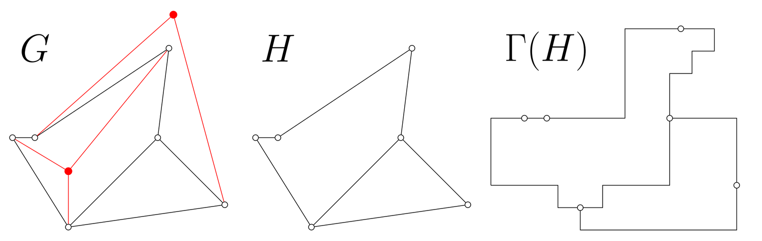
Extending Orthogonal Planar Graph Drawings is Fixed-Parameter Tractable

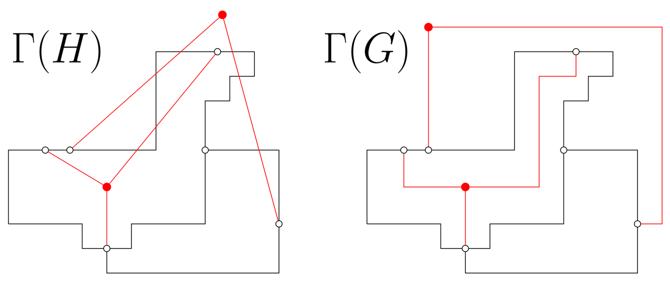
by Sujoy Bhore, Robert Ganian, Liana Khazaliya, Fabrizio Montecchiani, Martin Nöllenburg

Bend-minimal Orthogonal Extension Problem (BMOE)

INPUT: Graph G, an already fixed orthogonal drawing $\Gamma(H)$ for $H \subseteq G$, $\beta \in \mathbb{Z}$:
$\langle G, H \subseteq G, \Gamma(H)\rangle, \beta \in \mathbb{Z}$



TASK: Extend $\Gamma(H)$ to $\Gamma(G)$ using at most $\beta \geq 0$ additional bends



Definitions

The complement $X = V(G) \setminus V(H)$ is the *missing vertex set* of G, and $E_X = E(G) \setminus E(H)$ the *missing edge set*.

A planar orthogonal drawing $\Gamma(G)$ *extends* $\Gamma(H)$ if its restriction to the vertices and edges of H coincides with $\Gamma(H)$.

A *feature point* of an orthogonal drawing is a point representing either a vertex or a bend.

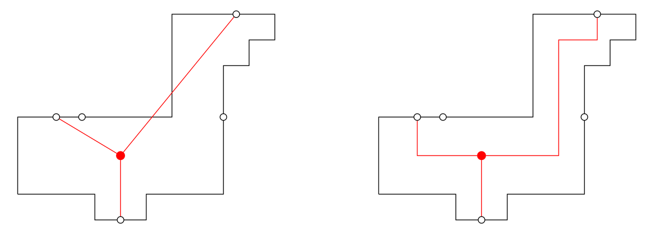
A vertex $a \in V(H)$ is called an *anchor* if it is incident to an edge in E_X.

A *port candidate* is a pair (a, d), i.e. for $ax \in E_X$ $a \in V(H)$, $d \in \{\downarrow, \uparrow, \leftarrow, \rightarrow\}$.

A *port-function* $\mathcal{P}$ is an ordered set of port candidates which contains precisely one port candidate for each missing edge $ax \in E_X, a \in V(H)$.

BMOE on just one Face (F-BMOE)

INPUT: Graph G_f (just one face), fixed orthogonal drawing $\Gamma(H_f)$ for $H_f \subseteq G_f$, set of missing vertices X_f, a port function for X_f:
$\langle G_f, H_f \subseteq G_f, \Gamma(H_f)\rangle$,
$X_f = V(G_f) \setminus V(H_f)$; port-function $\mathcal{P}$.



