Algorithmic Meta-Theorems

192.122

WS21/22

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ALGORITHMS AND COMPLEXITY GROUP

Informatics

Announcements

- Today last lecture, Friday was last exercise.
- 30 min overtime today.
- Feedback on Tuwel is highly appreciated (short anonymous survey).
- Oral exam: in January/February
- There will be a question session before the oral exam.
- Write email until end of year to dreier@ac.tuwien.ac.at if you want to participate in the oral exam.

Main Result for Sparse Graphs

Today, we finish the proof of the following theorem.

Theorem (Dvořák, Král, Thomas 2013)

Let $\mathcal{C}$ be a graph class with bounded expansion. There exists a function f such that for every FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n$.

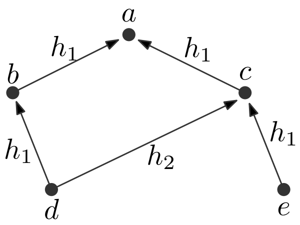
And then briefly talk about dense graphs.

Functional Graphs

A *functional graph* $\vec{G}$ is a structure with signature $\tau = \{h_1, h_2, \ldots, R_1, R_2, \ldots, Q_1, Q_2, \ldots\}$ where

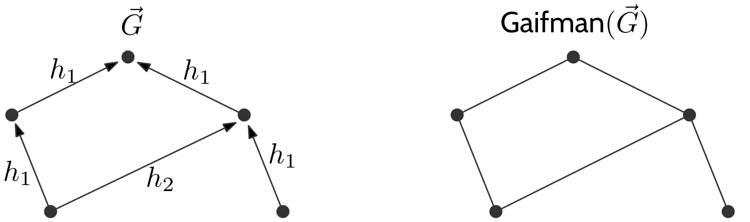
- $h_i \colon V \to V$ are unary functions
- $R_i \subseteq V$ are unary relations
- $Q_i \in \{0, 1\}$ are nullary relations

$$h_1(d) = b$$
$$h_2(d) = c$$



Classes

For a functional graph $\vec{G}$, the graph Gaifman$(\vec{G})$ has the same vertex set and edges uv iff $h_i(u) = v$ or $h_i(v) = u$ for some i.



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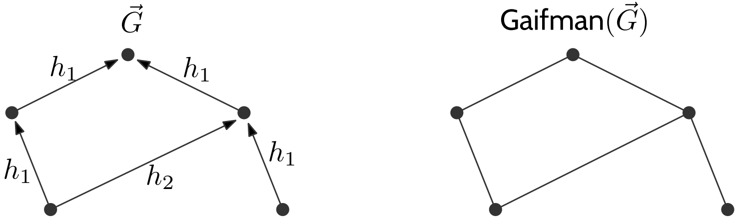

We say a class of functional graphs has *bounded expansion* if the class of their Gaifman graphs has.

Quantifier Elimination

$$\begin{array}{ccc} \vec{G} & & \vec{G}' \\ & \longrightarrow & \\ \exists y\, \varphi(y\bar{x}) & & \varphi'(\bar{x}) \end{array}$$

$$\begin{array}{ccc} \vec{G} & & \vec{G}' \\ \exists y\, \varphi(y\bar{x}) & \rightarrow & \varphi'(\bar{x}) \end{array}$$

Should still be short. Length depends only on φ.

Quantifier Elimination

Should still be sparse. Same Gaifman graph as $\vec{G}$.

$$\begin{matrix} \vec{G} & & \vec{G}' \\ \exists y\, \varphi(y\bar{x}) & \longrightarrow & \varphi'(\bar{x}) \end{matrix}$$

Should still be short. Length depends only on φ.

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We say there is a **quantifier elimination procedure** for a class $\mathcal{C}$ if the following holds.

