Upward and Orthogonal Planarity are W[1]-hard by Treewidth

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Giuseppe Liotta, Fabrizio Montecchiani, Kirill Simonov

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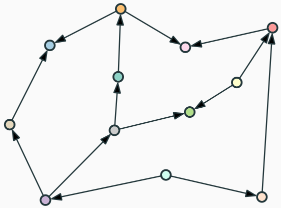
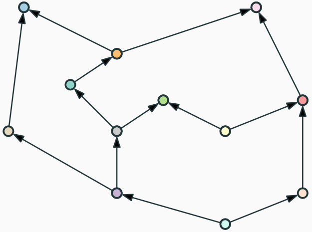
Classical variants of planarity

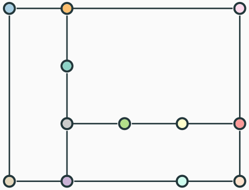
Directed graph $\vec{G}$



Upward planar drawing



Orthogonal drawing



Upward/Orthogonal Planarity Testing

With fixed embedding: `poly`-time solvable [Tamassia'87; BBLM'94]

With variable embedding: NP-complete [Garg, Tamassia'01]

Fixed-parameter tractability is a framework to deal with NP-hard problems:

- Choose a complexity parameter k independent of the input size n
- Find an OPT solution in time $f(k) \cdot n^{\mathcal{O}(1)}$ for some function f

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With variable embedding: NP-complete [Garg, Tamassia'01]

Develop algorithms for graphs which are large but simply structured

poly: SP-graphs (both); max deg < 4 (RP); single source (UP)

FPT: treedepth (UP), number of triconnected components (UP), number of sources (UP), number of vertices of degree 4 (RP)

Upward/Orthogonal Planarity Testing: treewidth

For the variable embedding: $n^{\mathcal{O}(\mathrm{tw})}$-algorithms

Orthogonal: [GD 2019, E. Di Giacomo, G. Liotta, F. Montecchiani]

Upward: [SoCG 2022, S. Chaplick et al.]

Question: [SoCG 2022, S. Chaplick et al.]

Is Upward Planarity W[1]-hard of FPT when parameterized by tw?

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Our Main Result:

Both Upward and Orthogonal Planarity testing are W[1]-hard.

Upward/Orthogonal Planarity Testing: treewidth

For the variable embedding: $n^{\mathcal{O}(\text{tw})}$-algorithms

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Our Main Result:

$\boxed{\text{Known } n^{\mathcal{O}(\text{tw})}\text{-algorithms cannot be improved to } n^{o(\text{tw})} \text{ under ETH.}}$

Overview [Key steps]

Outline

Multicolored Clique

All-or-Nothing Flow on <u>Planar</u> graphs

Circulating Orientation on Planar graphs

Orthogonal/Upward Planarity Testing

Concluding Remarks

Multicolored Clique to All-or-Nothing Flow

Multicolored Clique (MClique)

MULTICOLORED CLIQUE
Input: An undirected simple graph G and a partition of its vertex set into k sets $V_1, \ldots, V_k$, each consisting of N vertices.
Parameter: k.
Question: Does G contain a clique $C \subseteq V(G)$ such that $|C \cap V_i| = 1$ for each $i \in [k]$?

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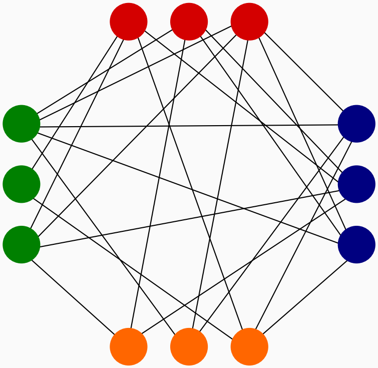

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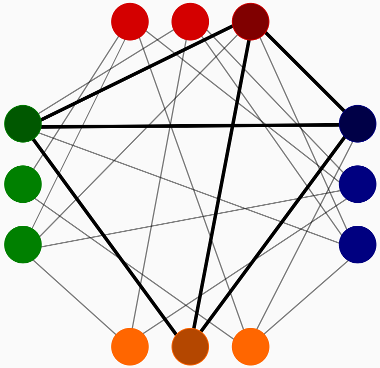

All-or-Nothing Flow[1] (AoNF)

ALL OR NOTHING FLOW
Input: A flow network (G, c, s, t) and a positive integer $\mathcal{F}$.
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[1]XNLP (at least W[1]-hard) when parameterized by tw: H. L. Bodlaender et al. Problems Hard for Treewidth but Easy for Stable Gonality, WG'22

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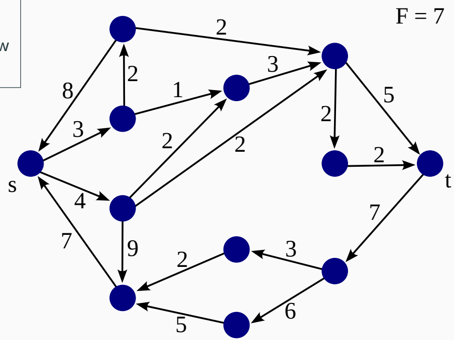

