}(\log^{7/8} n \cdot \log\log n)} \cdot \Delta^{\mathcal{O}(1)}\right)$$

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Brief complexity status

Parameterized complexity

- FPT with parameter k (number of crossings)

$$f(k) \cdot n^2, \quad f(k) = 2^{2^{2^{2^{\Omega(k)}}}}$$

[Grohe, STOC'01]

$$f(k) \cdot n, \quad f(k) = 2^{\mathcal{O}((k+\mathrm{tw})\log(k+\mathrm{tw}))}$$

[Verdiére and Magnard, ESA'21]

$$f(k) \cdot n, \quad f(k) = 2^{\mathcal{O}(k \log k)}$$

[Lokshtanov, Panolan, Saurabh, Sharma, Xue, Zehavi, SODA'25]

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And what about structural parameters? Surprisingly, nearly nothing

- FPT algorithm for $\mathrm{cr}(G)$ param. by the vertex cover [Sankaran and Hliněný, 2019]
- Poly alg. for $\mathrm{cr}(G)$ when G is maximal path-width 3 [Biedl, Chimani, Derka, and Mutzel, 2020]

Theorem [HK'24]

Crossing Number (G, k) is NP-complete even when a given graph is of path-width at most 12 and of tree-width at most 9.

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Theorem **[HK'24]**

Crossing Number (G, k) is NP-complete even when a given graph is of path-width at most 12 and of tree-width at most 9.

Part 2.

Crossing Number: NP-hardness

Result

Theorem [HK'24]

CROSSING NUMBER (G, k) is NP-complete even when a given graph is of path-width at most 12 and of tree-width at most 9.

Reduction.

SATISFABILITY: $(\mathcal{V}, \mathcal{C})$ $\rightarrow$ CROSSING NUMBER: (G, k)

$$\mathrm{pw}(G) = 9$$

$$\mathrm{tw}(G) = 12$$

Satisfability

SATISFABILITY
Input: A set of clauses $\mathcal{C} = \{C_1, \ldots, C_\ell\}$ over variables $\mathcal{V} = \{x_1, \ldots, x_n\}$
Question: Does there exist an assignment $\tau : \mathcal{V} \to \{\texttt{True}, \texttt{False}\}$ satisfying all clauses in $\mathcal{C}$?

`Reduction Idea.`

- a large "grid structure"
- small separators
- "flips" of some parts for the encoding
- clause edges that cause an equivalence

Color-encoding

Weighted crossing number

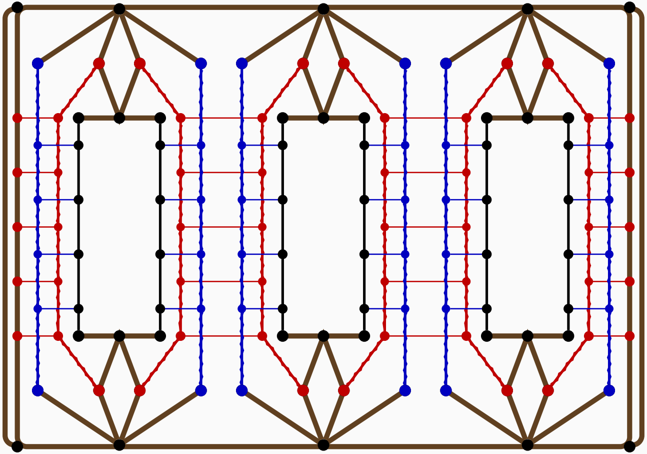
The ordinary crossing number, but

- an edge replaced by a bunch of several parallel;
- redraw the bunch tightly along the “cheapest”;
- a crossing contributes the product of edge weights.

Color	Weight
Heavy-brown (HB)	ω^8
Light-black (LB)	ω^6
Red (R)	$\omega^4 + \Theta_{n,\ell}(\omega^1)$
(R’)	ω^3
Blue (B)	$\omega^4 + \Theta_{n,\ell}(\omega^1)$
(B’)	ω^3
Cyan (C)	ω^2
Green (G)	$\omega^0 = 1$

Let $\omega = |E(G)|^2$, then one crossing of weight ω^{t+1} “outweighs” all crossings of G of weight ω^t.