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$$\vec{G} \models \exists y\, \varphi(y\bar{v}) \quad \text{iff} \quad \vec{G}' \models \varphi'(\bar{v}) \qquad \text{for all } \bar{v} \in V(\vec{G})^{|\bar{x}|}.$$

Quantifier Elimination

If we have a quantifier elimination procedure, we can do model-checking.

$$\vec{G} \models \exists x_1 \left(\qquad\qquad\qquad\qquad\qquad\qquad \right)$$

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$$\vec{G} \models \exists x_1 \left(\forall x_2 \left(\exists x_3 \overbrace{\varphi(x_1 x_2 x_3)}^{\text{quantifer-free}} \right) \right)$$

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$$\vec{G} \models \exists x_1 \left(\forall x_2 \left(\underbrace{\exists x_3 \; \overbrace{\varphi(x_1 x_2 x_3)}^{\text{quantifer-free}}}_{\text{replace with quantifier-free}} \right)\right)$$

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$$\vec{G}' \models \exists x_1 \left(\forall x_2 \; \overbrace{\varphi'(x_1 x_2)}^{\text{quantifier-free}} \right)$$

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$$\vec{G}'' \models \exists x_1 \quad \overbrace{\varphi''(x_1)}^{\text{quantifier-free}}$$

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Quantifier Elimination

If we have a quantifier elimination procedure, we can do model-checking.

$$\vec{G}''' \models \overbrace{\varphi'''}^{\text{quantifier-free}}$$

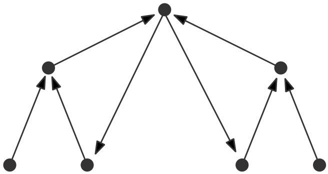
Roadmap

Roadmap: Construct quantifier elimination for graph classes of increasing complexity.

- forests of bounded depth ✓
- bounded treedepth
- bounded expansion

Bounded Treedepth

Let $\mathcal{C}$ be a class whose Gaifman graphs have *bounded treedepth*. Then $\mathcal{C}$ has a quantifier elimination procedure.

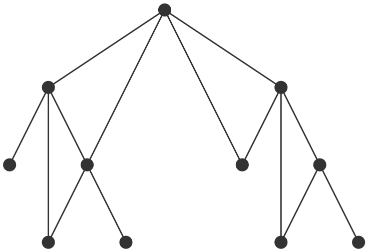


Bounded Treedepth

bounded
treedepth

$$\vec{G}$$

$$\exists y \, \varphi(y\bar{x})$$

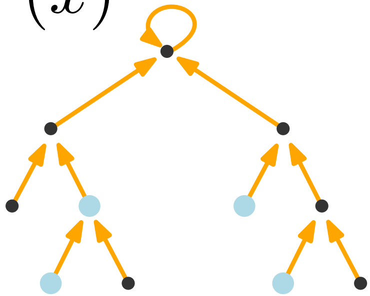


Bounded Treedepth

bounded treedepth → $\vec{G}$

forest of bounded depth → $\vec{G}'$

$$\exists y\, \varphi(y\bar{x}) \quad \longrightarrow \quad \exists y\, \varphi'(\bar{x})$$



Bounded Treedepth

bounded treedepth

$$\vec{G}$$

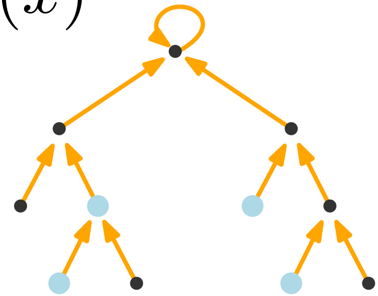
$$\longrightarrow$$

forest of bounded depth

$$\vec{G}''$$

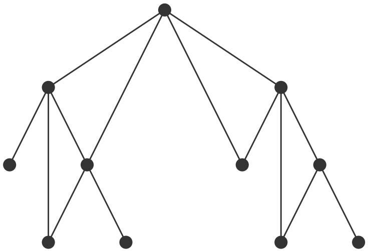
$$\exists y\, \varphi(y\bar{x}) \qquad \varphi''(\bar{x})$$

Use previous theorem for quantifier elimination on forests.