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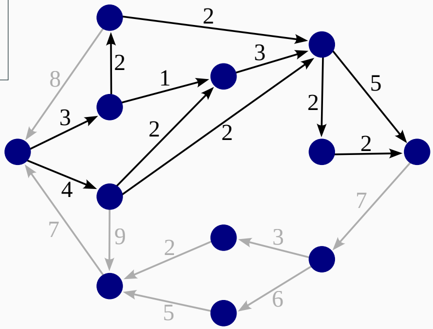
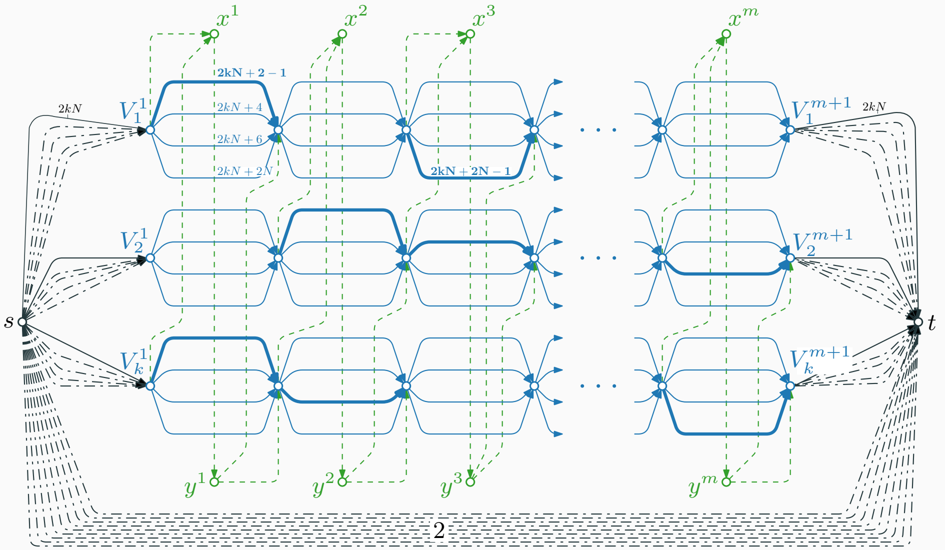
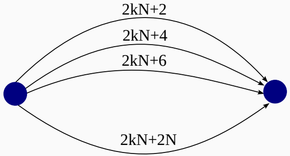


AoNF: (G', c, s, t) and $\mathcal{F} = k(2kN + 2N)$



MClique: $(G, (V_1, V_2, \ldots, V_k)), |V_i| = N$

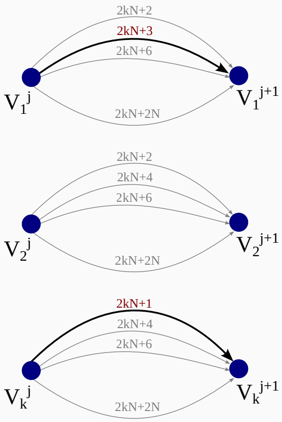
$V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,N}\}.$



$\texttt{Inflow} \in [2kN+2, 2kN+2N];$

$\texttt{Inflow}$ is even.

Non-edge $v_{1,2}v_{k,1}$ of G.



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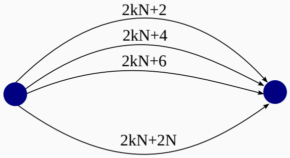

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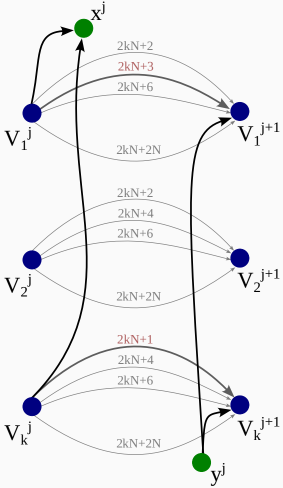

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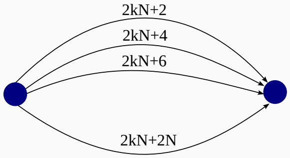

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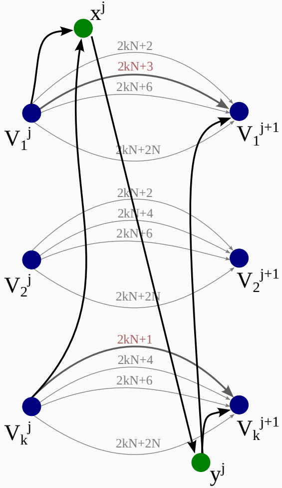
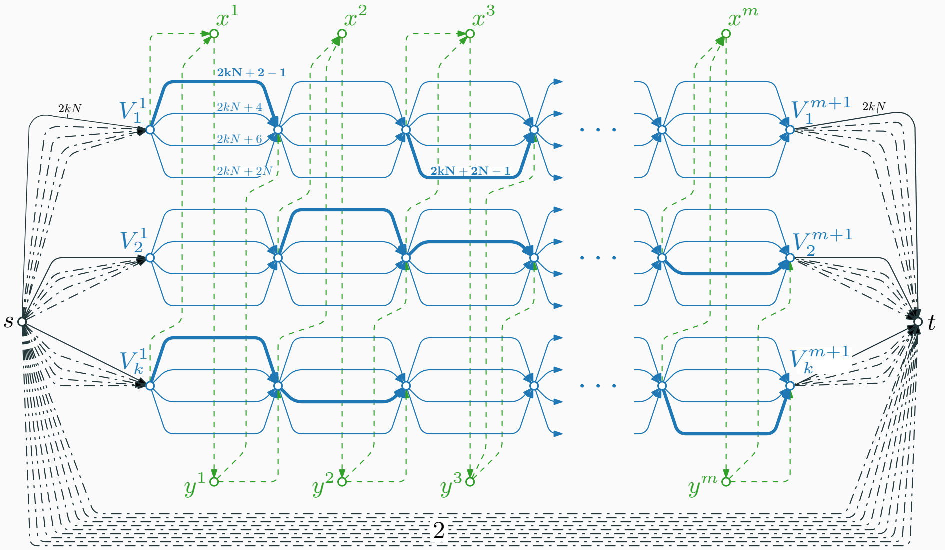
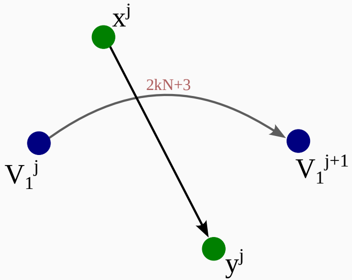


