Upward and Orthogonal Planarity are W[1]-hard by Treewidth

Bart M. P. Jansen, **Liana Khazaliya**, Philipp Kindermann, Giuseppe Liotta, Fabrizio Montecchiani, Kirill Simonov

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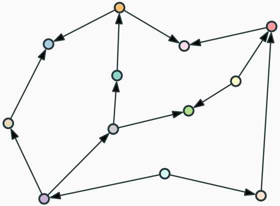
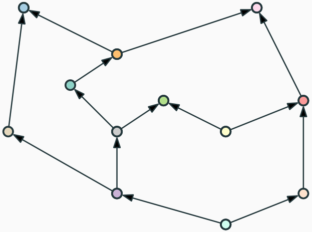
September 22, 2023

What we are interested in

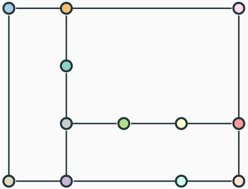
Directed graph $\vec{G}$



Upward planar drawing



Orthogonal drawing



Upward/Orthogonal Planarity Testing

With fixed embedding: poly-time solvable
With variable embedding: NP-complete

For the variable embedding: $n^{\mathcal{O}(\mathrm{tw})}$-algorithms

Orthogonal: [GD 2019, E. Di Giacomo, G. Liotta, F. Montecchiani]
Upward: [SoCG 2022, S. Chaplick et al.]

Question: [SoCG 2022, S. Chaplick et al.]

Is Upward Planarity W[1]-hard of FPT when parameterized by tw?

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Our Main Result:

Both Upward and Orthogonal Planarity testing are W[1]-hard.

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Our Main Result:

Known $n^{\mathcal{O}(\mathrm{tw})}$-algorithms cannot be improved to $n^{o(\mathrm{tw})}$ under ETH.

Overview [Key steps]

Outline

- Multicolored Clique

bounded pw ↘

- All-or-Nothing Flow

pw′ is $\mathcal{O}(\text{pw})$ ↙

- All-or-Nothing Flow on Planar graphs
- Circulating Orientation on Planar graphs

↗ pw of the triangulation

- Orthogonal/Upward Planarity Testing

\+ Concluding Remarks

Multicolored Clique to All-or-Nothing Flow

Multicolored Clique (MClique)

MULTICOLORED CLIQUE

Input: An undirected simple graph G and a partition of its vertex set into k sets $V_1, \ldots, V_k$, each consisting of N vertices.

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Question: Does G contain a clique $C \subseteq V(G)$ such that $|C \cap V_i| = 1$ for each $i \in [k]$?

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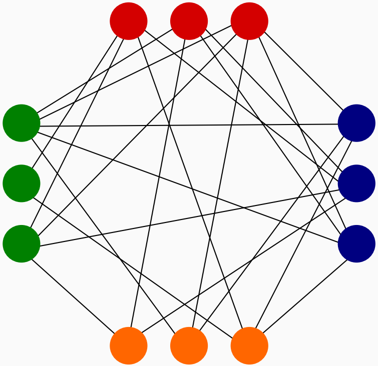

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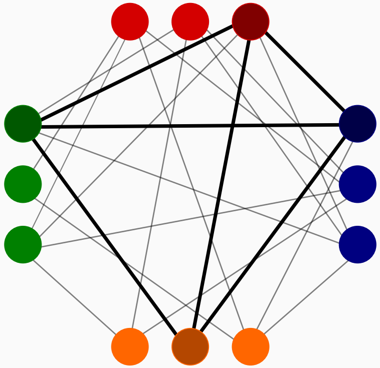

All-or-Nothing Flow[1] (AoNF)

All or Nothing Flow
Input: A flow network (G, c, s, t) and a positive integer $\mathcal{F}$.
Question: Does there exist an st-flow of value exactly $\mathcal{F}$, such that the flow through any arc $uv \in E(G)$ is either 0 or equal to $c(uv)$?