The frame and variable gadgets



The Frame with n Variable Gadgets for $n = 3$, $h = 4$

Color	Weight
HB	ω^8
LB	ω^6
R	$\omega^4 + \Theta_{n,\ell}(\omega^1)$
R' (hor)	ω^3
B	$\omega^4 + \Theta_{n,\ell}(\omega^1)$
B' (hor)	ω^3
C	ω^2
G	$\omega^0 = 1$

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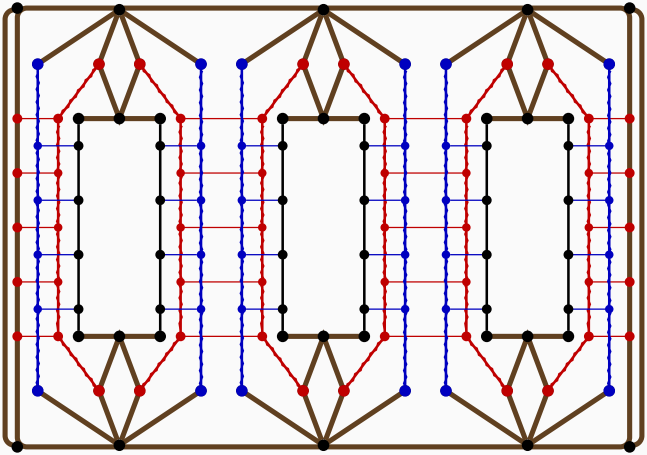

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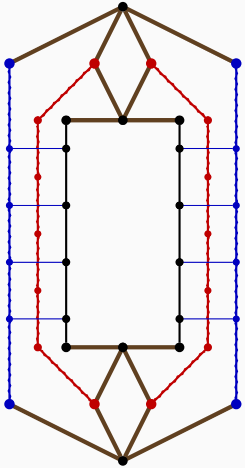
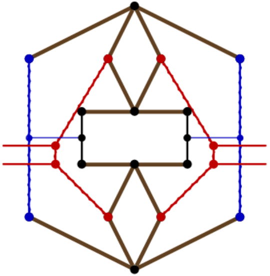


Edge-alternation

$g_j = \omega^4 + j(j+1)\omega$;

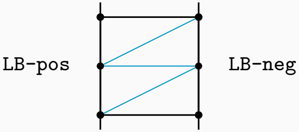
$s_j = \omega^4 + j(j+2)\omega$;

all horizontal ($\mathtt{R}/\mathtt{B}$) are of ω^3.

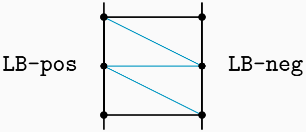


(a) $(2s_1 + g_3)\omega^3 = 3\omega^7 + 18\omega^4$ **(b)** $(s_1 + g_2 + s_2)\omega^3 = 3\omega^7 + 17\omega^4$ **(c)** $(g_1 + 2s_2)\omega^3 = 3\omega^7 + 18\omega^4$

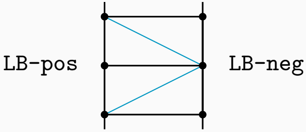
Encoding



$C_{\text{pos}}\colon\ x \in C$

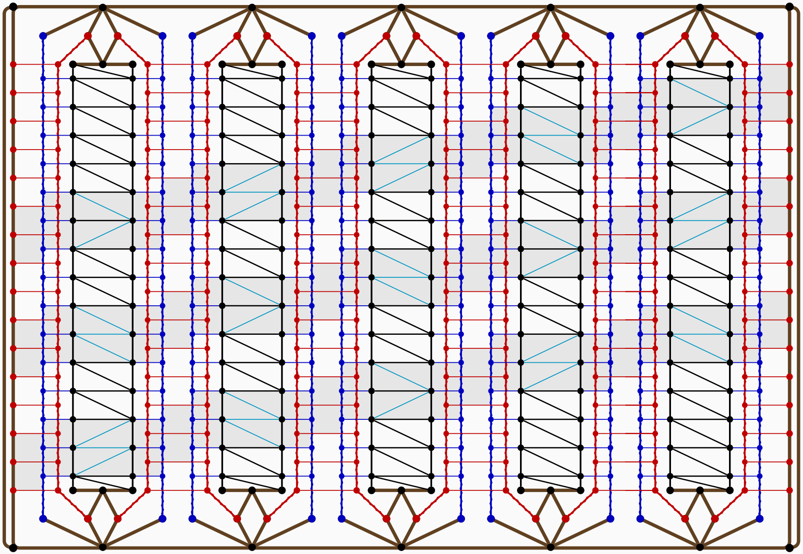


$C_{\text{neg}}\colon\ \overline{x} \in C$



C_{neut}

$\mathcal{C} = \{(x_1 \vee \overline{x_2} \vee x_4 \vee \overline{x_5}), (\overline{x_1} \vee \overline{x_3} \vee x_5), (x_2 \vee x_3 \vee \overline{x_4})\}$



