A SAT Approach to Branchwidth

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In memory of Helmut Veith

Branch Decomposition

Applications of Branch Decompositions:

- Introduced by Robertson and Seymour
- Similar to tree decomposition

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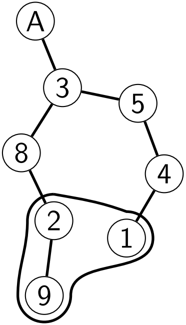
  - Hypergraphs
  - Matroids
  - Integer-valued symmetric submodular functions
  - Propositional CNF formulas
- Problems solved efficiently using dynamic programming on branch decomposition
  - Traveling salesman problem
  - #P-complete problem of propositional model counting
  - Generation of resolution refutations for unsatisfiable CNF formulas

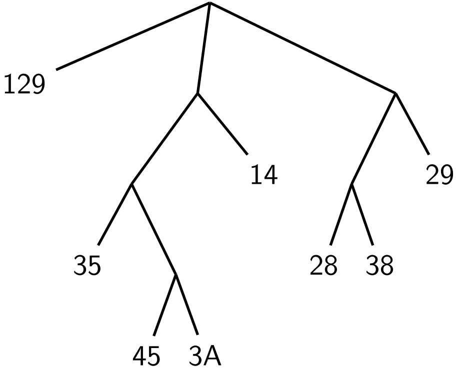
Branch Decomposition

Definition by Example:

Hypergraph G



Branch Decomposition of G



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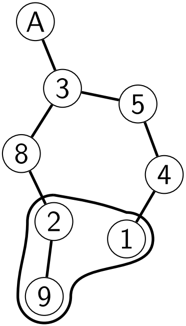

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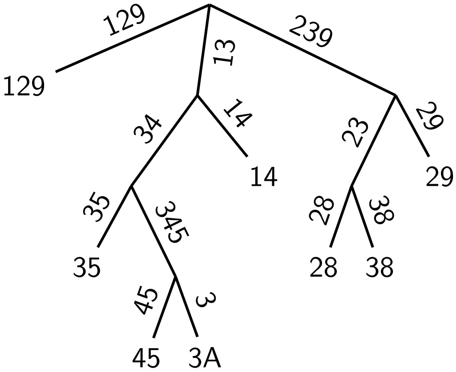

$Width = 3$

Branch Decomposition

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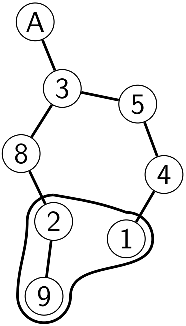

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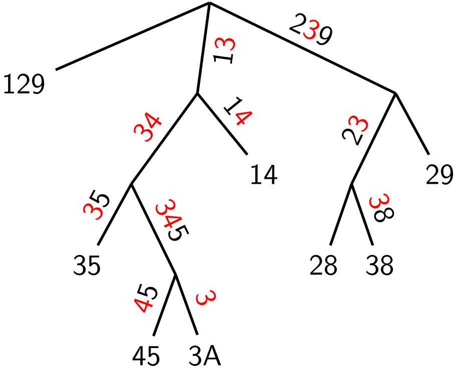

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Branchwidth

$$bw = \min_{\{\text{all decompositions}\}} \{\text{width of decomposition}\}$$

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Note: $bw \leq tw - 1 \leq \frac{3}{2} bw$

Why small widths?

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The following was noted by Kask et al. [2011]

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Task: Finding smallest width efficiently

Note: Finding optimal width is *NP*-hard

Existing Algorithms for Branch Decompositions

- Heuristics : splitting graph iteratively along a small cut

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- Heuristics : splitting graph iteratively along a small cut
- Exact :
  - Interger linear program (ILP) developed by Ulusal [2008]
  - An exponential-time tangle based algorithm (TANGLES), suggested by Robertson and Seymour [1991] and implemented by Hicks [2005]

Related work

Treewidth (tw)

- Samer and Veith [2009] introduced the first SAT encoding for treewidth.
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Clique width (cw)

- Heule and Szeider [2015] introduced the first SAT encoding for clique width (This is the first exact algorithm for clique width)

Our Contribution

- Efficient SAT-encoding based on new partition based characterization of branch decomposition
- Using local improvement techniques to avail use of SAT for large instances