[1] XNLP (at least W[1]-hard) when parameterized by tw: H. L. Bodlaender et al. Problems Hard for Treewidth but Easy for Stable Gonality, WG'22

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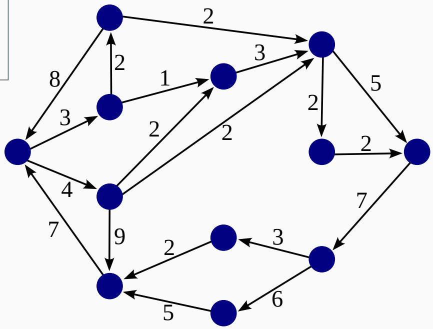

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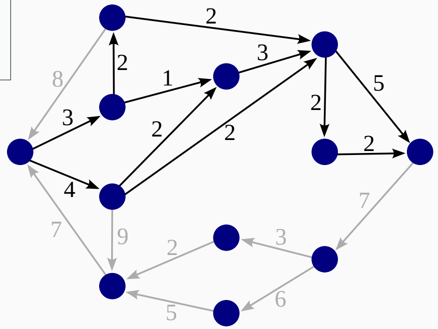
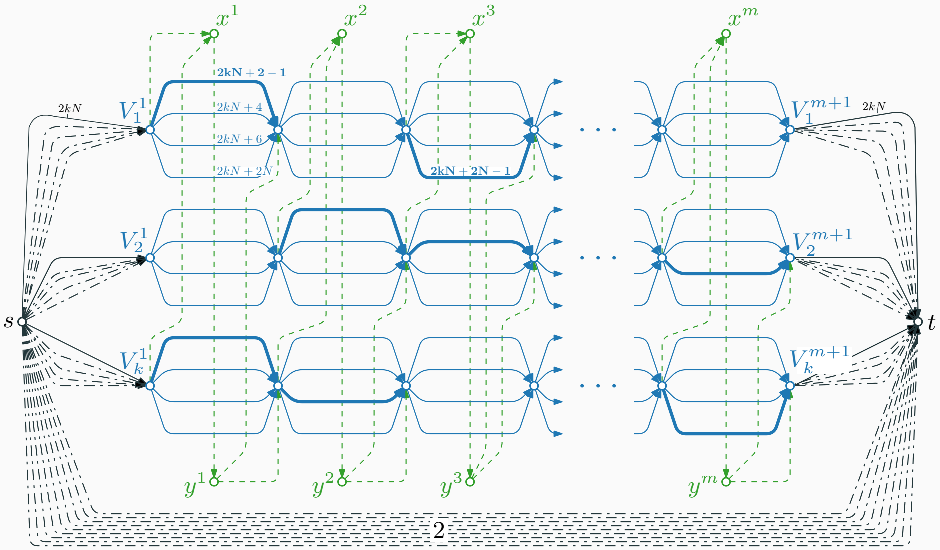
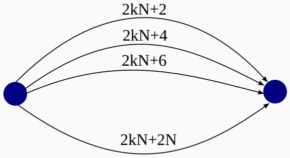


AoNF: (G', c, s, t) and $\mathcal{F} = k(2kN + 2N)$



MClique: $(G, (V_1, V_2, \ldots, V_k)), |V_i| = N$

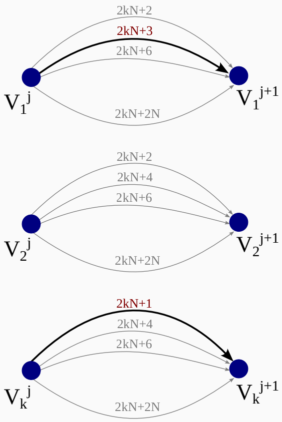
$V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,N}\}$.



`Inflow` $\in [2kN + 2, 2kN + 2N]$;

`Inflow` is even.

Non-edge $v_{1,2}v_{k,1}$ of G.



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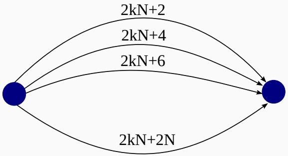

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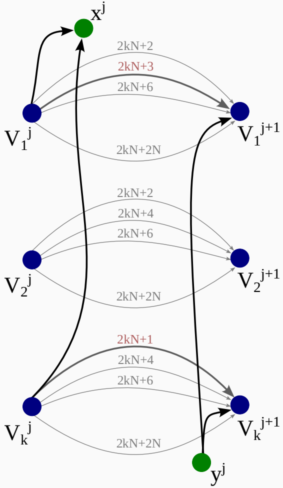

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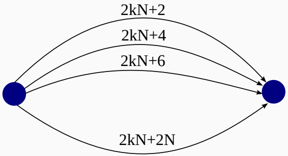