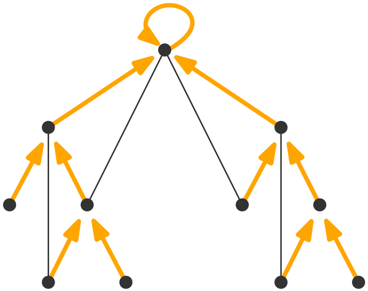
Bounded Treedepth

- Consider Gaifman graph of $\vec{G}$.



Bounded Treedepth

- Consider Gaifman graph of $\vec{G}$. Build depth-first search tree and make it functional.



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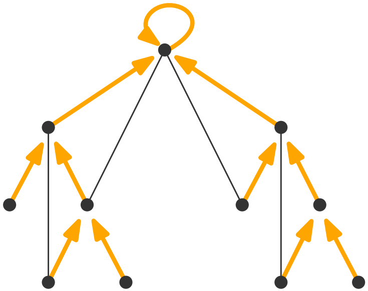
- Consider Gaifman graph of $\vec{G}$. Build depth-first search tree and make it functional. Graphs with treedepth d contain no paths longer than 2^d.



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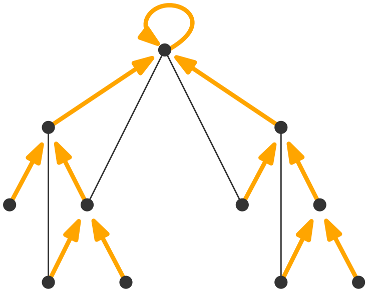
- Consider Gaifman graph of $\vec{G}$. Build depth-first search tree and make it functional. Graphs with treedepth d contain no paths longer than 2^d. We have a functional forest of depth at most 2^d.



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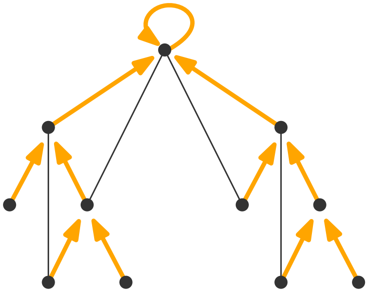

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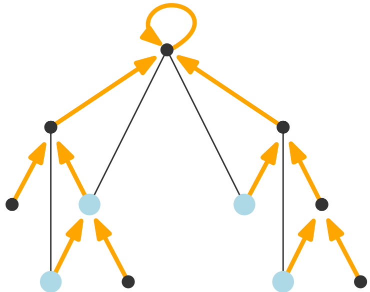
- All edges go between ancestors in the tree. Encode edges of $\vec{G}$ by predicates at lower endpoints.



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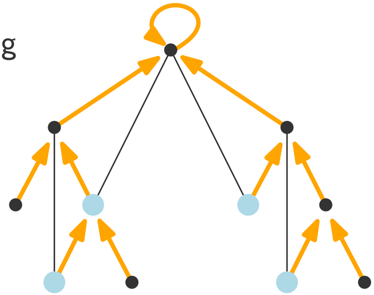
$P_{i,j,\updownarrow}(v)$: "There is edge from v to parent$^i(v)$ labeled with j going upwards/downwards".



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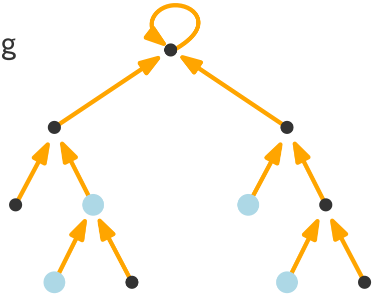

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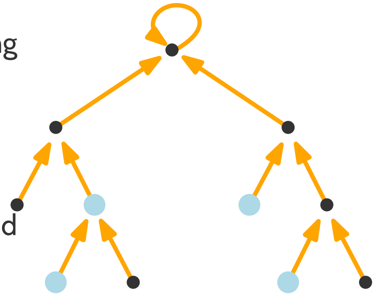
- This tree fully encodes $\vec{G}$.
- Replace atoms $f_j(x) = y$ by guessing tree-relationship and checking the new predicates.



