Problems in NP can Admit Double-Exponential Lower Bounds when Parameterized by Treewidth or Vertex Cover

Florent Foucaud, Esther Galby, Liana Khazaliya, Shaohua Li, Fionn Mc Inerney, Roohani Sharma, Prafullkumar Tale

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Part 1.
(In)tractability and Treewidth
Fixed-parameter tractability is a framework to deal with intractable problems:

- Choose a complexity parameter $k$ independent of the input size $n$
- Find an OPT solution in time $f(k) \cdot n^{O(1)}$ for some function $f$

Develop algorithms for graphs which are large but have a small solution size

...or simply structured
Intractable problems and approaches

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Develop algorithms for graphs which are large but have a small solution size

...or simply structured
**Def.** A tree decomposition of $G$ is a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where $T$ is a tree whose every node $t$ is assigned a vertex subset $X_t \subseteq V(G)$, called a bag, with following conditions:

$\mathcal{T}1$. $\bigcup_{t \in V(T)} X_t = V(G)$;

$\mathcal{T}2$. For every $vw \in E(G)$, there exists a node $t$ of $T$ such that bag $X_t$ contains both $v$ and $w$;

$\mathcal{T}3$. For every $v \in V(G)$, the set $T_v = \{t \in V(T) | v \in X_t\}$ induces a connected subtree of $T$.

**Def.** The width of $\mathcal{T}$ is $\max_{t \in V(T)} |X_t| - 1$.

**Def.** The treewidth $tw(G)$ is the minimum width over all tree decompositions of $G$. 
The treewidth of a graph $G$ is

$$\min \{ \omega(G^+) - 1 : G^+ \supseteq G \text{ and } G^+ \text{ is chordal} \}$$

The Cops-and-Robber Game

Treewidth is at most $t$ if and only if $t + 1$ cops can always catch the robber in $G$ in a monotone game if the robber is visible (to the cop player)

$$\text{tw}(K_n) = n - 1 \quad \text{tw}(P_n \times P_m) = \min(m, n) \quad \text{tw}(T) = 1$$
Many **NP-hard** problems are **FPT** parameterized by **treewidth** via dynamic programming on the tree decomposition.

For a given signature $\tau$, **monadic second order logic** has

- element-variables ($x, y, z, \ldots$) and set-variables ($X, Y, Z, \ldots$)
- relations $=$ (equation) and $x \in X$ (membership), as well as relations from $\tau$
- quantifiers $\exists$ and $\forall$, as well as operators $\land, \lor, \neg$

If $\varphi$ is a sentence, we write $G \models \varphi$ to indicate that $\varphi$ holds on $G$ (i.e., $G$ is a model of $\varphi$)

**Theorem** [[Courcelle’90]]

For a MSO$_1$ sentence $\varphi$ and graph $G$ one can decide whether $G \models \varphi$ in time $f(tw(G), |\varphi|)n$ for some function $f$. 
Conditional Lower Bounds

Exponential Time Hypothesis (ETH)  [Impagliazzo, Paturi, 1990]

Roughly, 3-SAT on $n$ variables cannot be solved in time $2^{o(n)}$.

Conditional lower bounds for tw are usually $2^{o(tw)}$, $2^{o(tw \log tw)}$ or $2^{o(poly(tw))}$.

Rarer results: Unless the ETH fails,

- QSAT with $k$ alternations admits a lower bound of a tower of exponents of height $k$ in the treewidth of the primal graph  PSPACE-complete  [Fichte, Hecher, Pfandler, 2020]
- $k$-Choosability and $k$-Choosability Deletion admit double- and triple-exponential lower bounds in treewidth, respectively  $\Pi^p_2$-complete and $\Sigma^p_3$-complete  [Marx, Mitsou, 2016]
- $\exists \forall$-CSP admits a double-exponential lower bound in the vertex cover number  $\Sigma^p_2$-complete  [Lampis, Mitsou, 2017]
Conditional Lower Bounds

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Conditional Lower Bounds

**Exponential Time Hypothesis (ETH)** [Impagliazzo, Paturi, 1990]

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**Rarer results:** Unless the ETH fails,

- **QSAT** with \( k \) alternations admits a lower bound of a tower of exponents of height \( k \) in the treewidth of the primal graph PSPACE-complete [Fichte, Hecher, Pfandler, 2020]

- **\( k \)-Choosability** and **\( k \)-Choosability Deletion** admit double- and triple-exponential lower bounds in treewidth, respectively \( \Pi^p_2 \)-complete and \( \Sigma^p_3 \)-complete [Marx, Mitsou, 2016]

- **\( \exists \forall \)-CSP** admits a double-exponential lower bound in the vertex cover number \( \Sigma^p_2 \)-complete [Lampis, Mitsou, 2017]
Question.
Does any NP-complete problem require at least double-exponential running time?