TASK: Compute the minimum number of bends needed to extend $\Gamma(H_f)$ to $\Gamma(G_f)$ and

(1) missing edges and vertices are only drawn in the face f;

(2) each edge $ax \in E_X$ where $x \in X_f$ connects to $a \in V(H_f)$ via its port candidate defined by $\mathcal{P}$; or

(3) determine that no such extension exists.

Complexity results for an extension problems

(1) planar, linear-time algorithm [Angelini et al., 2015]

(2) level planar, NP-hard [Brückner and Rutter, 2017]

(3) upward planar, NP-hard [Da Lozzo et al., 2020]

(4) bend-minimal orthogonal, NP-hard [Angelini et al., 2021]

Let $\kappa = |V(G) \setminus V(H)| + |E(G) \setminus E(H)|$, i.e. the number of missing elements.

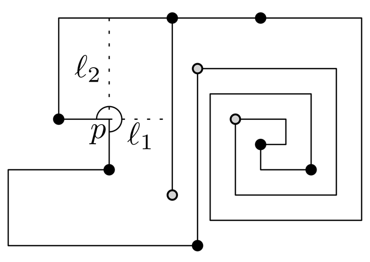
Main contribution. If H is connected, the BMOE problem parameterized by κ is Fixed-Parameter Tractable.

Preprocessing

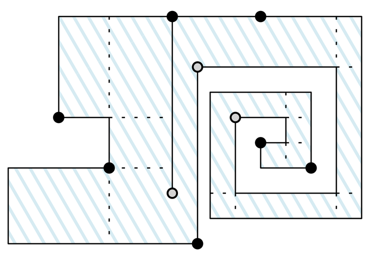
Branching

There is an algorithm that solves an instance of BMOE in time $2^{\mathcal{O}(\kappa)} \cdot T(|\mathcal{I}|, k)$, where $T(|\mathcal{I}|, k)$ is the time required to solve an instance $\mathcal{I}$ of F-BMOE with instance size $|\mathcal{I}|$ and parameter value k.

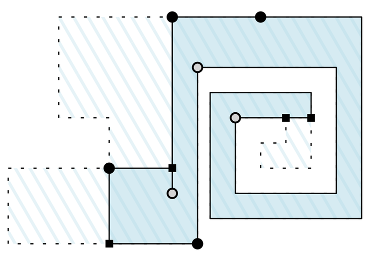
Prunning



A reflex corner p, projections ℓ_1 and ℓ_2.



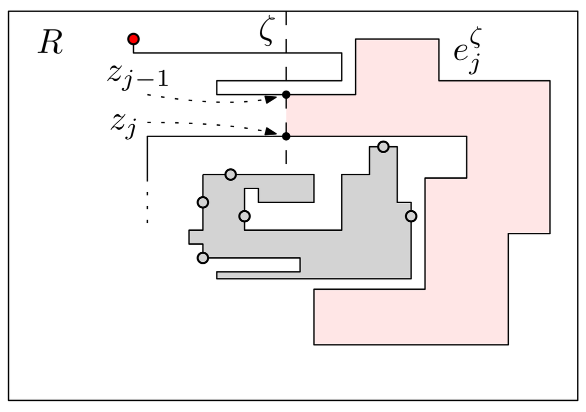
Non-essential reflex corners and projections (anchors — gray filling, non-anchors — solid).



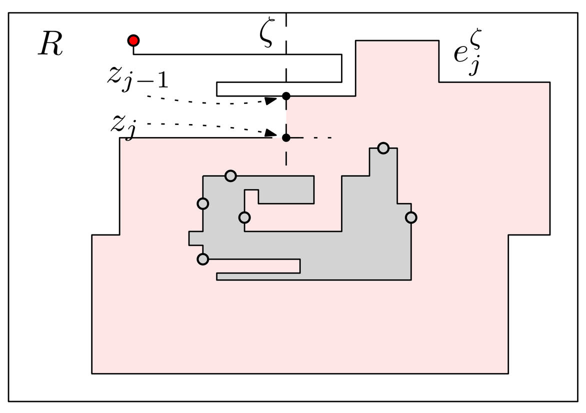
The corresponding clean instance.