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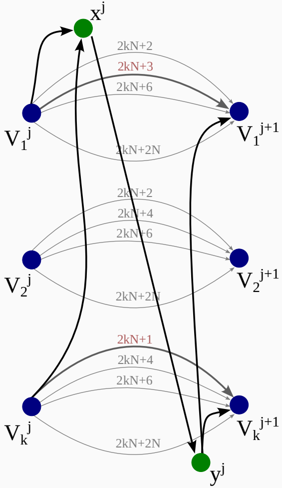
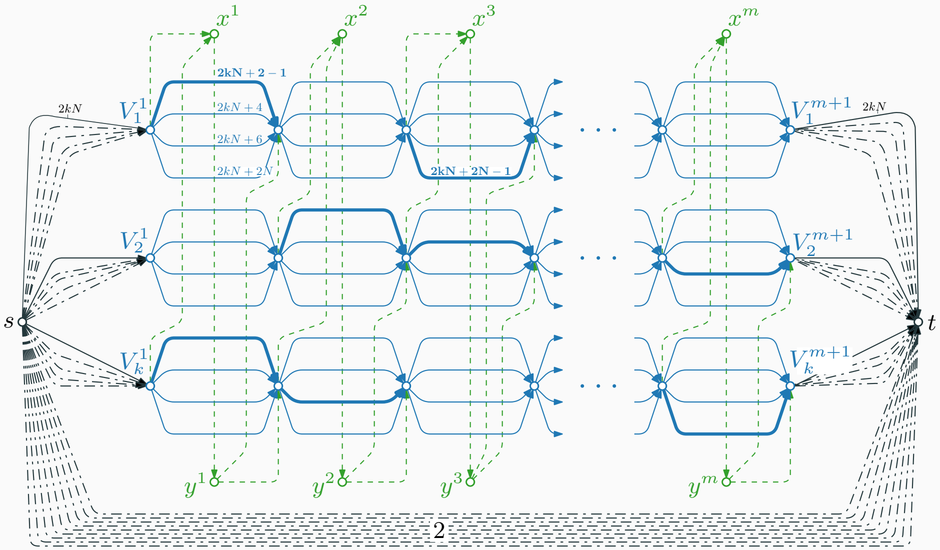
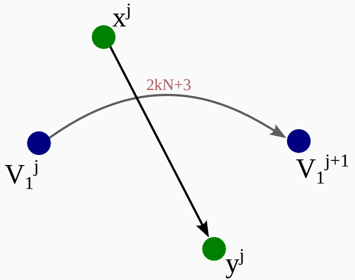


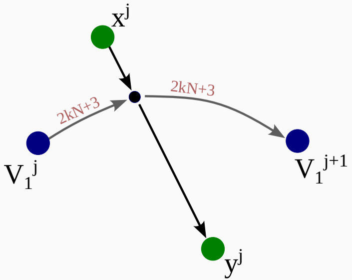
AoNF: (G', c, s, t) and $\mathcal{F} = k(2kN + 2N)$



