Uniquely Hamiltonian Graphs

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Retreat Talk

Basic Definitions

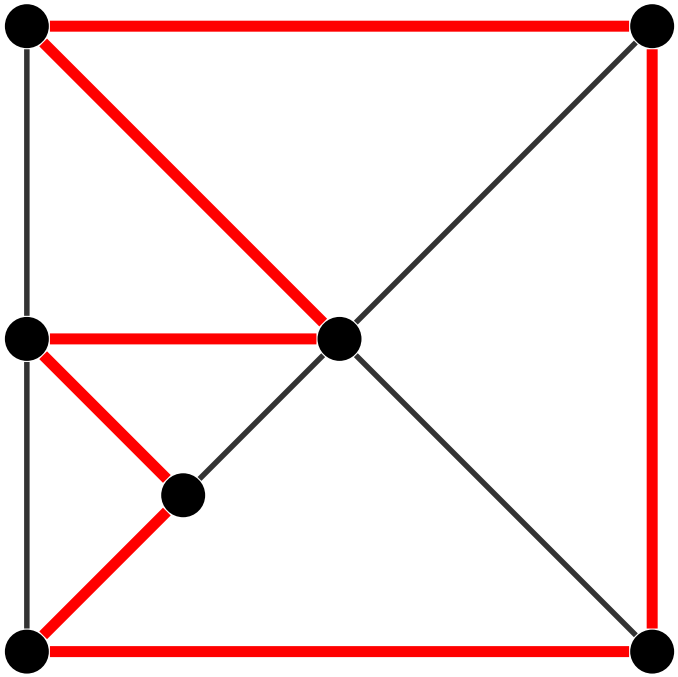
Definition (Hamiltonian Graph)

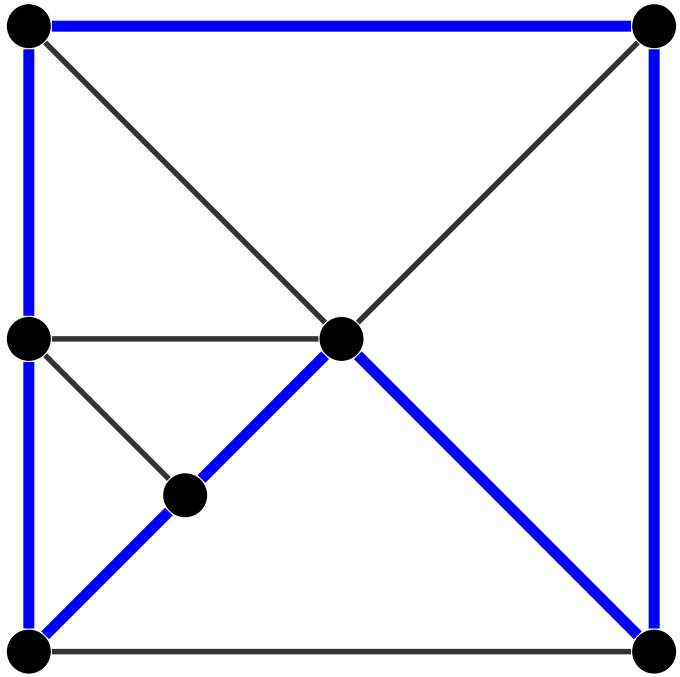
Let G be a (multi)graph. A *hamiltonian cycle* in G is a cycle in G which visits each vertex exactly once. A graph that contains a hamiltonian cycle is called a *hamiltonian graph*.

Definition (UHG)

If a graph contains exactly one hamiltonian cycle it is called a *uniquely hamiltonian graph* (UHG).

Example





Theoretical Results

Theorem

Let G be a loopless graph which contains only odd vertices, that is every vertex has odd degree. Then for every edge e in G, the number of hamiltonian cycles going through e is even. Therefore, G is not uniquely hamiltonian.

Theorem

There is no r-regular loopless UHG if $r > 22$.

Theorem

There is a 4-regular uniquely hamiltonian multigraph without loops.
There is a simple UHG with minimal degree four.

Theorem

Every simple planar UHG has at least two nodes with degree less than four.

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Conjectures

Sheehan's Conjecture

There is no 4-regular simple UHG.

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Simple Planar Graphs of Minimum Degree 3

Goal: Find a simple planar uniquely hamiltonian graph with minimum degree 3 and therefore disprove the conjecture of Bondy and Jackson.

Idea: Use graph transformations to weaken the requirements for the graph. Then find/construct a graph with the weaker requirements and apply the transformations.