Outer face

Lemma. BMOE instance admits a solution with no ζ-handles and at most $4k(k+1)$ ζ-spirals.



ζ-handle



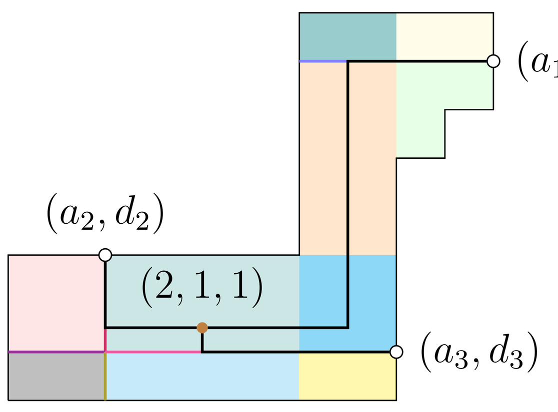
ζ-spiral

Lemma. F-BMOE for an outer face could be solved in time $2^{\mathcal{O}(k^2 \log k)} \cdot T(|\mathcal{I}|, k)$, where $T(|\mathcal{I}|, k)$ is the time to solve an instance of F-BMOE for the inner face.

Discretizing the Instances

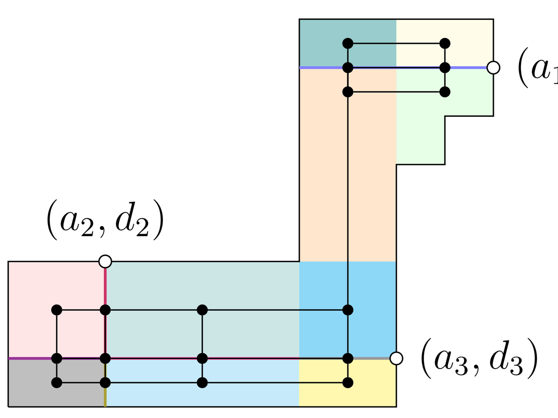
Sectors and the Sector Graph

For a point $p \in f$, the *bend distance* $\mathrm{bd}(p, (a, d))$ to a port candidate (a, d) is the minimum integer q such that there exists an orthogonal polyline with q bends connecting p and a in the interior of f which arrives to a from direction d.



For each point $p \in f$ and a port-function $\mathcal{P} = ((a_1, d_1), \ldots, (a_q, d_q))$, a bend-vector of the point p is the tuple $\mathrm{vect}(p) = (\mathrm{bd}(p, (a_1, d_1)), \ldots, \mathrm{bd}(p, (a_q, d_q)))$.

Given a port-function $\mathcal{P}$, a *sector* F is a maximal connected set of points with the same bend-vector w.r.t. $\mathcal{P}$.



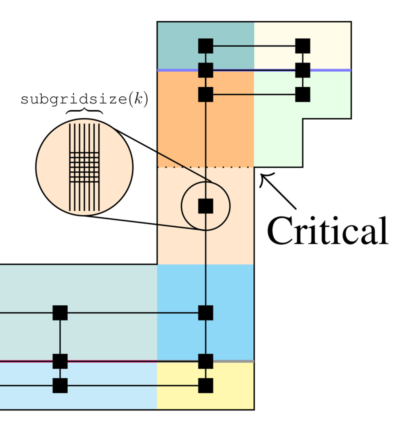
Sectors A and B are *adjacent* if there exists a point p in A and a direction $d \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}$ such that the first point outside of A hit by the ray starting from p in direction d is in B.

Observation. The number of vertices in $\mathcal{G}$ is upper-bounded by $9x^2$, where x is the number of feature points in $\Gamma(H_F)$.

The Sector-Grid

Our aim is to construct a "universal" point-set with the property that there exists a solution which places feature points only on these points

A reflex corner is *critical* for a sector S if it is incident to at least two distinct sectors, and (S, d)-*critical* if it is also can be reached by a ray from some point in S traveling in direction d.