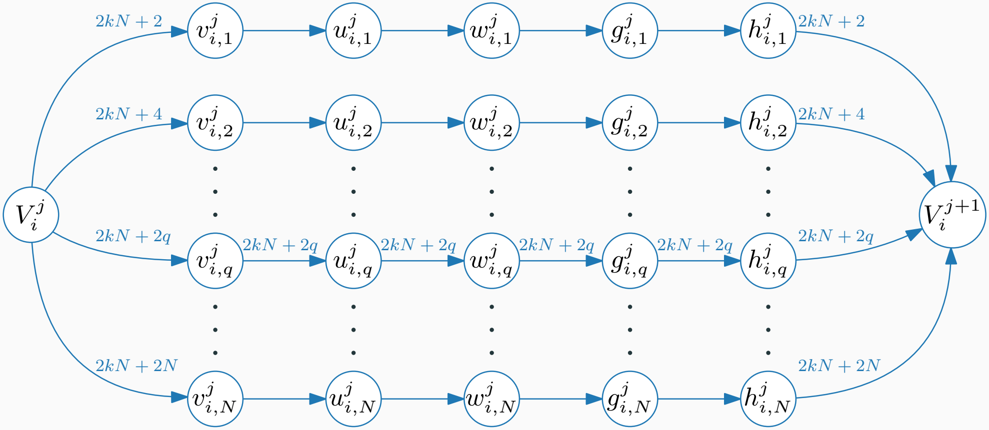
Planarization of the AoNF

Observation

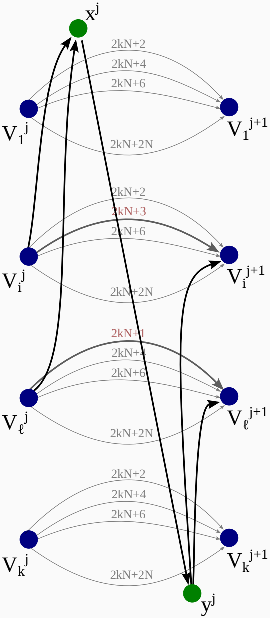




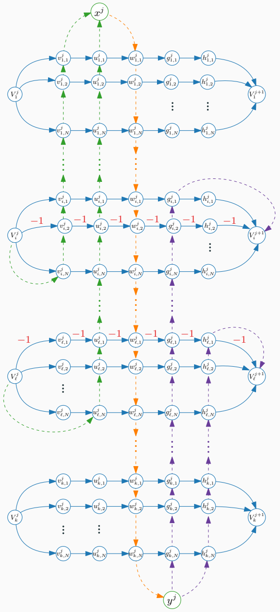
Planarizing a crossing of two edges via a degree-4 vertex does not change the answer, when the capacities of the edges differ.



V_i^j
$2kN+2$
$2kN+4$
$2kN+2q$
$2kN+2N$
$v_{i,1}^j$
$u_{i,1}^j$
$w_{i,1}^j$
$g_{i,1}^j$
$h_{i,1}^j$
$v_{i,2}^j$
$u_{i,2}^j$
$w_{i,2}^j$
$g_{i,2}^j$
$h_{i,2}^j$
$v_{i,q}^j$
$u_{i,q}^j$
$w_{i,q}^j$
$g_{i,q}^j$
$h_{i,q}^j$
$v_{i,N}^j$
$u_{i,N}^j$
$w_{i,N}^j$
$g_{i,N}^j$
$h_{i,N}^j$
$2kN+2q$
$2kN+2q$
$2kN+2q$
$2kN+2q$
$2kN+2$
$2kN+4$
$2kN+2q$
$2kN+2N$
V_i^{j+1}



x^j
2kN+2
2kN+4
2kN+6
2kN+2N
V_1^j
V_1^{j+1}
2kN+2
2kN+3
2kN+6
2kN+2N
V_i^j
V_i^{j+1}
2kN+1
2kN+4
2kN+6
2kN+2N
V_ℓ^j
V_ℓ^{j+1}
2kN+2
2kN+4
2kN+6
2kN+2N
V_k^j
V_k^{j+1}
y^j