Rarer results: Unless the ETH fails,

- QSAT with $k$ alternations admits a lower bound of a tower of exponents of height $k$ in the treewidth of the primal graph \( \text{PSPACE-complete} \) \cite{fichte2020}

- $k$-Choosability and $k$-Choosability Deletion admit double- and triple-exponential lower bounds in treewidth, respectively \( \Pi^P_2 \text{-complete} \) and \( \Sigma^P_3 \text{-complete} \) \cite{marx2016}

- $\exists \forall$-CSP admits a double-exponential lower bound in the vertex cover number \( \Sigma^P_2 \text{-complete} \) \cite{lampis2017}
Part 2.

Metric Graph Problem(s)
**Def.** A resolving set is a $S \subseteq V(G)$ such that $\forall u, v \in V$, $\exists z \in S$ with $d(z, u) \neq d(z, v)$.

**Def.** The minimum size of a resolving set of $G$ is the metric dimension of $G$. 

![Graph with vertices and edges]
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![Diagram](attachment:image.png)
**Metric Dimension**

[Slater '75, Harary, Melter '76]

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Vertices 4 and 6 are not resolved by 5 nor 8.
**Def.** A **resolving set** is a $S \subseteq V(G)$ such that $\forall u, v \in V$, $\exists z \in S$ with $d(z, u) \neq d(z, v)$.

**Def.** The **minimum size** of a resolving set of $G$ is the **metric dimension** of $G$.

**Observation.** For any twins $u, v \in V(G)$ and any resolving set $S$ of $G$, $S \cap \{u, v\} \neq \emptyset$.
**Metric Dimension (MDim)**

**Metric Dimension**

**Input:** An undirected simple graph $G$ and a positive integer $k$

**Question:** Is $\text{md}(G) \leq k$?

<table>
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<td>[Epstein et al'15]</td>
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<td>Interval</td>
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<td>[Foucaud et al'17]</td>
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</table>
Parameterized complexity of Metric Dimension

A lower parameter is upper bounded by a function of the higher one

- FPT $(f(k) \cdot n^{O(1)})$-time algorithm
- XP $(n^k)$-time algorithm
- W[1]-hard (not FPT unless FPT = W[1])
- para-NP-hard (not XP unless P = NP)

$n$: size of input
$k$: size of parameter
Parameterized complexity of Metric Dimension

From NP-hardness results on previous slide
Parameterized complexity of Metric Dimension

Q1: Complexity parameterised by Feedback Vertex Set?

Q2: Complexity parameterised by treewidth?

W[2]-hard parameterised by solution size [Hartung, Nichterlein '13]

**FPT** \( f(k) \cdot n^{O(1)} \)-time algorithm

**XP** \( n^f(k) \)-time algorithm

**W[1]-hard** (not FPT unless FPT = W[1])

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Parameterized complexity of Metric Dimension

- **Minimum Clique Cover**
  - Distance to Clique
  - Distance to Co-Cluster
  - Distance to Cluster
  - Distance to Disjoint Paths
  - Feedback Edge Set
  - Treedepth
  - Bandwidth
  - Treewidth

- **Maximum Independent Set**
  - Distance to Cograph
  - Distance to Interval
  - Feedback Vertex Set
  - Pathwidth
  - Maximum Degree

- **Max Leaf Number**
  - Vertex Cover

- **Q1**: Complexity parameterised by Feedback Vertex Set? [Hartung, Nichterlein '13]

- **Q2**: Complexity parameterised by treewidth? [Eppstein '15], [Belmonte et al '17], [Diaz et al '17]

  Q2 answered first by [Bonnet, Purohit '21]. Then, improved by [Li, Pilipczuk '22]

- **Q1** answered for the combined parameter Feedback Vertex Set + Pathwidth [Galby, Khazaliya, Mc Inerney, Sharma, Tale '23]

- **FPT** \((f(k) \cdot n^{O(1)})\)-time algorithm
- **XP** \((n^k)\)-time algorithm
- **W[1]-hard** (not FPT unless \(\text{FPT} = \text{W}[1]\))
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[Eppstein '15]
Parameterized complexity of Metric Dimension

- Distance to Clique
- Distance to Co-Cluster
- Distance to Cluster
- Distance to Disjoint Paths
- Feedback Edge Set
- Treedepth
- Bandwidth
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- Distance to Perfect
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[Epstein et al '15]
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### Complexity Parameterised by

- **Feedback Vertex Set**
  - Q1: [Hartung, Nichterlein '13]
- **treewidth**
  - Q2: [Eppstein '15], [Belmonte et al '17], [Diaz et al '17]

  Q2 answered first by [Bonnet, Purohit '21].