First Attempt

Encoding Branch Decomposition Tree

- Encoded the Branch Decomposition Tree to CNF formula

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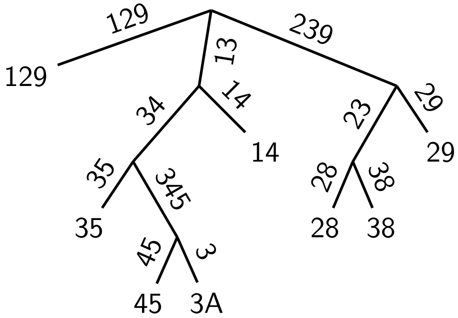
Encoding Branch Decomposition Tree

- Encoded the Branch Decomposition Tree to CNF formula
- This method could not solve even up to 30 edges

New Approach

Partition Based Characterization



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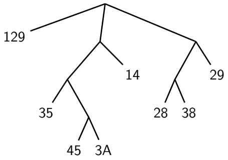

$$\Big\{\{129\}, \{35\}, \{45\}, \{3A\}, \{14\}, \{28\}, \{38\}, \{29\}\Big\}$$

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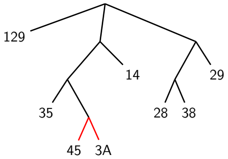

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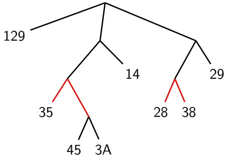

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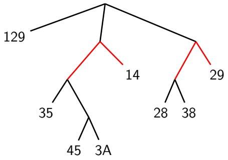

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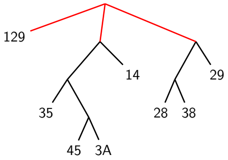

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SAT Encoding

Tools for encoding partition based characterization

- Equivalence classes $s(e, f, i)$
  To encode refinement as underlying tree is implicitly represented by refinement of partitions

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Tools for encoding partition based characterization

- Equivalence classes $s(e, f, i)$
- Cut along the edges $c(e, u, i)$
  Set for every vertex that is cut vertex

SAT Encoding

Tools for encoding partition based characterization

- Equivalence classes $s(e, f, i)$
- Cut along the edges $c(e, u, i)$
- Cardinality constraints
  - To bound the number of cut vertices
  - Sequential unary counter

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Tools for encoding partition based characterization

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- Cardinality constraints

Output

For a (hyper) graph G and an integer k we produce formula $F(G, k)$ which is satisfiable iff G has a branch decomposition of width k.

Experimental setup

- SAT solver: Glucose 4.0
- System: 4-core Intel Xeon CPU E5649, 2.35GHz, 72GB RAM, Ubuntu 14.04
- Memory limit: 8GB
- Timeout: 6hours
- Instances: Famous named graphs

First Results

Graph	$\lvert V \rvert$	$\lvert E \rvert$	w
Watsin	50	75	6
Kittell	23	63	6
Holt	27	54	9
Shrikhande	16	48	8
Errera	17	45	6
Brinkmann	21	42	8
Clebsch	16	40	8
Folkman	20	40	6
Paley13	13	39	7
Poussin	15	39	6
Robertson	19	38	8

Limits

- We have observed a limit of 70 edges
  - Timeout
  - The size of encoding (some cases exceeded 1GB)
  - Same limit is observed for *tw* and *cw*

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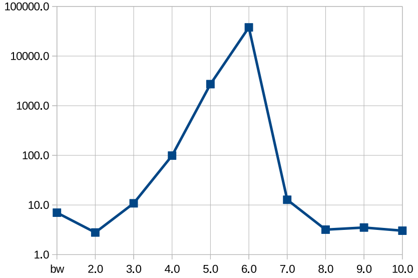
- We have observed a limit of 70 edges
  - Timeout
  - The size of encoding (some cases exceeded 1GB)
  - Same limit is observed for *tw* and *cw*
- But we can provide upper bounds for graphs with up to 150 edges in reasonable time



Branchwidth = 7

What do we do now?

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Extend SAT to large instances?

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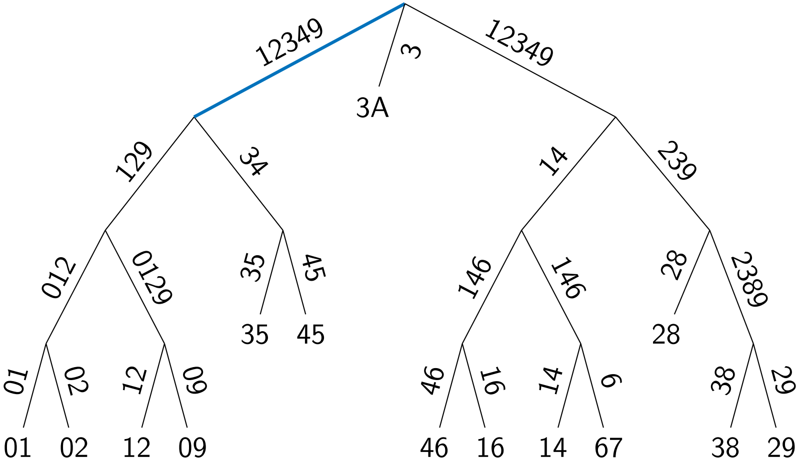
Use it locally!