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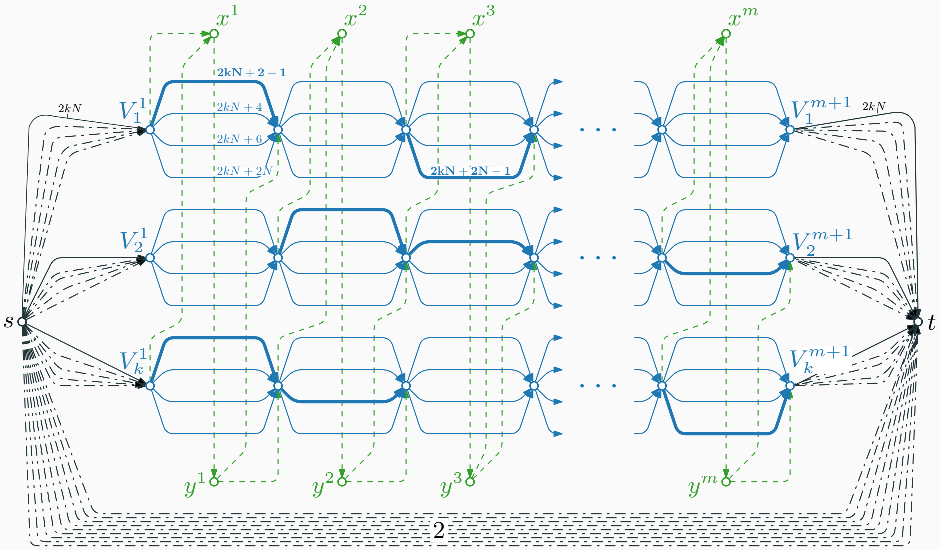
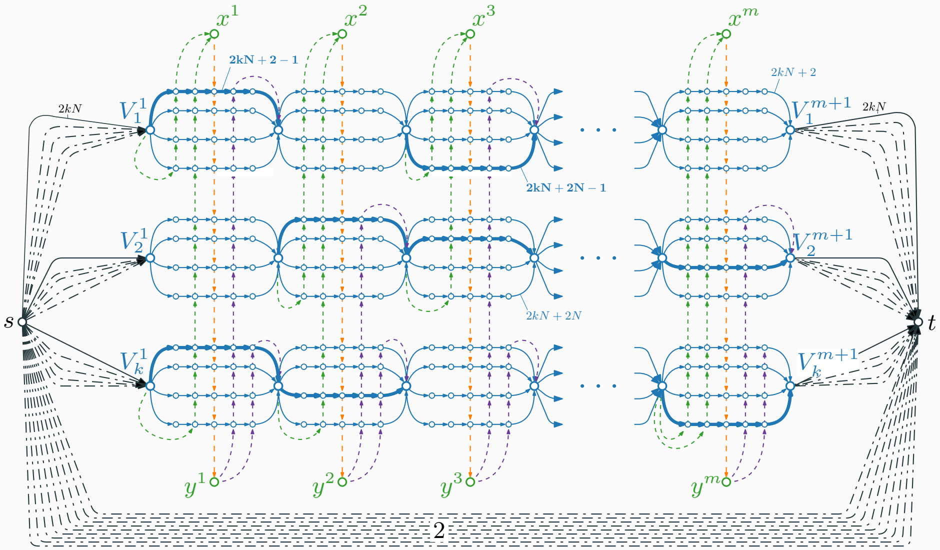
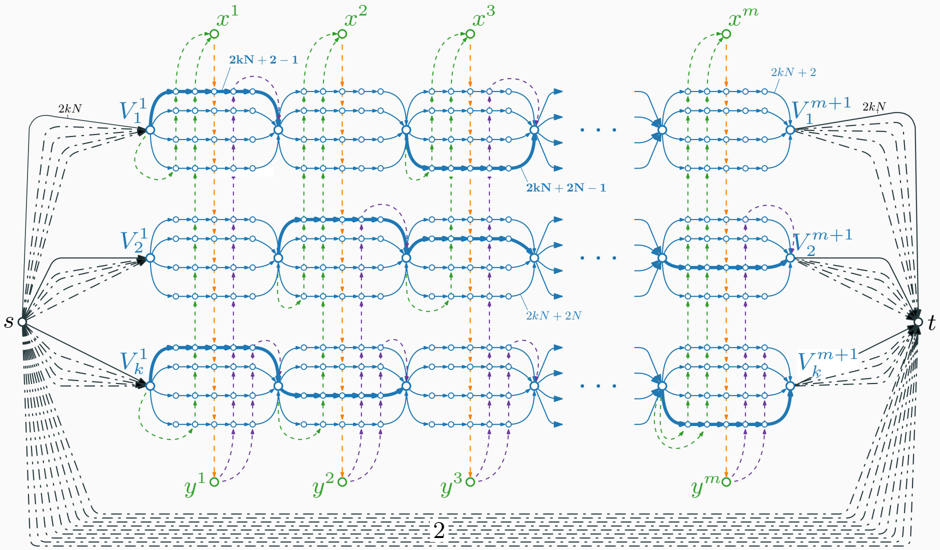


Planar AoNF: (G'', c, s, t) and $\mathcal{F} = k(2kN + 2N)$



First remark: bounded pathwidth



All-or-Nothing Flow (planar) to Circulating Orientation

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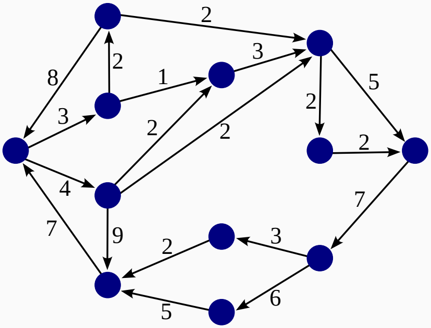

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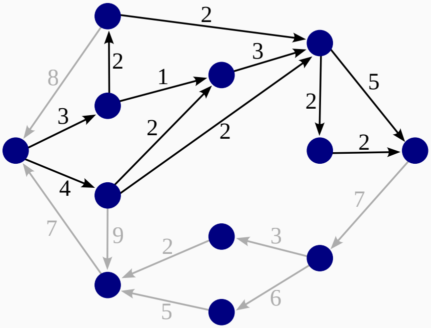

Circulating Orientation (CO)

Circulating Orientation

Input: An undirected graph G with an edge-capacity function $c\colon E(G) \to \mathbb{Z}_{\geq 0}$.

Question: Is it possible to orient the edges of G, such that for each vertex $v \in V(G)$ the total capacity of edges oriented into v is equal to the total capacity of edges oriented out of v? (Such an orientation is called a circulating orientation.)

