Non-Clashing Teaching in Graphs: Algorithms, Complexity, and Bounds

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Notation and Definitions

Let G be a simple, finite, and undirected graph.

For an integer $r \geq 0$ and a vertex $v \in V(G)$, the *ball* $B_r(v)$ is the set of all vertices at distance at most r from its *center* v.

Let $\mathcal{B}$ be a set of balls of G.

A (*positive*) *teaching map* T for $\mathcal{B}$ is a mapping which assigns to each ball $B \in \mathcal{B}$ a *teaching set* $T(B) \subseteq V(G)$ (pos: $T(B) \subseteq B$).

The *dimension* of T is $\max_{B \in \mathcal{B}} |T(B)|$.

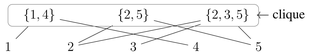
A teaching map T is *non-clashing* for $\mathcal{B}$ if, for each pair of distinct balls $B^1, B^2 \in \mathcal{B}$, there exists a vertex $w \in T(B^1) \cup T(B^2)$ such that $w \in B^1 \cup B^2$ and $w \notin B^1 \cap B^2$.

A *closed neighborhood* $N[v]$ of $v \in G$ is a ball of radius 1, *i.e.* $N[v] = B_1(v)$.

Non-Clashing Teaching: NCTD (NCTD$^+$)

INPUT: A graph G, a set $\mathcal{B}$ of closed neighborhoods in G, a positive integer k.

QUESTION: Is there a (positive) non-clashing teaching map for $\mathcal{B}$ with dimension at most k?



Graph G for a binary concept class $\mathcal{C} = \{\{1,4\},\{2,5\},\{2,3,5\}\}$ and $\mathcal{B} = \{N[C]\}_{C \in \mathcal{C}}$.

Overview of the Results & Other Related Work:

	NCTD$^+$ Balls	NCTD$^+$ CN	NCTD Balls	NCTD CN
Solution size k	NP-h if $k=1$	NP-h if $k=1$	NP-h if $k=1$	NP-h if $k=1$
Vertex cover	FPT [1]	FPT	?	FPT
Vertex integrity	FPT [1]	FPT	?	?
Treedepth	?	FPT	?	?
Treewidth	W[1]-hard [1]	?	?	?
Alg. up. bound	$2^{\mathcal{O}(k \cdot d \cdot \lvert V\rvert \cdot \log \lvert V\rvert)}$ [1]	$2^{\mathcal{O}(\lvert E\rvert)}$	$2^{\mathcal{O}(k \cdot d \cdot \lvert V\rvert \cdot \log \lvert V\rvert)}$ [1]	$2^{\mathcal{O}(k \cdot \lvert V\rvert \cdot \log \lvert V\rvert)}$ [1]
Alg. low. bound	$2^{o(k \cdot d \cdot \lvert V\rvert)}$ [1]	$2^{o(f(k) \cdot (\lvert V\rvert + \lvert E\rvert))}$	$2^{o(f(k) \cdot \lvert V\rvert)}$	$2^{o(f(k) \cdot \lvert V\rvert)}$
Planar	Unbounded by cycle [2]	≤ 7	≤ 615 [2]	≤ 5
Unit square	Unbounded by cycle [2]	≤ 4	≤ 615 [2]	≤ 4

NP-h, CN, [1], and [2] denote NP-hard, closed neighborhoods, Ganian et al. 2025, and Chalopin et al. 2024, resp.
Blue-shaded cells indicate the results presented in this paper.

Introduced the concept
[Kirkpatrick et al., 2019; Fallat et al., 2023]

Most efficient batch teaching model satisfying Goldman-Mathias collusion-avoidance criterion
[Kirkpatrick et al., 2019; Fallat et al., 2023]

Initial study of NCTD and NCTD$^+$ for Balls
[Chalopin et al., 2024]

Follow-up study of NCTD$^+$ for Balls
[Ganian et al., 2025]

Intractability and Running Time Bounds

Theorem.
For any computable function f, there is no $2^{o(f(k) \cdot (|V(G)| + |E(G)|))}$ time algorithm for NCTD$^+$ assuming ETH*.

*The *Exponential Time Hypothesis (ETH)*: There is a constant $c > 0$ such that there is no 2^{cn} algorithm for 3-SAT.

[Impagliazzo & Paturi, 2001]

Proof: a reduction from the 3-SATISFIABILITY problem.