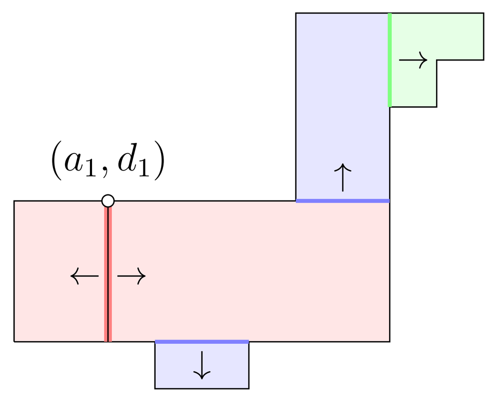
Lemma. For each sector $S \in \mathcal{F}_f$ and for each direction $d \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}$, there are at most $4k$ (S, d)-*critical* reflex corners.

Corollary. Given an instance $\mathcal{I}$ of F-BMOE we can construct a point-set (called a *sector grid*) in time $\mathcal{O}(|\mathcal{I}|)$ with the following properties:
(1) $\mathcal{I}$ admits a solution whose feature points all lie on the sector grid, and
(2) each sector contains at most $\texttt{gridsize}(k)$ points of the sector grid.

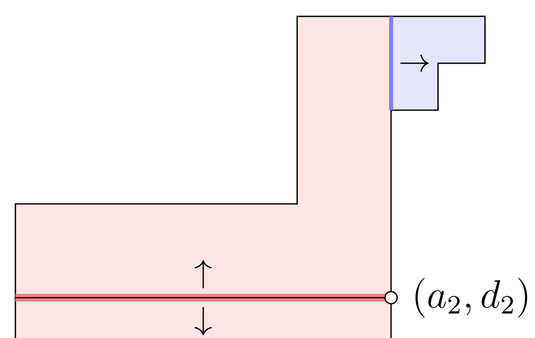
$$\texttt{gridsize}(k) = \texttt{subgridsize}^2(k) \cdot (8k)^2, \quad \texttt{subgridsize}(k) = 112k^3 + 202k^2 + 85k$$

Exploiting the Treewidth of Sector Graphs

Sector Graphs Are Tree-Like



Sector graph for the first port candidate.

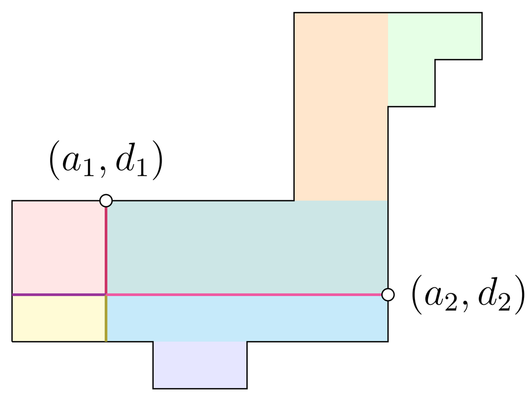


Sector graph for the second port candidate.

Let $\mathcal{P} = ((a_1, d_1), \ldots, (a_q, d_q))$ be the port-function for the considered face f ($q \leq 4k$).

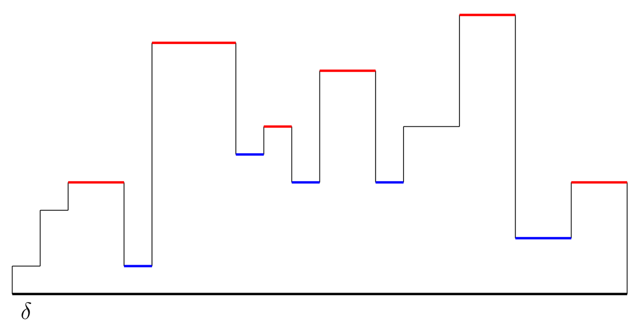
For each $1 \leq i \leq q$, let
$\mathcal{P}_i = ((a_1, d_1), \ldots, (a_i, d_i))$ be a prefix of $\mathcal{P}$;
$\mathcal{F}_i$ be the set of sectors for $\mathcal{P}_i$.
$\mathcal{G}_i$ be the sector graph for $\mathcal{P}_i$.