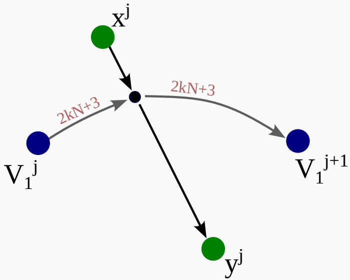
AoNF: (G', c, s, t) and $\mathcal{F} = k(2kN + 2N)$



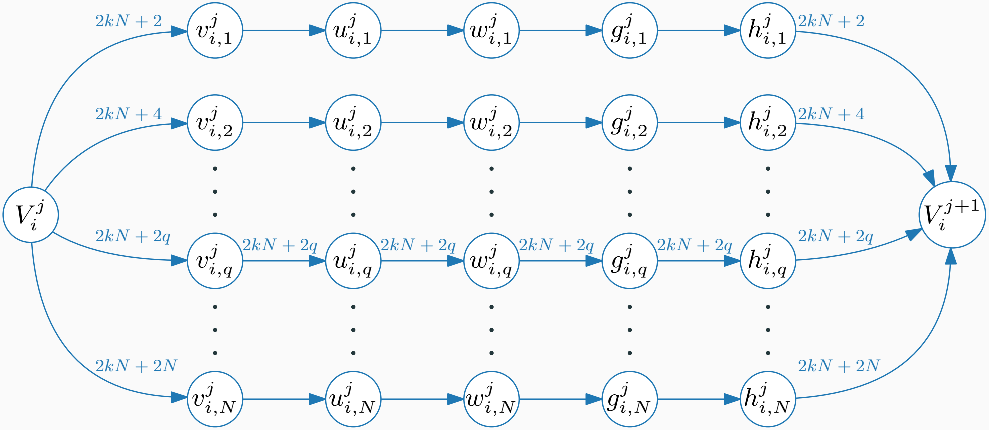
Planarization of the AoNF

Observation





Planarizing a crossing of two edges via a degree-4 vertex does not change the answer, when the capacities of the edges differ.



V_i^j
$2kN+2$
$2kN+4$
$2kN+2q$
$2kN+2N$
$v_{i,1}^j$
$u_{i,1}^j$
$w_{i,1}^j$
$g_{i,1}^j$
$h_{i,1}^j$
$v_{i,2}^j$
$u_{i,2}^j$
$w_{i,2}^j$
$g_{i,2}^j$
$h_{i,2}^j$
$v_{i,q}^j$
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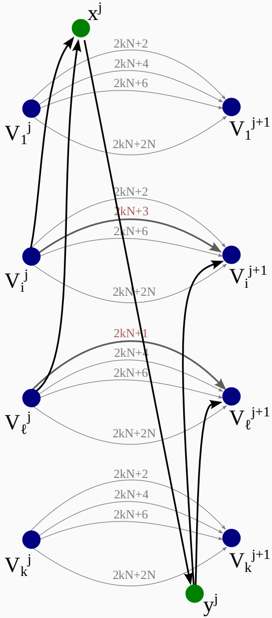
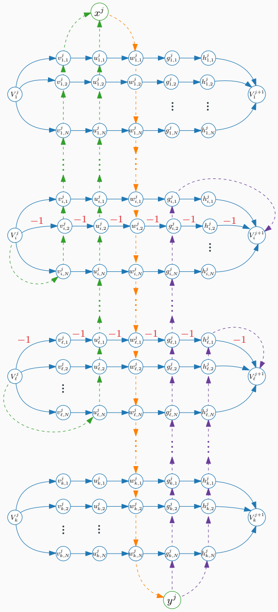
$g_{i,N}^j$
$h_{i,N}^j$
$2kN+2$
$2kN+4$
$2kN+2q$
$2kN+2N$
V_i^{j+1}



x^j
2kN+2
2kN+4
2kN+6
2kN+2N
V_1^j
V_1^{j+1}
2kN+2
2kN+3
2kN+6
2kN+2N
V_i^j
V_i^{j+1}
2kN+1
2kN+4
2kN+6
2kN+2N
V_ℓ^j
V_ℓ^{j+1}
2kN+2
2kN+4
2kN+6
2kN+2N
V_k^j
V_k^{j+1}
y^j