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$$\vec{G}$$

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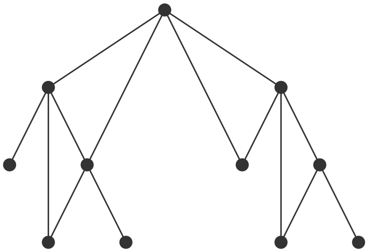

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$$\vec{G} \longrightarrow \vec{G}'$$

forest of bounded depth

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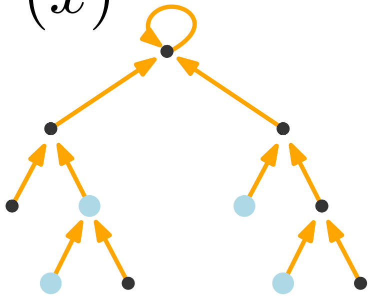

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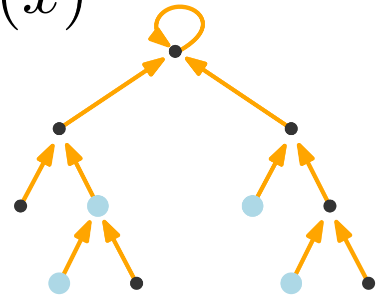

The last step.

Let $\mathcal{C}$ be a class with bounded expansion. Then $\mathcal{C}$ has a quantifier elimination procedure.

Low Treedepth Colorings

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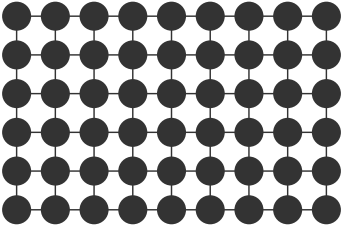

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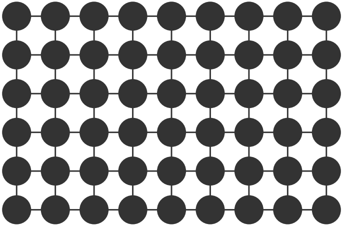

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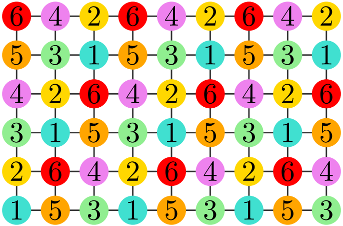

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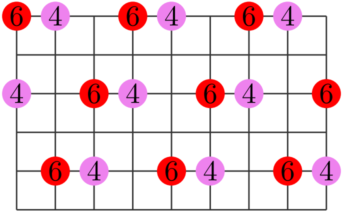

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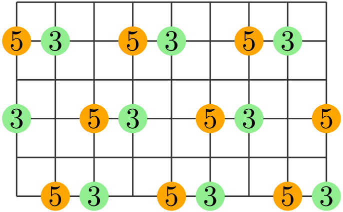

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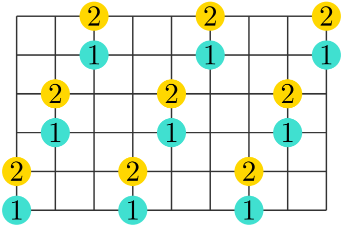

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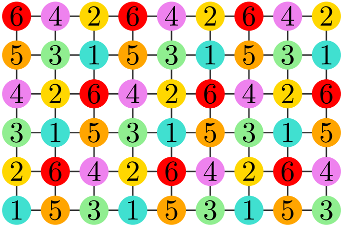

Bounded Expansion

We are given a formula $\exists y \varphi(\bar{x})$ where φ is quantifier-free.

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Replace all function applications such as $p(g(x_i)) = x_j$ with directed labeled edges such as $\exists a\, x_i \underset{g}{\rightsquigarrow} a \wedge a \underset{p}{\rightsquigarrow} x_j$.

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Replace all function applications such as $p(g(x_i)) = x_j$ with directed labeled edges such as $\exists a \, x_i \underset{g}{\rightsquigarrow} a \wedge a \underset{p}{\rightsquigarrow} x_j$.

The result is a formula $\exists \bar{y} \, \psi(\bar{y}\bar{x})$ on a normal (non-functional) directed edge-labeled graph G.