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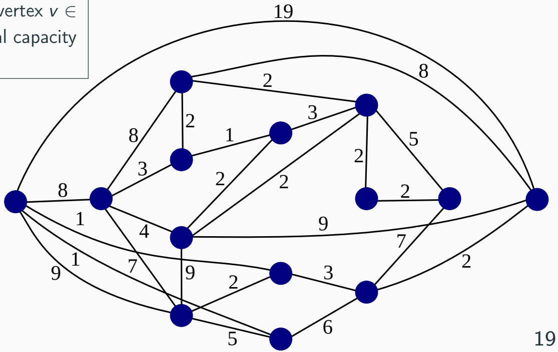
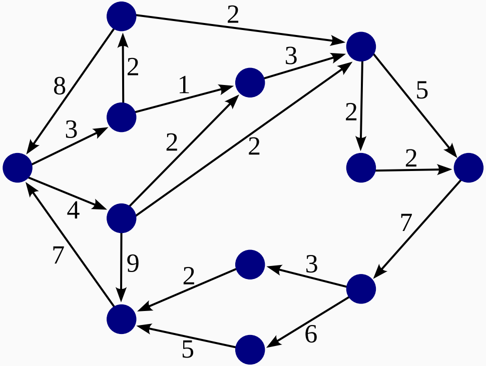


AoNF to CO



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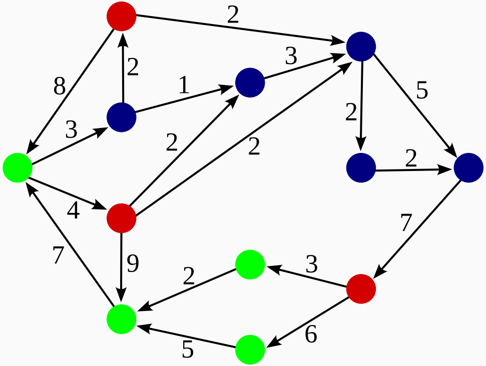

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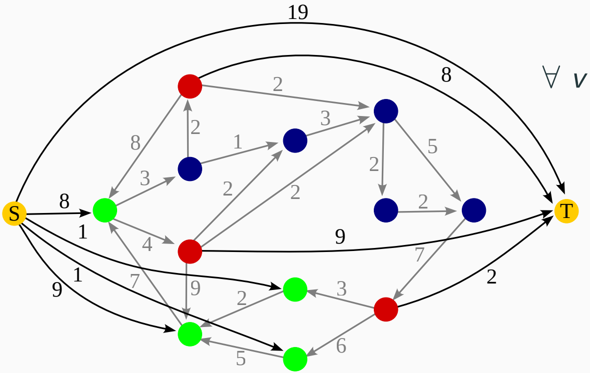


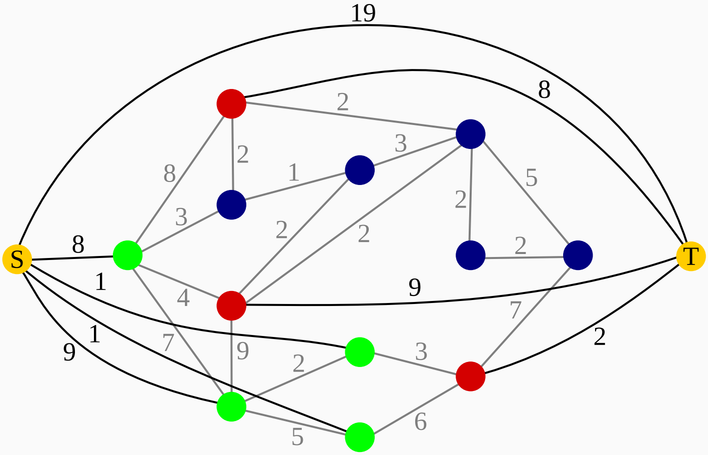
$$\forall\ v \in V(G) \setminus \{s, t\}\colon\ d_G^+(v) = d_G^-(v),$$

the source has no incoming arcs,

the sink has no outgoing arcs,

and $\mathcal{F} = d_G^+(s)/2 = d_G^-(t)/2$.