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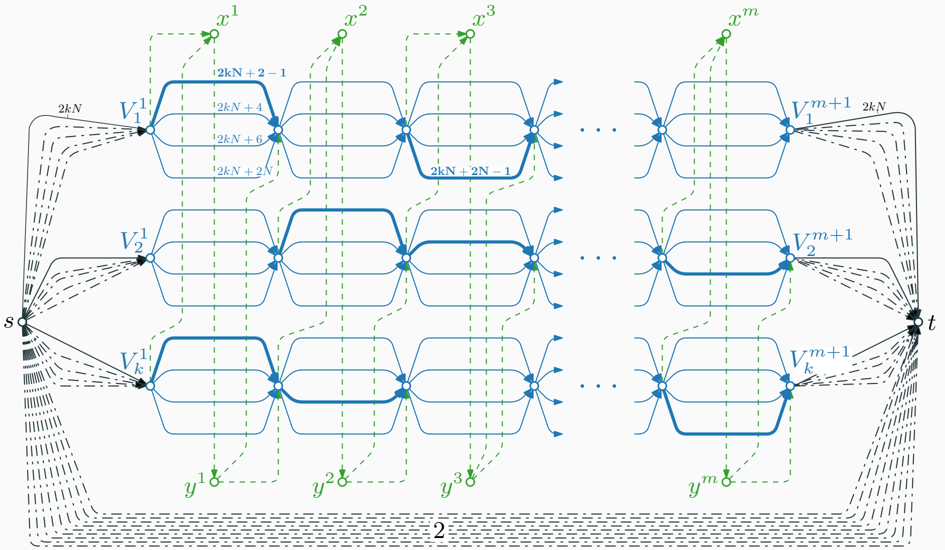
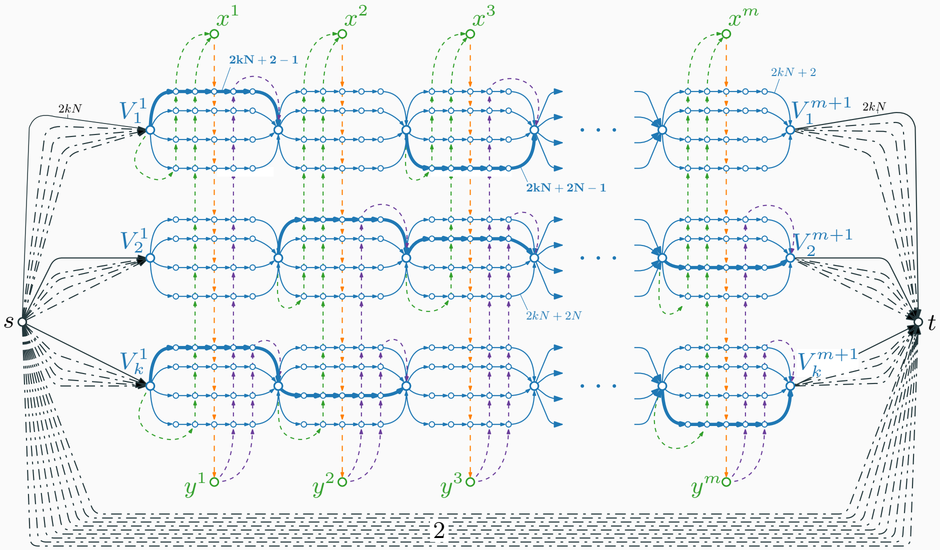
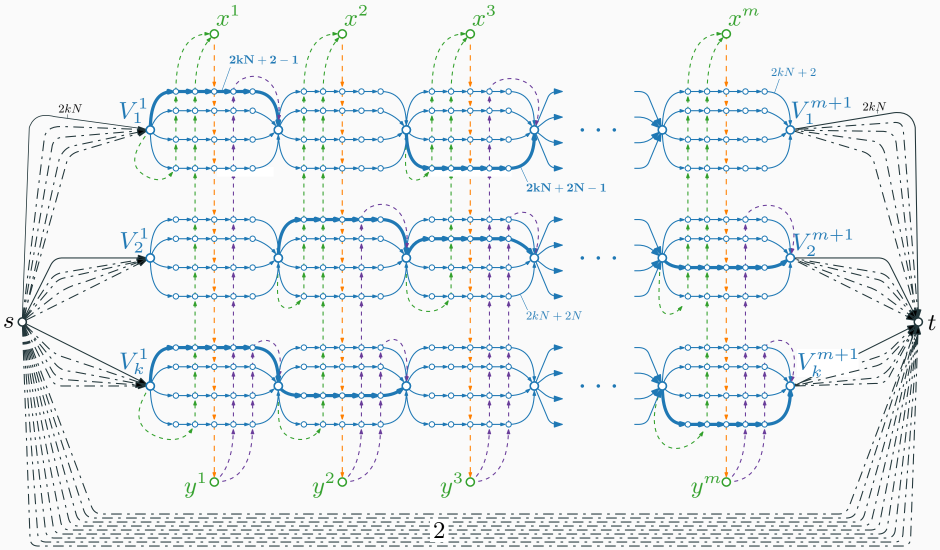