Bounded Expansion

For every $\bar{v} \in V(G)^{|\bar{x}|}$

$$G \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

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For every $\bar{v} \in V(G)^{|\bar{x}|}$

$$G \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

$$\bigvee_{\substack{S \subseteq \Lambda \\ |S|=p}} \text{colors}(\bar{v}) \subseteq S \wedge$$

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$$\bigvee_{\substack{S \subseteq \Lambda \\ |S| = p}} \text{colors}(\bar{v}) \subseteq S \wedge G[S] \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

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For all graphs $G[S]$ (of treedepth $\leq p$) we perform quantifier elimination.

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$$\bigvee_{\substack{S \subseteq \Lambda \\ |S|=p}} \text{colors}(\bar{v}) \subseteq S \wedge \vec{G}'_S \models \psi_S(\bar{v}) \quad \text{iff}$$

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For all graphs $G[S]$ (of treedepth $\leq p$) we perform quantifier elimination. And stack the graphs on top of each other.

For every $\bar{v} \in V(G)^{|\bar{x}|}$

$$G \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

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$$\vec{G}' \models \bigvee_{\substack{S \subseteq \Lambda \\ |S|=p}} \text{colors}(\bar{v}) \subseteq S \wedge \psi_S(\bar{v}).$$

Finally, pull out the graph.

Summary

This completes the proof. We have solved the model-checking problem on bounded expansion by performing quantifier elimination on trees and lifting it up using low treedepth colorings.

A Note on Nowhere Dense Classes

General rule: Things that work for bounded expansion also work for nowhere dense, but in an uglier way.

Comparison

Bounded Expansion

Nowhere Dense

Comparison

Bounded Expansion

Nowhere Dense

p-treedepth colorings with $f(p)$ colors ✓

Comparison

Bounded Expansion

p-treedepth colorings with $f(p)$ colors ✓

Nowhere Dense

p-treedepth colorings with $f(\varepsilon, p)n^{\varepsilon}$ colors for all $\varepsilon > 0$ ✓

Comparison

Bounded Expansion

p-treedepth colorings with $f(p)$ colors ✓

Enumerate p-subsets in time $\binom{f(p)}{p}$ ✓

Nowhere Dense

p-treedepth colorings with $f(\varepsilon, p)n^{\varepsilon}$ colors for all $\varepsilon > 0$ ✓

Comparison

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p-treedepth colorings with $f(p)$ colors ✓

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Enumerate p-subsets in time $\binom{f(\varepsilon,p)n^\varepsilon}{p}$

Comparison

Bounded Expansion

p-treedepth colorings with $f(p)$ colors ✓

Enumerate p-subsets in time $\binom{f(p)}{p}$ ✓

Nowhere Dense

p-treedepth colorings with $f(\varepsilon, p)n^\varepsilon$ colors for all $\varepsilon > 0$ ✓

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Are there any more interesting sparse graph classes beyond nowhere dense?

We show the answer is no! We have found the most general sparse graph classes on which first-order problems are tractable!

Precise Chacaterization

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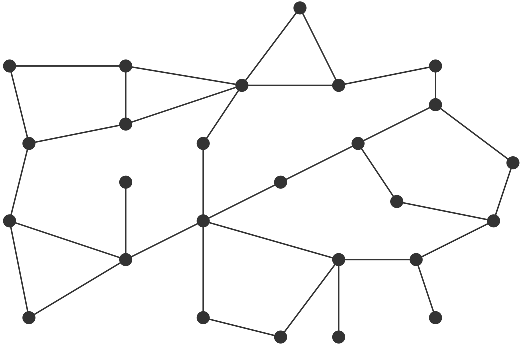
We discussed the forward implication already. We will prove the backward implication.