AoNF to CO



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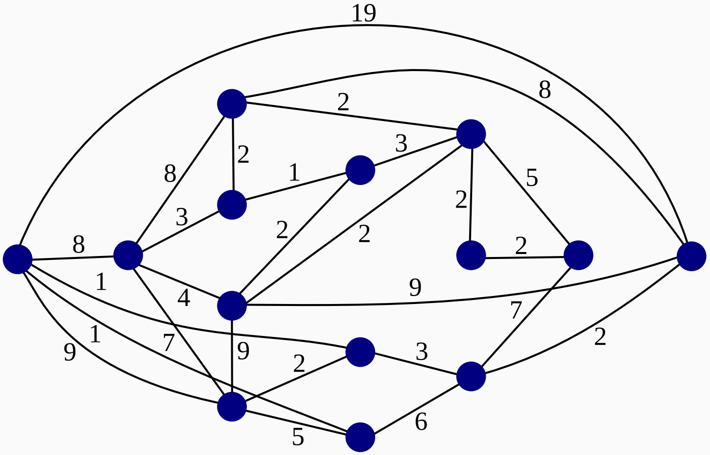
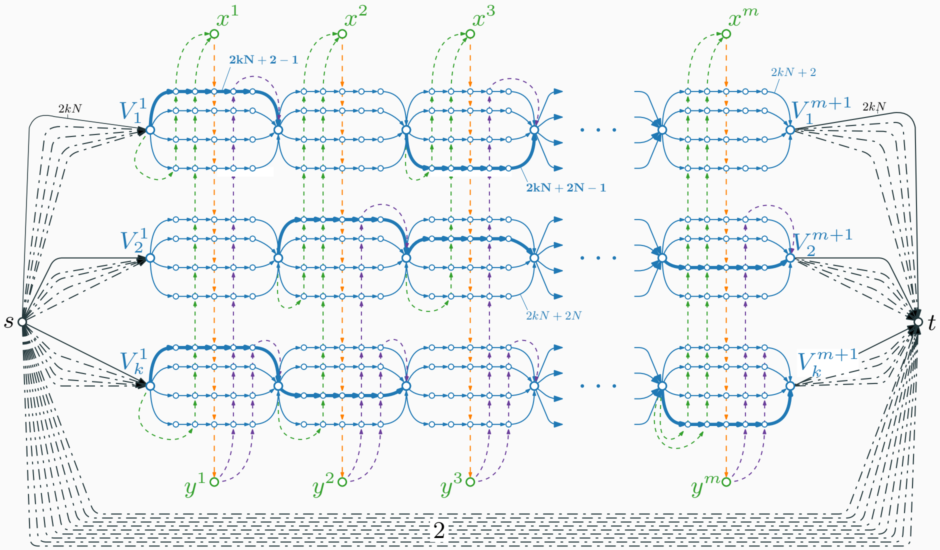


Second remark: a nice embedding



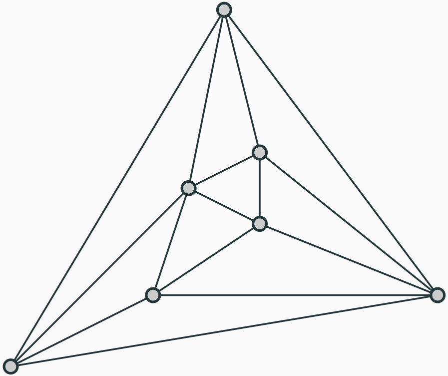
Circulating Orientation to Upward Planarity Testing

Black box

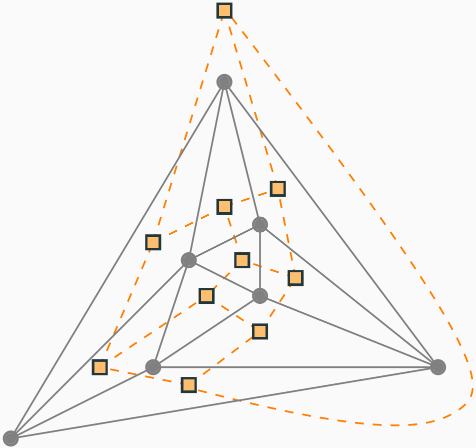
Theorem (Biedl'16)

There is a polynomial-time algorithm that, given a simple planar graph G of pathwidth k on at least three vertices, outputs a plane triangulation G' of G such that $\mathrm{pw}(G') \in \mathcal{O}(k)$.