Planar AoNF: (G'', c, s, t) and $\mathcal{F} = k(2kN + 2N)$



First remark: bounded pathwidth



All-or-Nothing Flow (planar) to Circulating Orientation

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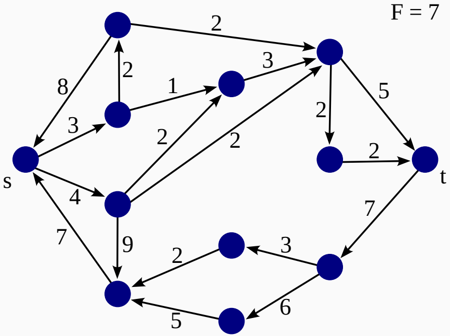

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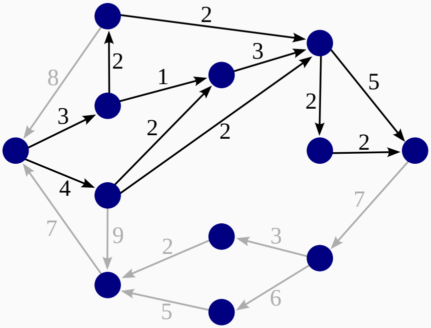

Circulating Orientation (CO)

CIRCULATING ORIENTATION

Input: An undirected graph G with an edge-capacity function $c\colon E(G) \to \mathbb{Z}_{\geq 0}$.

Question: Is it possible to orient the edges of G, such that for each vertex $v \in V(G)$ the total capacity of edges oriented into v is equal to the total capacity of edges oriented out of v? (Such an orientation is called a circulating orientation.)

Circulating Orientation (CO)

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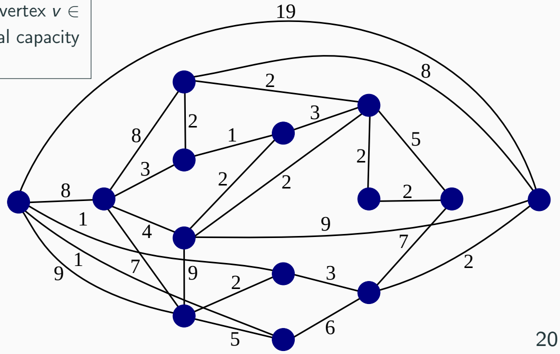
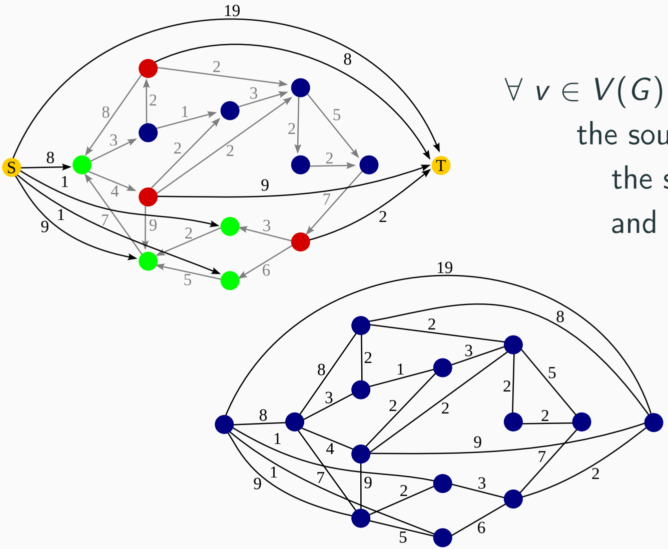


AoNF to CO



$\forall\ v \in V(G) \setminus \{s,t\}\colon\ d_G^+(v) = d_G^-(v)$,

the source has no incoming arcs,

the sink has no outgoing arcs,

and $\mathcal{F} = d_G^+(s)/2 = d_G^-(t)/2$.

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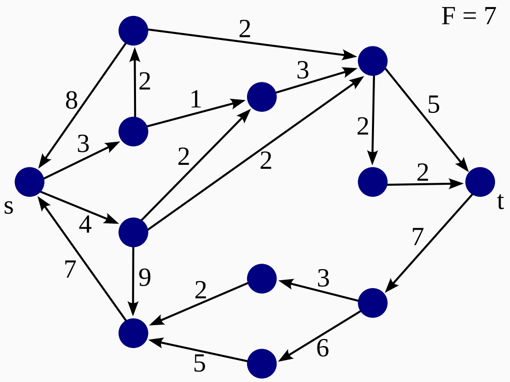

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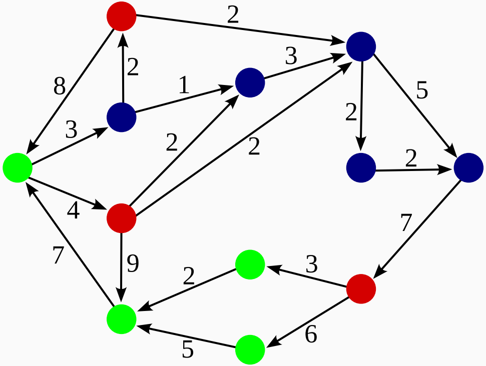