$h = 4\ell + n - 2$

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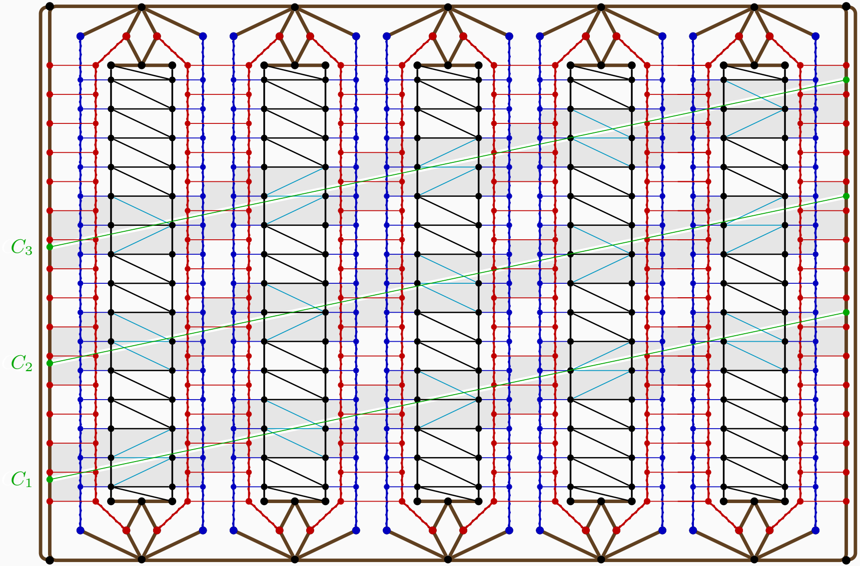

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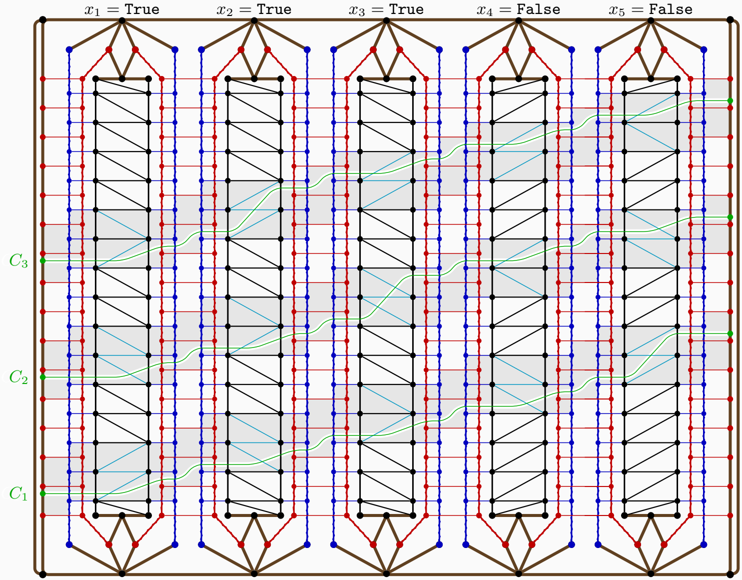