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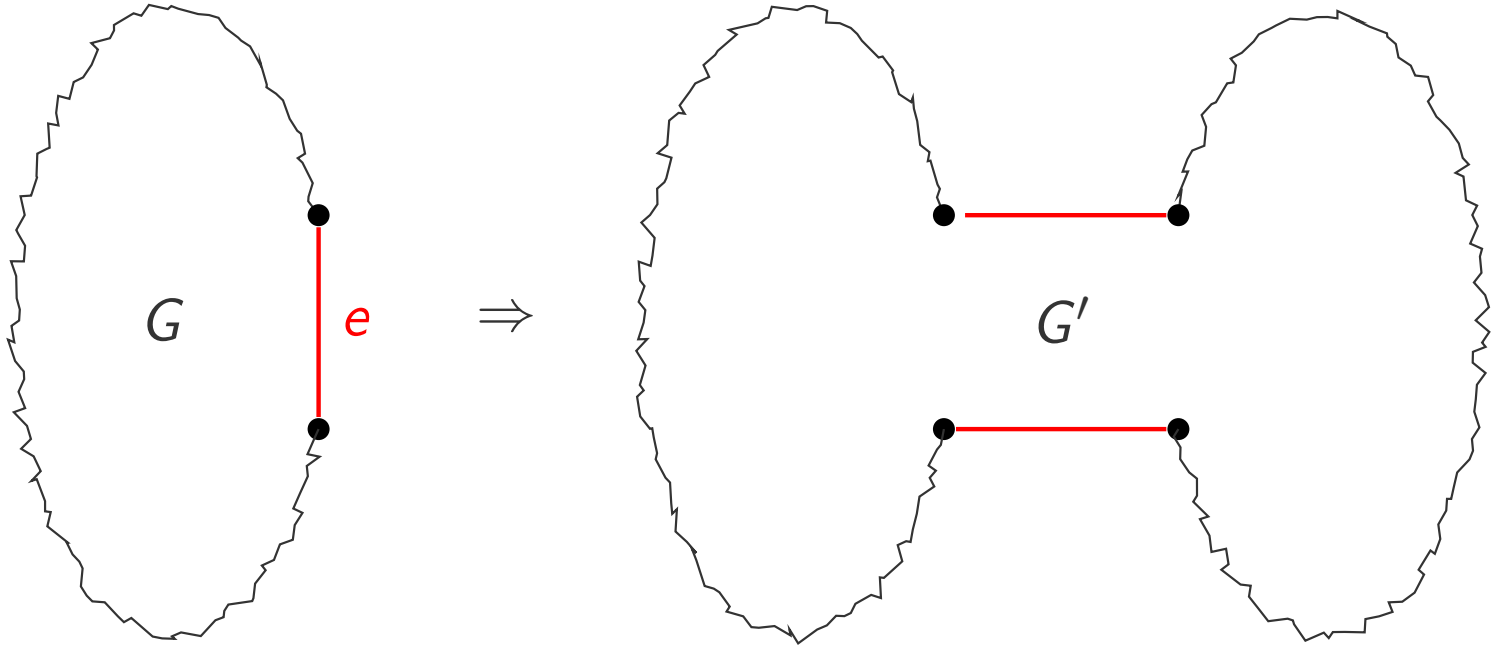
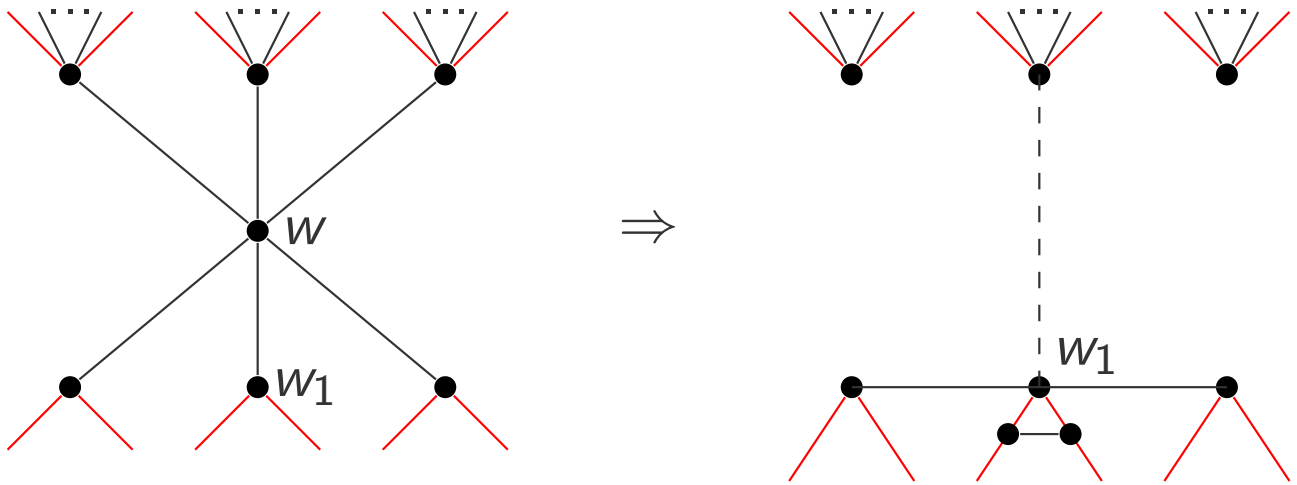
Fixing one Edge

Let G be a simple planar graph with minimal degree 3 and e an edge of G.
If G contains a uniuqe hamiltonian cycle containing the edge e we can construct a simple planar UHG G' with minimal degree 3.



Transformation for one Missed Node

Let G be a graph with a unique maximal cycle which misses w.
If all neighbors of w have degree higher than 3 we can simply remove w and all adjacent edges and get a simple planar hamiltonian graph with minium degree 3.
Let w_1 be a neighbor of w with degree 3. We transform G into a hamiltonian graph as follows.



The dashed edge is only used if w_1 is the only degree 3 neighbor.

Ultimative goal: Find a simple planar graph with minimum degree 3 and a unique maximal dominating cycle containing some edge e.
Algorithmic Approaches: Until now two algorithms, which can only test for unique maximal cycles missing exactly one vertex where developed:

- depth-first search algorithm by Fabian Leder
- integer linear program by Andreas Chvatal

About 200 hypohamiltonian graphs got already tested with these algorithms.