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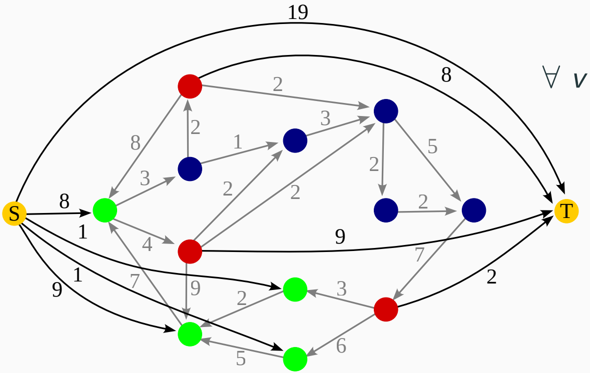

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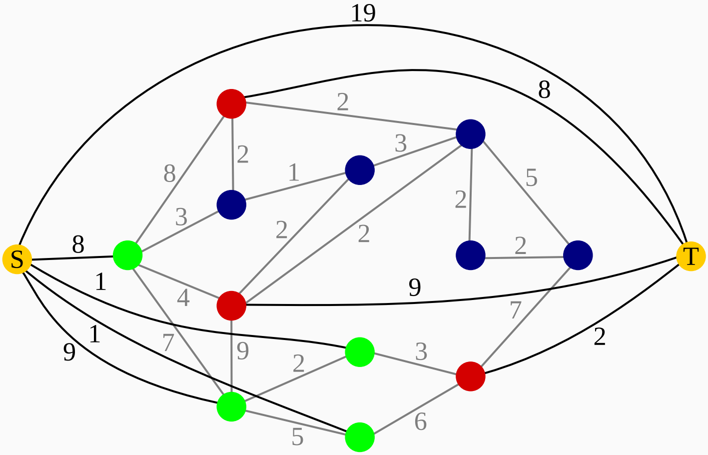
AoNF to CO



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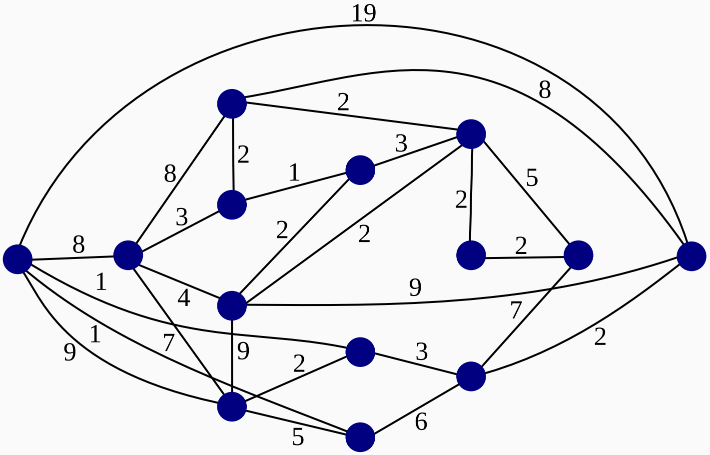
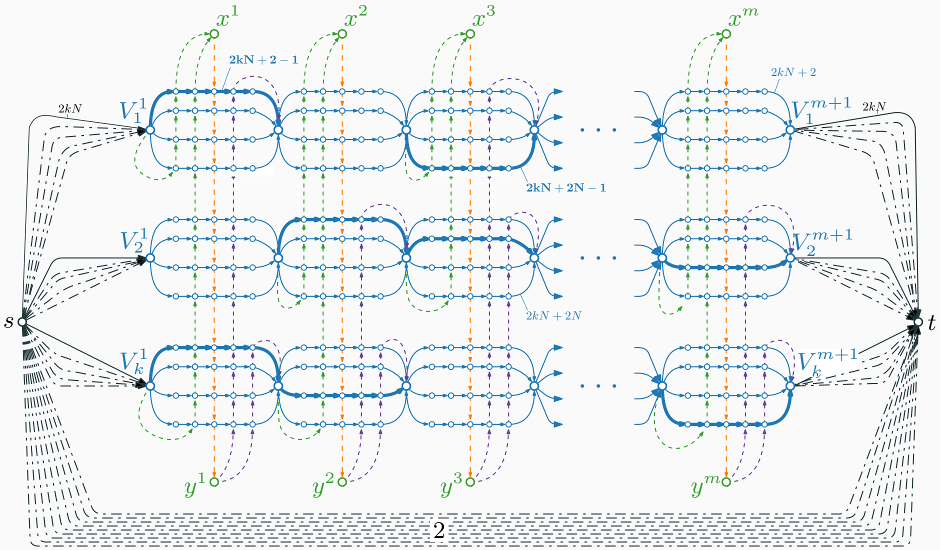


Second remark: a nice embedding



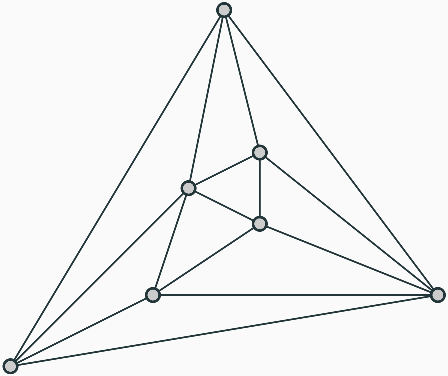
Circulating Orientation to Upward Planarity Testing