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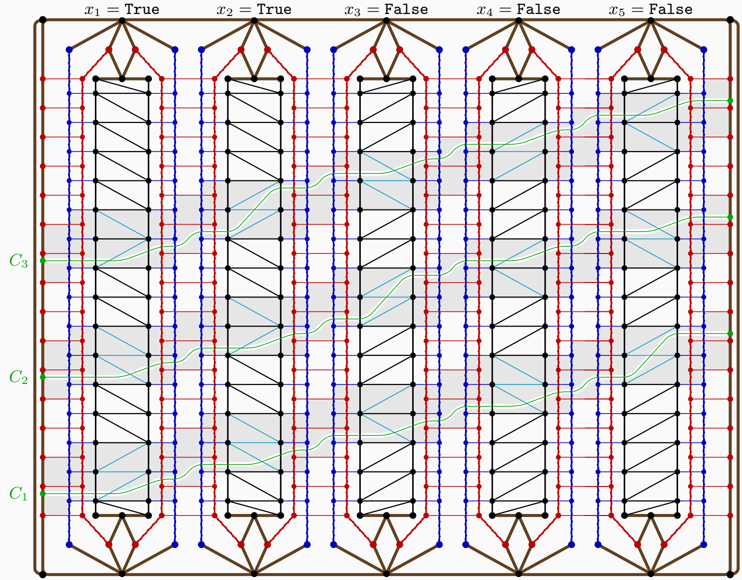

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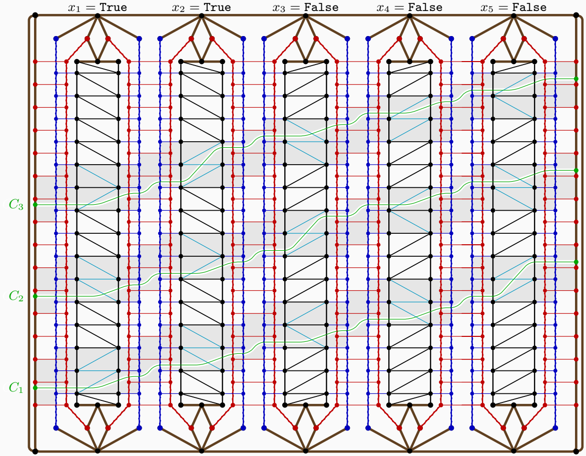

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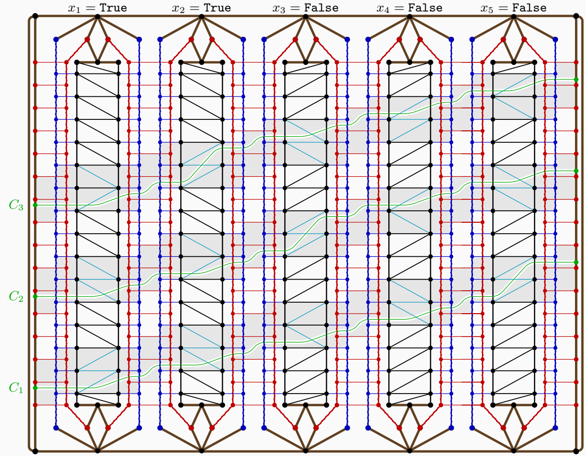
C	ω^2
G	$\omega^0 = 1$

$$k = 2n(2h+1)\omega^7 + 2n\ell\omega^6 + 4n\ell\omega^4 + 2n\sum_{j=2}^{h+1} j(j+1)\omega^4 + 2n\sum_{j=1}^{h+1} j(j+2)\omega^4 + n\ell\omega^2 + (\omega^2 - 1)$$

$$\mathcal{C} = \{(x_1 \vee \overline{x_2} \vee x_4 \vee \overline{x_5}), (\overline{x_1} \vee \overline{x_3} \vee x_5), (x_2 \vee x_3 \vee \overline{x_4})\}$$



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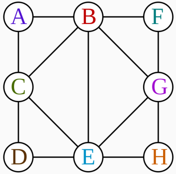
Part 3.

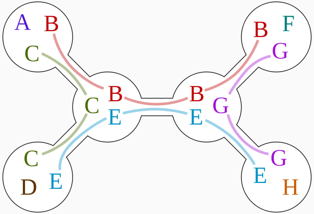
Tree- and Path-width

Tree-width

Def. A tree decomposition of G is a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a bag, with following conditions:



- $\mathcal{T}1$. $\bigcup_{t \in V(T)} X_t = V(G)$;
- $\mathcal{T}2$. For every $vw \in E(G)$, there exists a node t of T such that bag X_t contains both v and w;
- $\mathcal{T}3$. For every $v \in V(G)$, the set $T_v = \{t \in V(T) | v \in X_t\}$ induces a connected subtree of T.