How?

1. Generate heuristic decomposition (we used heuristics provided by Hicks)
2. Pick local branch decomposition around large cut (using specialized DSF procedure)
3. Use SAT to improve local branch decomposition and plug it back in

Repeat till no more improvement possible or timeout

By Example



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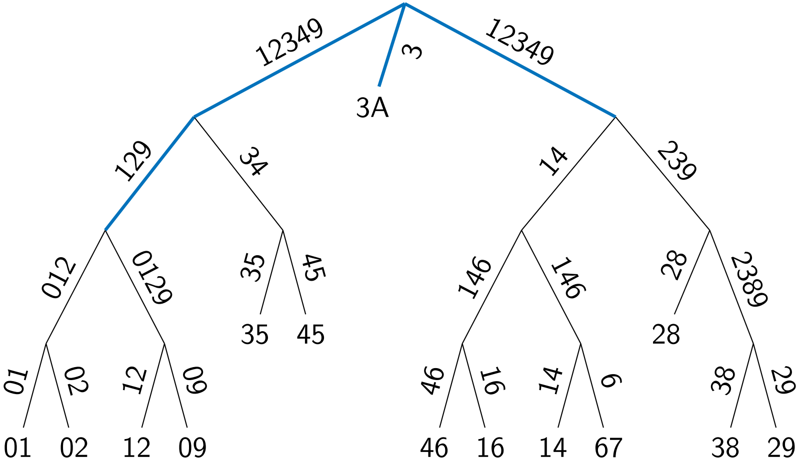
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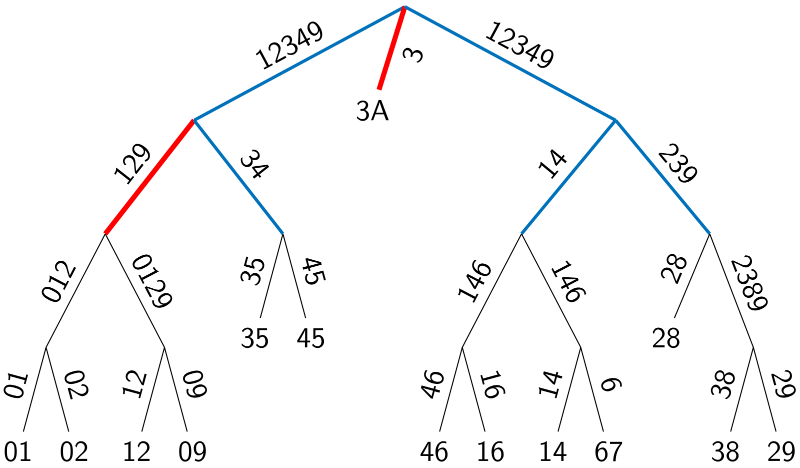
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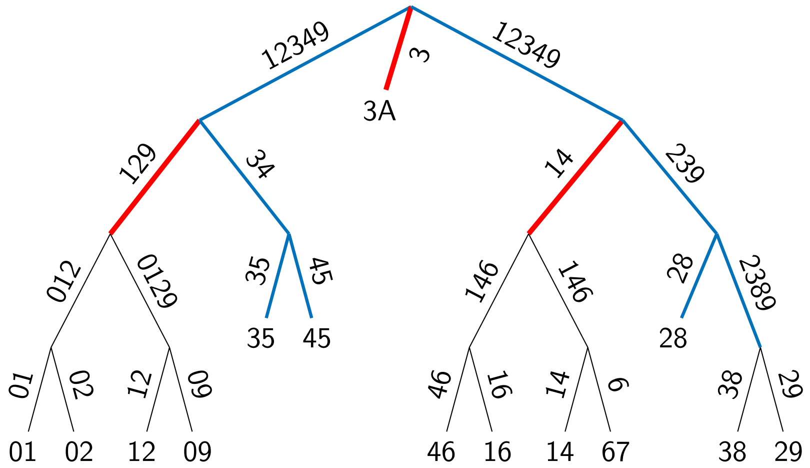