Black box

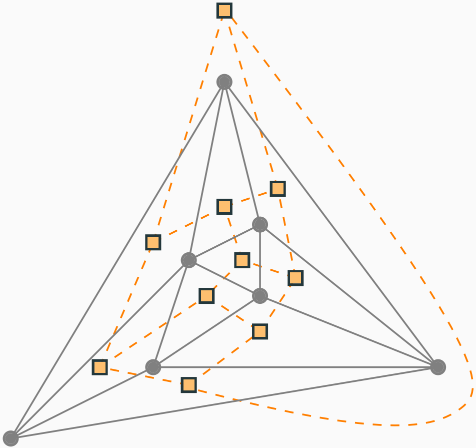
Theorem (Biedl'16)

There is a polynomial-time algorithm that, given a simple planar graph G of pathwidth k on at least three vertices, outputs a plane triangulation G' of G such that $\mathrm{pw}(G') \in \mathcal{O}(k)$.

Triangulated instance of CO



Dual Graph

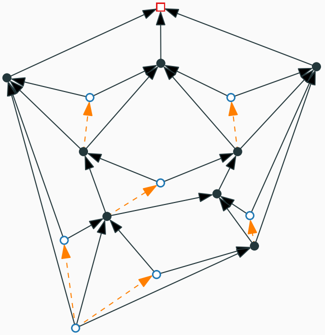


Black Box #2

Theorem (Amini, Huc, and Pérennes'09)

For a triconnected planar graph G, $\mathrm{pw}(G^*) \leq 3\,\mathrm{pw}(G) + 2$, where G^* is the dual graph of G.

st-Planar graph



A digraph G is an st-planar graph if it admits a planar embedding such that:

(1) it contains no directed cycle;

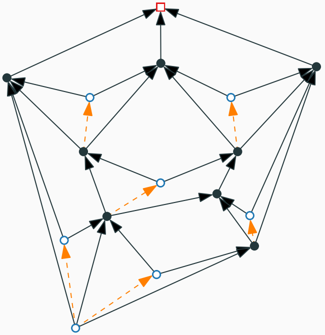
(2) it contains a single source vertex s and a single sink vertex t;

(3) s and t both belong to the external face of the planar embedding.

A digraph G is upward if and only if G is a subgraph of an st-planar graph.

A triconnected st-planar graph has a unique upward planar embedding (up to its outer face).

st-Planar graph



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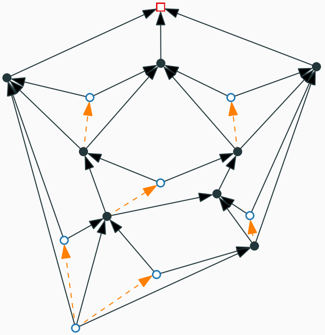

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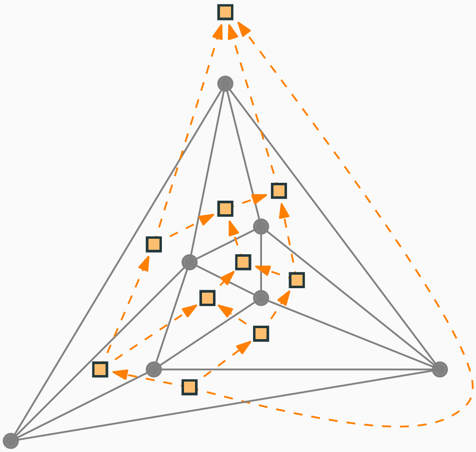
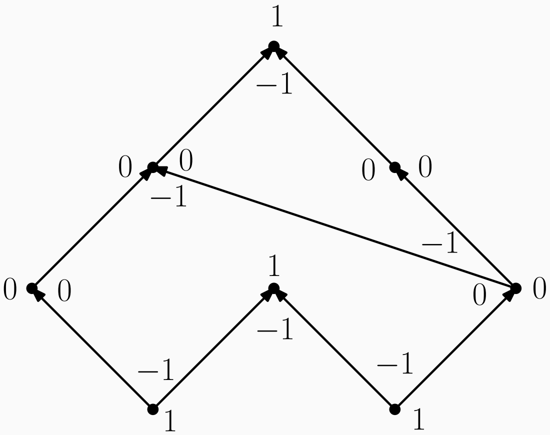
A triconnected st-planar graph has a unique upward planar embedding (up to its outer face).