  Then, improved by [Li, Pilipczuk '22].

- **Feedback Vertex Set + Pathwidth** [Galby, Khazaliya, Mc Inerney, Sharma, Tale '23]

### FPT parameterised by

- treelength + max degree [Belmonte et al '17]
- and clique-width + diameter [Gima et al '21]
Parameterized complexity of Metric Dimension

Q1: Complexity parameterised by Feedback Vertex Set? [Hartung, Nichterlein ’13]
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Part 3.
Our Technique and MDim
**Results**

**Theorem**

**Metric Dimension and Geodetic Set**

- can be solved in $2^{\text{diam}O(tw)} \cdot n^{O(1)}$ time
- no $2^{f(\text{diam})o(tw)} \cdot n^{O(1)}$ time algorithm assuming ETH

**Strong Metric Dimension**

- can be solved in $2^{2^{O(vc)}} \cdot n^{O(1)}$ time, admits $2^{O(vc)}$ kernel
- no $2^{2^{o(vc)}} \cdot n^{O(1)}$ time algorithm, or $2^{o(vc)}$ kernel, assuming ETH

*Source: [FGKLMST, 2024]*
### Theorem

**Metric Dimension and Geodetic Set**

- can be solved in $2^{\text{diam}^O(tw)} \cdot n^O(1)$ time
- no $2^{f(\text{diam})^o(tw)} \cdot n^O(1)$ time algorithm assuming ETH

#### Reduction.

3-Partitioned 3-SAT: $\varphi \rightarrow$ Metric Dimension: $(G, k)$

- $\text{tw}(G) = \log(n)$
- $\text{diam}(G) = \text{const}$
3-Partitioned 3-SAT

Input: 3-CNF formula $\varphi$ with a partition of its variables into 3 disjoint sets $X^\alpha$, $X^\beta$, and $X^\gamma$ such that $|X^\alpha| = |X^\beta| = |X^\gamma| = n$ and each clause contains at most one variable from each of $X^\alpha$, $X^\beta$, and $X^\gamma$.

Question: Is $\phi$ satisfiable?

Theorem [Lampis, Melissinos, Vasilakis, 2023]

3-Partitioned 3-SAT: no $2^{o(n)}$ time algorithm assuming ETH
Encode SAT with small separator

\[(x_1^\alpha \lor x_3^\beta \lor \overline{x_4^\gamma}) \land ((x_1^\alpha \lor x_4^\gamma) \land (x_3^\beta \lor \overline{x_4^\gamma}))\]

\[t_{2i}^\alpha \text{ represents } x_i^\alpha\]

\[f_{2i-1}^\alpha \text{ represents } \overline{x_i^\alpha}\]
Set-Representation Gadget

\((x_1^\alpha \lor x_3^\beta \lor x_4^\gamma) \land (\overline{x_1^\alpha} \lor x_4^\gamma) \land (\overline{x_3^\beta} \lor \overline{x_4^\gamma})\)
Let $F_p$ be the collection of subsets of \{1, \ldots, 2p\} that contain exactly $p$ integers.

No set in $F_p$ is contained in another set in $F_p$ (Sperner family).

There exists $p = O(\log n)$ s.t. $\binom{2p}{p} \geq 2n$.

We define a 1-to-1 function $\text{set-rep} : \{1, \ldots, 2n\} \rightarrow F_p$.

$t_2^\alpha$ is the only vertex in $A^\alpha$ that does not share a common neighbour with $c_1 = (x_1^\alpha \lor x_3^\beta \lor \overline{x_4}^\gamma)$.
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We define a 1-to-1 function

$$\text{set-rep} : \{1, \ldots, 2n\} \rightarrow F_p.$$  

$t_2^\alpha$ is the only vertex in $A^\alpha$ that does not share a common neighbour with $c_1 = (x_1^\alpha \lor x_3^\beta \lor \overline{x_4}^\gamma)$.
**Observation.** For any twins $u, v \in V(G)$ and any resolving set $S$ of $G$, $S \cap \{u, v\} \neq \emptyset$.