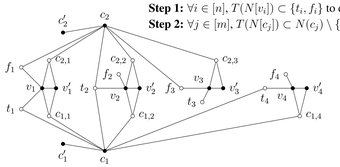
3-SATISFIABILITY (3-SAT)

INPUT: A CNF formula over a set of clauses $\mathcal{C} = \{c_1, \ldots, c_m\}$ containing variables from $\mathcal{X} = \{x_1, \ldots, x_n\}$, where each clause has exactly 3 literals.

QUESTION: Is there a variable assignment $\tau : \mathcal{X} \to \{\texttt{True}, \texttt{False}\}$ satisfying each clause in $\mathcal{C}$?

Example. The graph G obtained by applying our reduction on the 3-SAT instance
$\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{C} = \{(\overline{x_1} \vee x_2 \vee \overline{x_4}), (x_1 \vee \overline{x_2} \vee x_3)\}$.

Step 1: $\forall i \in [n], T(N[v_i]) \subset \{t_i, f_i\}$ to distinguish $N[v_i]$ & $N[v_i']$
Step 2: $\forall j \in [m], T(N[c_j]) \subset N(c_j) \setminus \{c_j'\}$ to distinguish $N[c_j]$ & $N[c_j']$



Only the closed neighborhoods of filled vertices are in $\mathcal{B}$.

Twins Lemma.
Let G be a graph with 4 pairwise *false twins** $u_1, \ldots, u_4$ with $N[u_1], \ldots, N[u_4] \in \mathcal{B}$. Then, for any non-clashing teaching map T of size 1 for $\mathcal{B}$, there exists $i \in [4]$ such that $T(N[u_i]) = \{u_i\}$.

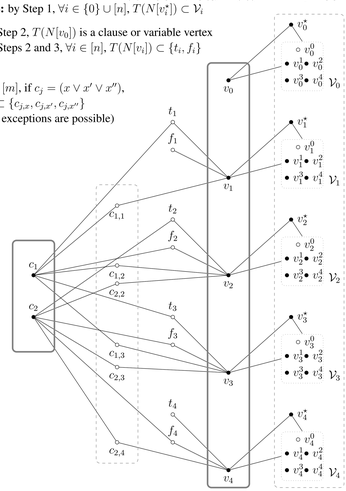
*Given a graph G, two distinct vertices $u, v \in V(G)$ are called *false twins* if and only if $N[v] \setminus \{v\} = N[u] \setminus \{u\}$.

Theorem.
For any computable function f, there is no $2^{o(f(k) \cdot |V(G)|)}$ time algorithm for NCTD assuming ETH.

Proof: a reduction from 3-SATISFIABILITY.

Example. The graph G obtained by applying our reduction on the 3-SAT instance
$\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{C} = \{(\overline{x_1} \vee x_2 \vee \overline{x_3}), (\overline{x_2} \vee x_3 \vee x_4)\}$.

Step 1: by Twins Lemma, $\forall i \in \{0\} \cup [n], T(N[v_i^p]) = \{v_i^p\}$ for some $p \in [4]$
Step 2: by Step 1, $\forall i \in \{0\} \cup [n], T(N[v_i^*]) \subset \mathcal{V}_i$
Step 3: by Step 2, $T(N[v_0])$ is a clause or variable vertex
Step 4: by Steps 2 and 3, $\forall i \in [n], T(N[v_i]) \subset \{t_i, f_i\}$
Step 5: by Step 4, $\forall j \in [m]$, if $c_j = (x \vee x' \vee x'')$, then $T(N[c_j]) \subset \{c_{j,x}, c_{j,x'}, c_{j,x''}\}$ (however, some exceptions are possible)



The two bold rectangles together form a clique. The edges going from vertices in one of the two large dashed rectangles to vertices in the other are omitted for legibility. Only the closed neighborhoods of filled vertices are in $\mathcal{B}$.

Tractability* for Structural Parameterizations

*In *parameterized complexity*, the running-times of algorithms are studied with respect to a parameter $p \in \mathbb{N}$ and input size n. A parameterized problem is *fixed-parameter tractable* (FPT) if it can be solved by an algorithm running in time $f(p) \cdot n^{\mathcal{O}(1)}$, where f is a computable function.