Shallow Topological Minors

H is an *depth-r topological minor* of G ($H \preccurlyeq_r^{\mathrm{top}} G$) if H can be built from G by



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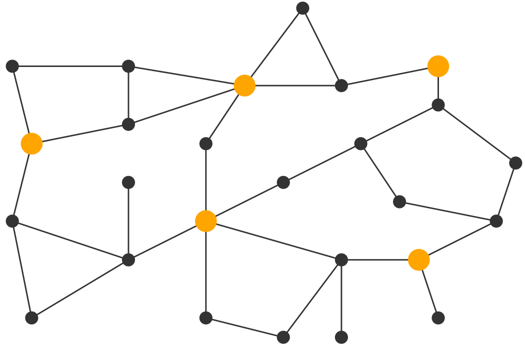

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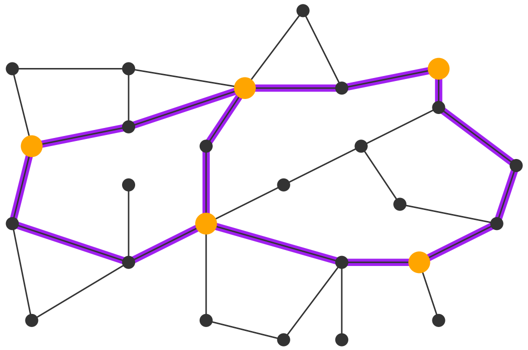

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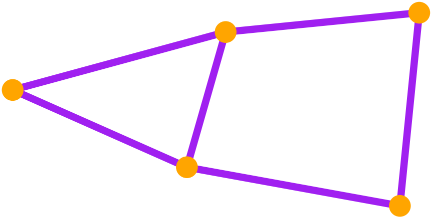
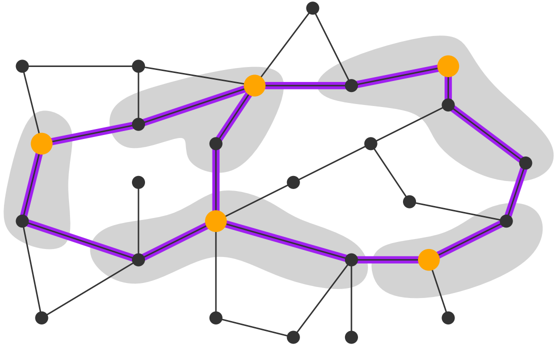


If H is an depth-r topological minor then it also is a depth-r minor.



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We can also measure sparsity using topological minors.

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Surprisingly, also

- $\nabla_r(G) \leq 2^{d^2+3d+3}(\lceil\tilde{\nabla}_r(G)\rceil)^{(d+2)^2}$
- $\omega_r(G) \leq 1 + (\tilde{\omega}_{3r+1}(G))^{2d+2}$

Equivalent Definitions

Therefore it does not matter if we consider normal or topological shallow minors.

Bounded Expansion

A graph class $\mathcal{C}$ has bounded expansion if there exists a function $f(r)$ such that for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$ we have $\nabla_r(G) \leq f(r)$.

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We can slightly restate this.

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We say a graph class is *somewhere dense* if it is not nowhere dense:

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Subdivisions of Arbitrary Subgraphs

We show the following.

Lemma

If a graph class is monotone and somewhere dense then first-order model-checking on this class is AW[$*$]-hard.

Subdivisions of Arbitrary Subgraphs

We need Ramsey's Theorem.

Lemma

For every $t, r \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that an edge-coloring of K_k with r colors contains a monochromatic K_t.

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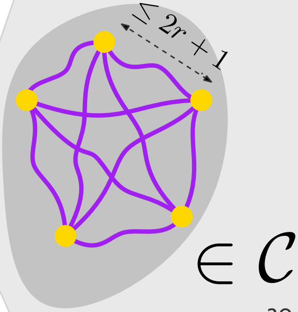

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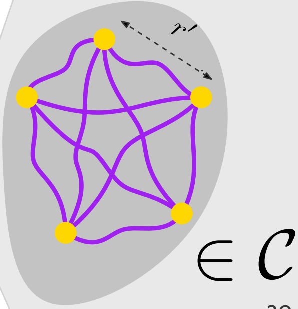

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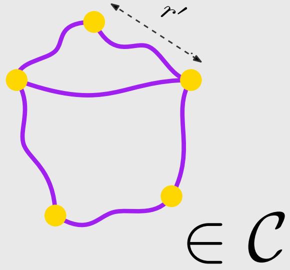

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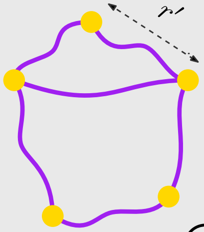

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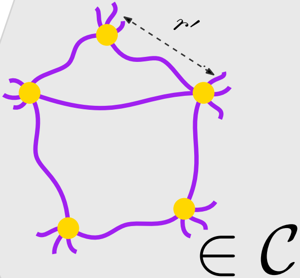
- We may even assume that the nails have some "hair" to distinguish them.