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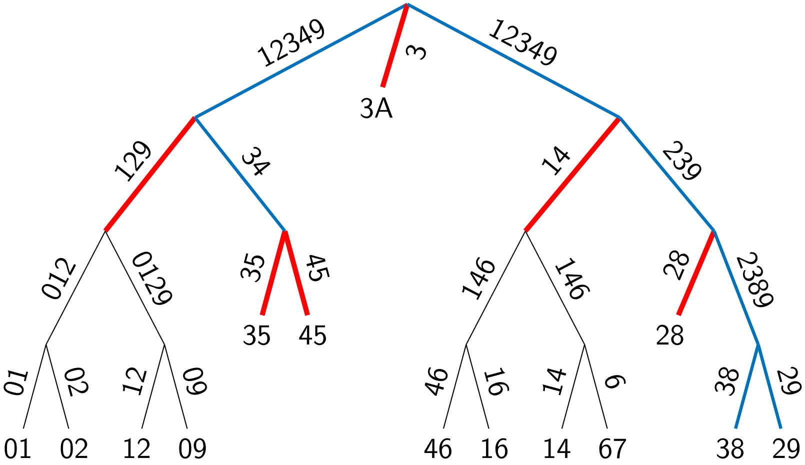

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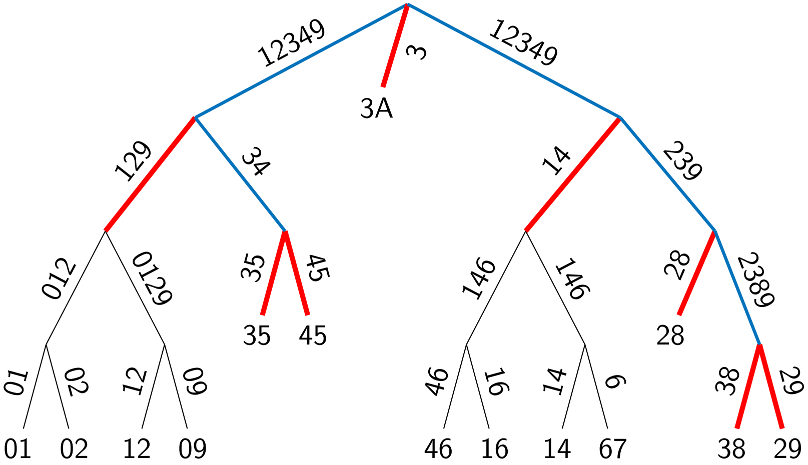
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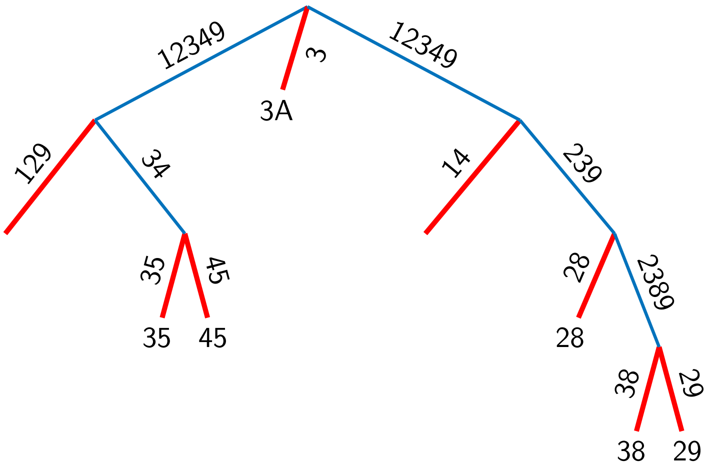

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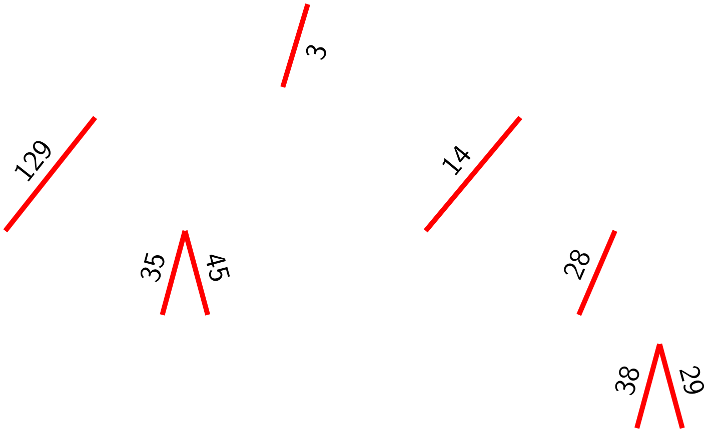