Triangulated instance of CO



Dual Graph

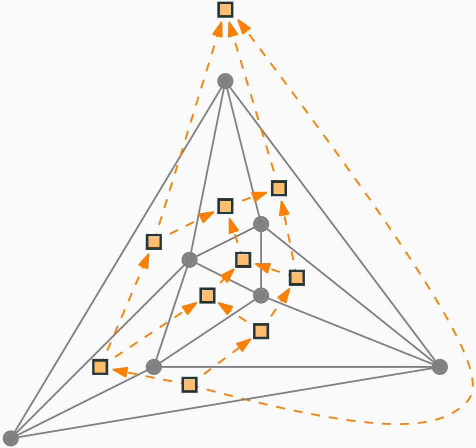


Black Box #2

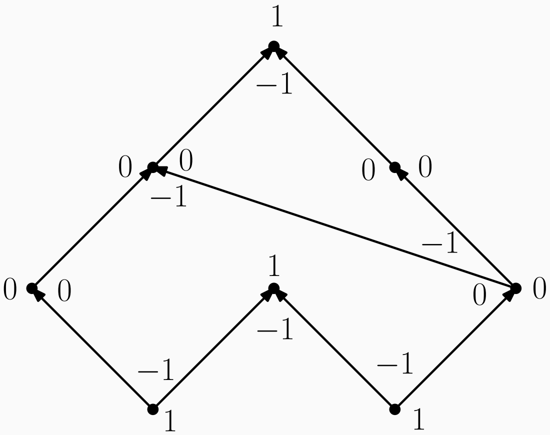
Theorem (Amini, Huc, and Pérennes'09)

For a triconnected planar graph G, $\mathrm{pw}(G^*) \leq 3\,\mathrm{pw}(G) + 2$, where G^* is the dual graph of G.

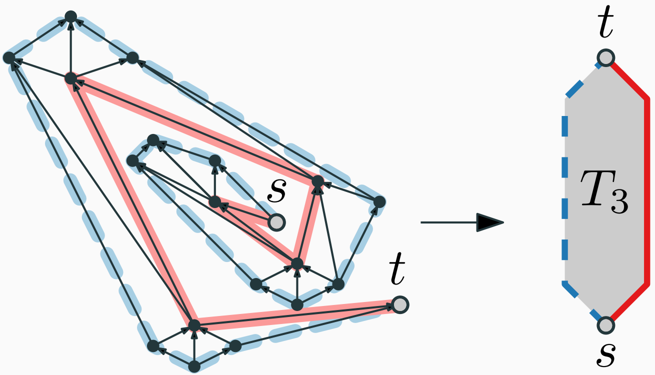
Testing a Dual Graph for Upward Planarity



Angle Assignment

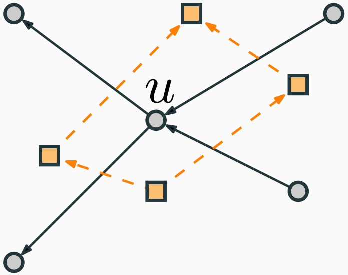


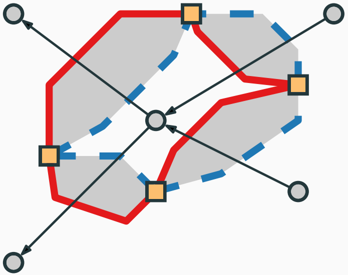
Tendril[2] Gadget



[2] A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

Reduction Idea: Face Balancing





… and Orthogonal Planarity Testing

Differences

- Important that we start with a triangulated graph
- Subdivision of edges to allow an orthogonal embedding
- Orthogonal Tendril[3]

[3] A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

Concluding remarks

Remarks

We have proved that

Known $n^{\mathcal{O}(\mathrm{tw})}$-algorithms cannot be improved to $n^{o(\mathrm{tw})}$ under ETH.

What other points are also one might find interesting:

- Alternative[4] proof of NP-completeness
- Hardness extends for cutwidth of the primal

[4] A. Garg and R. Tamassia, On the Computational Complexity of Upward and Rectilinear Planarity Testing, SIAM J. Computing, 1994

Further

- Membership in XNLP[5] of both Upward and Orthogonal Planarity Testing: can be solved nondeterministically in time $f(k)n^{\mathcal{O}(1)}$ and space $f(k)log(n)$?
- FPT or W[1]-hard for taking as a parameter the cutwidth of the dual graph
- More restrictive parameterizations may yield FPT algorithms

[5] H. L. Bodlaender et al. Parameterized Problems Complete for Nondeterministic FPT time and Logarithmic Space, FOCS'21

Thanks for attention!

Further directions

- Membership in XNLP
- Cutwidth of the dual graph
- Other parameterizations

Contents