Lemma [and an induction base].
The sector graph for a single port is a tree.

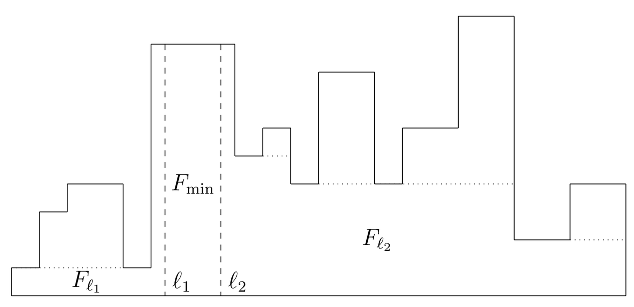


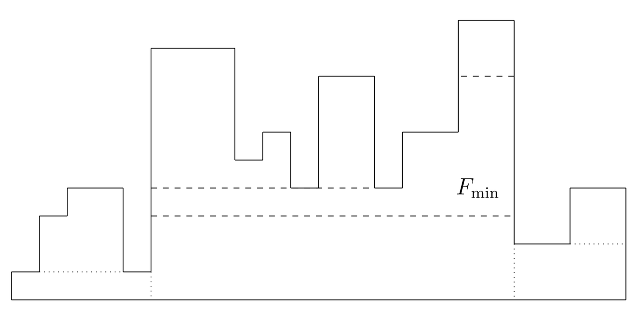
Adding the second port to the first and how sectors being subdivided after.

Baseline Argument



Red (blue) segments are local maxima (minima); the segment δ is a baseline.





A line-segment δ on the boundary of a sector F is a *baseline* if

(1) each point in F can be reached by a ray starting at and orthogonal to δ, and

(2) the line-segment δ touches F on one side and points in $f \setminus F$ on the other side.

Lemma. Each sector in $\mathcal{F}_t$, $1 \leq t \leq q$, admits at least one baseline.

Lemma. After adding one new port, each existing sector splits to at most the number of its local maxima (up to a constant) many subsectors.

Lemma. For each sector $F \in \mathcal{F}_q$, the number of local maxima is upper-bounded by $4k$.

Theorem.
Let $\mathcal{G}$ be a sector graph of a face f of the drawing $\Gamma(G)$. Then
$\mathrm{tw}(\mathcal{G}) \leq (4 + 4k)^{4k}$.

Dynamic Programming

An instance $\mathcal{I} = \langle G_f, H_f, \Gamma(H_f), \mathcal{P}\rangle$ of F-BMOE with $k = |V(G_f) \setminus V(H_f)|$:

(1) admits a sector graph $\mathcal{G}$;

(2) treewidth of $\mathcal{G}$ is at most $(4 + 4k)^{4k}$;

(3) a bend-minimal extension of $\Gamma(H_f)$ to $\Gamma(G_f)$ only contain feature points on the sector-grid;

(4) there are at most $\texttt{gridsize}(k)$ sector-grid points per sector;

(5) $\texttt{gridsize}(k) = \mathcal{O}(k^6)$.

Lemma. F-BMOE can be solved in time

$$2^{k^{\mathcal{O}(1)}} \cdot |V(G_f)|.$$

Theorem.
BMOE can be solved in time

$$2^{\kappa^{\mathcal{O}(1)}} \cdot n,$$

where n is the number of feature points of $\Gamma(H)$.

Concluding Remarks

The Bend-Minimal Orthogonal Extension Problem is Fixed-Parameter Tractable in the number of missing elements.

- What if H is not connected?
- The approach can be adjusted to minimize the number of bends per edge.
- Can we extend the result to planar drawings using a fixed number of slopes?

arXiv:2302.10046