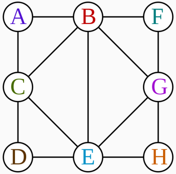
Def. The width of $\mathcal{T}$ is $\max_{t \in V(T)} |X_t| - 1$.

Def. The tree-width $\mathrm{tw}(G)$ is the **minimum** width over all tree decompositions of G.

Tree-width

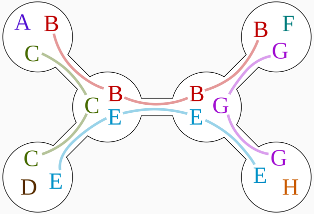
The tree-width of a graph G is

$$\min\{\omega(G^+) - 1 : G^+ \supseteq G \text{ and } G^+ \text{ is chordal}\}$$



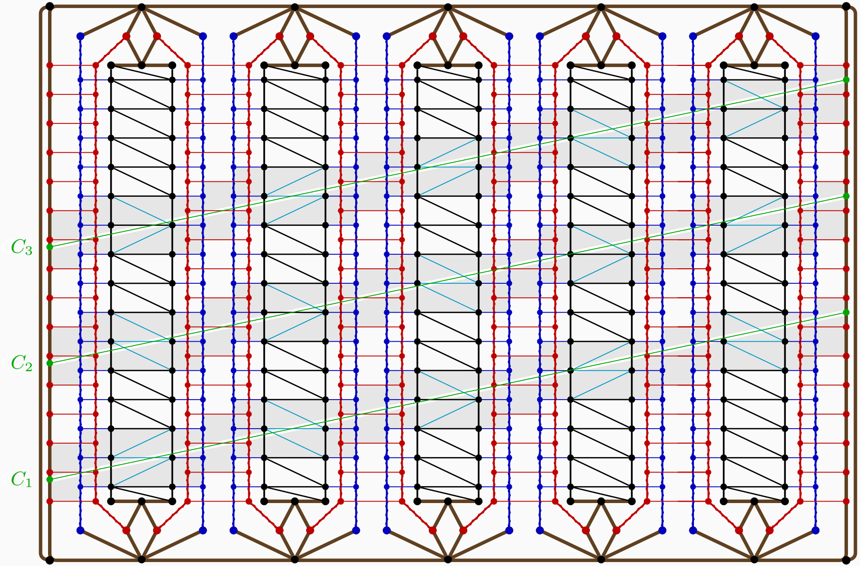
The Cops-and-Robber Game

Tree-width [path-width] is at most t if and only if $t+1$ cops can always catch the robber in G in a monotone game if the robber is *visible* [*invisible*] (to the cop player)



$\mathrm{tw}(K_n) = n - 1$ $\qquad$ $\mathrm{tw}(P_n \times P_m) = \min(m, n)$ $\qquad$ $\mathrm{tw}(T) = 1$

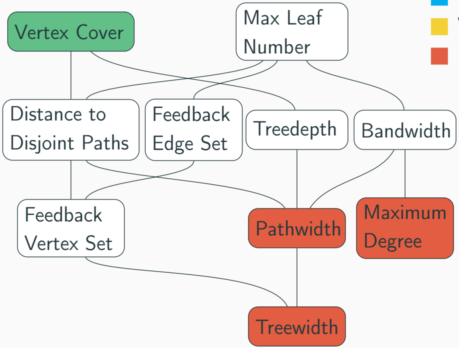
Back to the construction



Part 4.

Conclusion

Question: [In]tractability for structural parameters

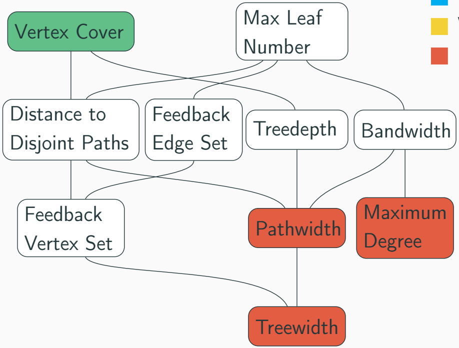


- FPT   ($f(k) \cdot n^{O(1)}$-time algorithm)
- XP   ($n^{f(k)}$-time algorithm)
- W[1]-hard   (not FPT unless FPT = W[1])
- para-NP-hard   (not XP unless P = NP)

n: size of input

k: size of parameter

Thank you for your attention!



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