- For any resolving set $S$, $|S \cap \text{bits}(X)| \geq \log(|X|) + 1$
- $|S \cap \text{bits}(X)|$ distinguishes each vertex in $X \cup \text{bit-rep}(X)$ from every other vertex in $G$
- $\text{nullifier}(X)$ guarantees that the rest part of $V(G)$ does not affected by the gadget

Purple edges represent all possible edges
Lower bound for Metric Dimension parameterized by $tw$

nullifier($X^\alpha$) nullifier($A^\alpha$) nullifier($V^\alpha$) nullifier($C$)

bit-rep($X^\alpha$) bit-rep($A^\alpha$) bit-rep($V^\alpha$) bit-rep($C$)

$X^\alpha$ $A^\alpha$ $V^\alpha$ $C$

Purple — all possible edges
Blue — set-rep
Red — complementary to blue

Note: $tw(G) = \log(n)$
$diam(G) = \text{const}$

**Theorem** [FGKLMST, 2024]

**Metric Dimension**: no $2^{f(diam)^{o(tw)}} \cdot n^{O(1)}$ time algorithm assuming ETH
Lower bound for Metric Dimension parameterized by $tw$

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**Theorem**

**Metric Dimension:** no $2^{f(diam)^{o(tw)}} \cdot n^{O(1)}$ time algorithm assuming ETH

Note: $tw(G) = \log(n)$

$diam(G) = \text{const}$
Part 4.
Other Results and Applications
**Geodetic Set**

**Input:** An undirected simple graph $G$

**Question:** Does there exist $S \subseteq V(G)$ such that $|S| \leq k$ and, for any vertex $u \in V(G)$, there are two vertices $s_1, s_2 \in S$ such that a shortest path from $s_1$ to $s_2$ contains $u$?

**Theorem** [FGKLMST, 2024]

**Geodetic Set**

- no $2^{f(\text{diam})^{o(\text{tw})}} \cdot n^{O(1)}$ time algorithm assuming ETH
Strong Metric Dimension

**Input:** An undirected simple graph $G$

**Question:** Does there exist $S \subseteq V(G)$ such that $|S| \leq k$ and, for any pair of vertices $u, v \in V(G)$, there exists a vertex $w \in S$ such that either $u$ lies on some shortest path between $v$ and $w$, or $v$ lies on some shortest path between $u$ and $w$?

**Theorem** [FGKLMST, 2024]

Strong Metric Dimension

- no $2^{\omega(v_c)} \cdot n^{O(1)}$ time algorithm, or $2^{o(v_c)}$ kernel, assuming ETH
### Theorem

**Metric Dimension and Geodetic Set**

- Can be solved in $2^{\text{diam}O(^{tw})} \cdot n^{O(1)}$ time
- No $2^{f(\text{diam})o(^{tw})} \cdot n^{O(1)}$ time algorithm assuming ETH

---

**Theorem**

**Strong Metric Dimension**

- Can be solved in $2^{2O(^{vc})} \cdot n^{O(1)}$ time, admits $2^{O(^{vc})}$ kernel
- No $2^{2o(^{vc})} \cdot n^{O(1)}$ time algorithm, or $2^{o(^{vc})}$ kernel, assuming ETH
### Applications of the Technique

**Theorem**  
[Chalopin, Chepoi, Mc Inerney, Ratel, COLT 2024]

**Positive Non-Clashing Teaching Dimension** for Balls in Graphs

- no $2^{o(vc)} \cdot n^{O(1)}$ time algorithm, or $2^{o(vc)}$ kernel, assuming ETH

<table>
<thead>
<tr>
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<tr>
<td><strong>Positive Non-Clashing Teaching Dimension</strong> for Balls in Graphs</td>
<td><strong>Locating-Dominating Set (resp., Test Cover)</strong></td>
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Part 5.
Open Problems
Open Questions

Q1: Are there certain properties shared by distance-based graph problems, that imply such running times? Is there a possible way to generalize our approach to a broader class of problems.

Q2: For which classic problems in NP are the best known FPT algorithms parameterized by $tw$, $vc$ (or other parameters) double-exponential?

Q3: For which classic problems do the best known kernelization algorithms output a kernel with $2^{O(vc)}$ vertices?
... and for Metric Dimension

**Q4:** XP or para-NP-hard parameterised by Feedback Vertex Set?

**Q5:** W[1]-hard or FPT parameterised by Feedback Edge Set?

**Q6:** Distance to Disjoint Paths? Bandwidth?
Further directions

Q1: Are there certain properties shared by distance-based graph problems, that imply such running times? Is there a possible way to generalize our approach to a broader class of problems.

Q2: For which classic problems in NP are the best known FPT algorithms parameterized by $tw$, $vc$ (or other parameters) double-exponential?

Q3: For which classic problems do the best known kernelization algorithms output a kernel with $2^{O(vc)}$ vertices?

For Metric Dimension:

Q4: XP or para-NP-hard parameterised by Feedback Vertex Set?

Q5: W[1]-hard or FPT parameterised by Feedback Edge Set?

Q6: Distance to Disjoint Paths? Bandwidth?