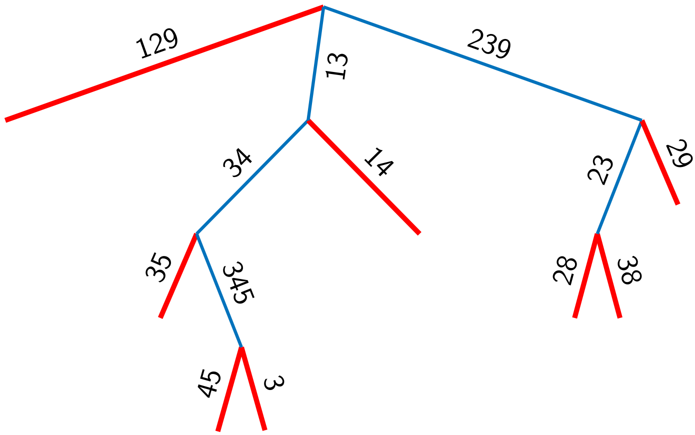
Hypergraph consisting of all boundary edges

By Example

Input:
The hypergraph (H) consisting every boundary **edge**

Theorem

Improved branch decomposition of hypergraph (H) can be plugged in to the original decomposition.



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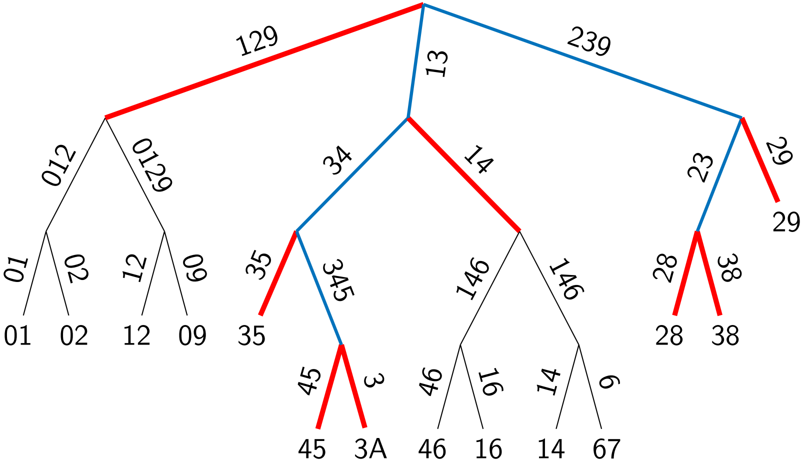
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- System: 4-core Intel Xeon CPU E5649, 2.35GHz, 72GB RAM, Ubuntu 14.04
- Memory limit: 8GB
- Timeout: 6hours
- Timeout per SAT call: 600sec
- Size of local branch decomposition: 80edges
- Instances: TreewidthLIB

Results

Graph	$\lvert V\rvert$	$\lvert E\rvert$	iw	w	diff
inithx.i.2-pp	363	8897	55	45	10
fpsol2.i.2-pp	333	7910	48	39	9
graph13	458	1877	141	134	7
bn_31-pp	1148	3317	40	36	4
bn_4	100	574	42	38	4
celar08-pp-003	76	421	20	16	4
fpsol2.i.2	451	8691	53	49	4
graph05-wpp	94	397	28	24	4
graph04-pp	179	678	52	48	4
u724.tsp	724	2117	29	26	3
water-wpp	22	96	11	8	3
celar05-pp	80	426	18	15	3
mulsol.i.2-pp	116	2468	62	59	3

How are we compared to others?

Exact Encodings:

- TANGLES[1]
  - Exponential time and space algorithm ($\mathcal{O}(m^k)$)
  - Limit of branchwidth 8
  - Cannot deal with large graphs

[1] These results are based on older hardware and software, thus the comparison can not be concrete

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- ILP[1]
  - Limit of 13 edges for hypergraphs

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The SAT-based local improvement

- Scales to large instance with several thousands edges
- High branchwidth upper bounds

Future Work

1. Extending the encoding to obtain specialized decomposition to aid local improvement
2. Encoding various other parameters such as boolean width, rank width (similar decomposition scheme)
3. Extending the branch decomposition approach to apply in field of knowledge compilation
4. Extending current approach with incremental and MAXSAT solving

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- SAT-based local improvements for branch decompositions
  - Provides the means for scaling the SAT-approach to much larger instances
  - New application field of SAT solvers

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Thank you.

References

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