Orienting the Dual Graph



Angle Assignment



Characterization of UP-graphs

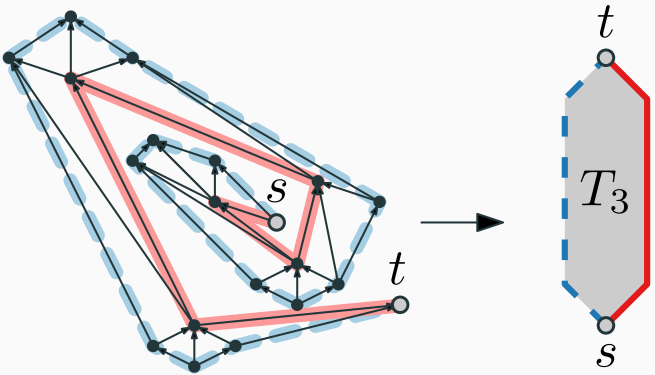
Theorem (BBLM'94, DGL'09)

Let $\mathcal{E}$ be a planar embedding of the underlying graph of G, and λ be an assignment of each angle of each face in $\mathcal{E}$ to a value in $\{-1, 0, 1\}$. Then $\mathcal{E}$ and λ define an upward planar embedding of G if and only if the following properties hold:

- ***UP0*** *If α is a switch angle, then $\lambda(\alpha) \in \{-1, 1\}$, and if α is a flat angle, then $\lambda(\alpha) = 0$.*
- ***UP1*** *If v is a switch vertex of G, then $n_{\mathbf{1}}(v) = 1$, $n_{-\mathbf{1}}(v) = \deg(v) - 1$, $n_{\mathbf{0}}(v) = 0$.*
- ***UP2*** *If v is a non-switch vertex of G, then $n_{\mathbf{1}}(v) = 0$, $n_{-\mathbf{1}}(v) = \deg(v) - 2$, $n_{\mathbf{0}}(v) = 2$.*
- ***UP3*** *If f is a face of G, then*

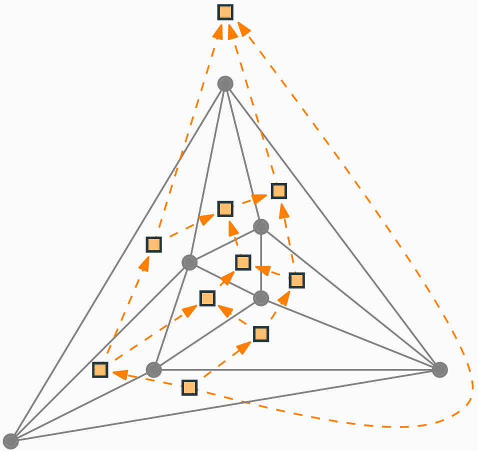
$$n_1(f) - n_{-1}(f) = \begin{cases} -2 & \text{if } f \text{ is an internal face,} \\ +2 & \text{if } f \text{ is the outer face.} \end{cases}$$

Tendril[2] Gadget

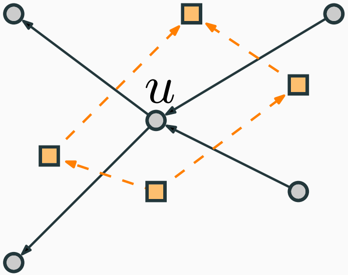


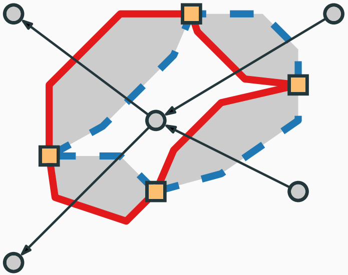
[2] A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

Orienting the Dual Graph



Reduction Idea: Face Balancing





… and Orthogonal Planarity Testing

Differences

- Important that we start with a triangulated graph
- Subdivision of edges to allow an orthogonal embedding
- Orthogonal Tendril[3]

[3] A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

Concluding remarks

Remarks

We have proved that

$$\boxed{\text{Known } n^{\mathcal{O}(\mathrm{tw})}\text{-algorithms cannot be improved to } n^{o(\mathrm{tw})} \text{ under ETH.}}$$

What other points are also one might find interesting:

- Alternative[4] proof of NP-completeness
- Hardness extends for cutwidth of the primal

[4] A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

Further

- Membership in XNLP[5] of both Upward and Orthogonal Planarity Testing: can be solved nondeterministically in time $f(k)n^{\mathcal{O}(1)}$ and space $f(k)log(n)$?
- FPT or W[1]-hard for taking as a parameter the cutwidth of the dual graph
- More restrictive parameterizations may yield FPT algorithms

[5] H. L. Bodlaender et al. Parameterized Problems Complete for Nondeterministic FPT time and Logarithmic Space, FOCS'21

Thank you for your attention!

Further directions

- Membership in XNLP
- Cutwidth of the dual graph
- Other parameterizations

Contents

(Parameterized) Space Complexity Classes

XNLP was introduced as $N[f\,\text{poly}, f\,\log]$ by [Elberfeld et al., IPEC 2012]

L	deterministic space $\mathcal{O}(\log n)$	time $n^{\mathcal{O}(1)}$
NL	nondeterministic space $\mathcal{O}(\log n)$	time $n^{\mathcal{O}(1)}$
XL	deterministic space $f(k) \cdot \log n$	<u>no</u> $f(k) \cdot n^{\mathcal{O}(1)}$ time
XNL	nondeterministic space $f(k) \cdot \log n$	<u>no</u> $f(k) \cdot n^{\mathcal{O}(1)}$ time
XNLP	nondeterministic space $f(k) \cdot \log n$	and time $f(k) \cdot n^c$

Slice-wise Polynomial Space Conjecture [Pilipczuk and Wrochna 2018]

XNLP-hard problems do not have an algorithm that runs in $n^{f(k)}$ time and $f(k) \cdot n^{\mathcal{O}(1)}$ space.