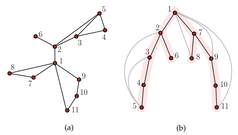
[Downey & Fellows, 1999; Cygan et al., 2015]

Theorem.
NCTD is fixed-parameter tractable parameterized by the vertex cover number* of G.

*$\texttt{vc}(G)$: the *vertex cover number* of G is the cardinality of a smallest vertex subset $X \subset V(G)$ such that $G - X$ is edgeless.

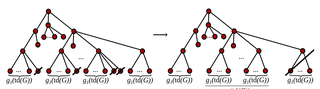
Theorem.
NCTD$^+$ is fixed-parameter tractable parameterized by the treedepth* of G.

*$\texttt{td}(G)$: the *treedepth* of a connected graph G is the minimum height of a rooted tree $\mathcal{T}$ such that $V(\mathcal{T}) = V(G)$ and, for all $uv \in E(G)$, either u is an ancestor of v or v is an ancestor of u in $\mathcal{T}$.
The tree $\mathcal{T}$ is a *treedepth decomposition* of G.

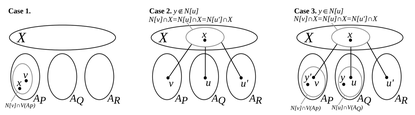


(a) A graph G. (b) A treedepth decomposition $\mathcal{T}$ of G witnessing $\texttt{td}(G) \leq 4$.

Proof: by *pruning* of a treedepth decomposition.



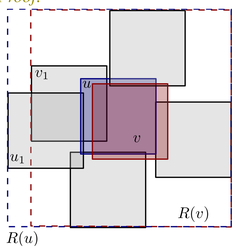
Reduction Rule. Let G be a graph, $X \subseteq V(G)$ a subset of its vertices, and $A = \{A_1, \ldots, A_\ell\}$ a subset of the connected components of $G - X$ such that $\max_{i \in [\ell]} |A_i| = t$. If $\ell > (|X| + t) \cdot 2^{(|X|+t)^2} \cdot 2^{2t+|X|+1}$, then delete a particular connected component in A.



Combinatorial Upper Bounds

Theorem.
For any unit square graph G, if $\mathcal{B} = \{N[v] \mid v \in V(G)\}$, then NCTD$^+(\mathcal{B}) \leq 4$.

Proof:



$T(N[v])$ is the vertices represented by the leftmost, rightmost, topmost, and bottommost squares in $R(v)$ (they may not be unique).

$R(u) \neq R(v)$: $\exists x \in T(N[u]) \cup T(N[v])$ that distinguishes $N[u]$ and $N[v]$;

$R(u) = R(v)$: $N[u] = N[v]$ as each square in $R(v)$ ($R(u)$, resp.) intersects v (u, resp.).

In the proof, the minimum rectangles $R(u)$ (blue) and $R(v)$ (red) enclosing $N[u]$ and $N[v]$, respectively. The leftmost square in $R(u)$ ($u_1 \in T(N[u])$) is not contained in $R(v)$.

Theorem.
For any planar graph G, if $\mathcal{B} = \{N[v] \mid v \in V(G)\}$, then NCTD$^+(\mathcal{B}) \leq 7$.

Theorem.
For any planar graph G, if $\mathcal{B} = \{N[v] \mid v \in V(G)\}$, then NCTD$(\mathcal{B}) \leq 5$.

Proof:
For all $v \in V(G)$, if $d(v) \leq 4$, then set $T(N[v]) := N[v]$.
Otherwise, for each $v \in V(G)$, place any 3 neighbors (v_1, v_2, v_3) of v in $T(N[v])$. There is at most one other vertex $u \in V(G)$ such that $v_1, v_2, v_3 \in N(u)$, since G is planar.

If there exists $N[v], N[y] \in \mathcal{B}$ such that $d(v) \geq 5$, $N[v] \neq N[y]$, and T does not currently satisfy the non-clashing condition, then there exists such a vertex $w \in (N[v] \setminus N[y]) \cup (N[y] \setminus N[v])$ that will be added to $T(N[v])$ or $T(N[y])$ to distinguish the pair.

Now, T is a positive non-clashing teaching map for $\mathcal{B}$ and for each $v \in V(G)$ with $d(v) \geq 5$. But at most 2 vertices were added to any teaching set during the above process as the vertices in $\{u, v, v_1, v_2, v_3\}$ cannot form a K_5 (if $v_2 v_3 \notin E(G)$, then $y \in \{u, v_1\}$).