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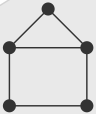

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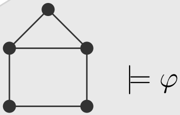

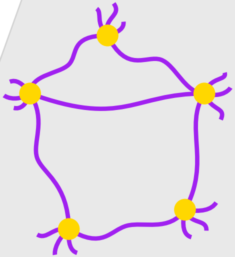

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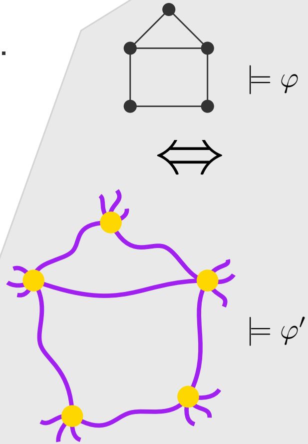

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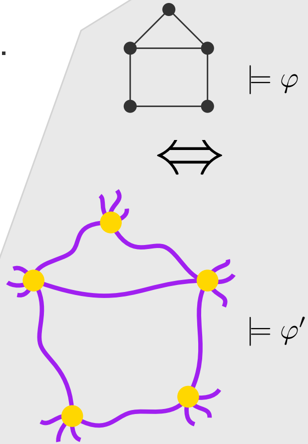
- Since the modified graph is in $\mathcal{C}$, the modified formula can be evaluated in fpt time.



Summary

For monotone graph classes, we cannot go beyond nowhere dense classes. Nowhere dense classes are a natural definition of “sparse graphs”.

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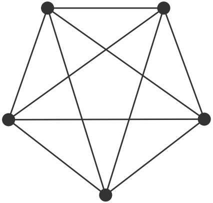

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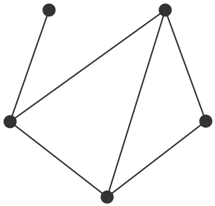

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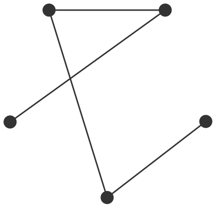

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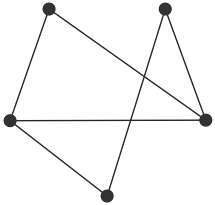

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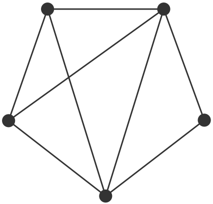

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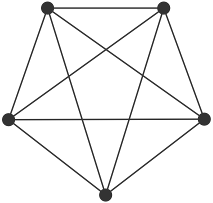

Beyond Sparsity

Goal: Find tractable classes beyond sparsity.

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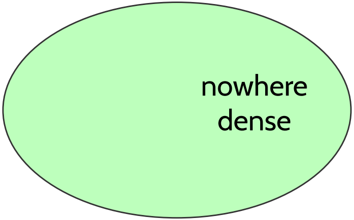
Today, we discuss some results, as well as possible candidates.

Classes with FPT first-order model-checking?



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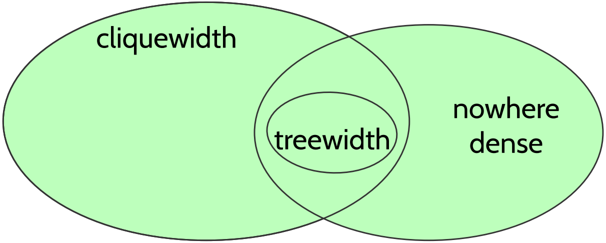
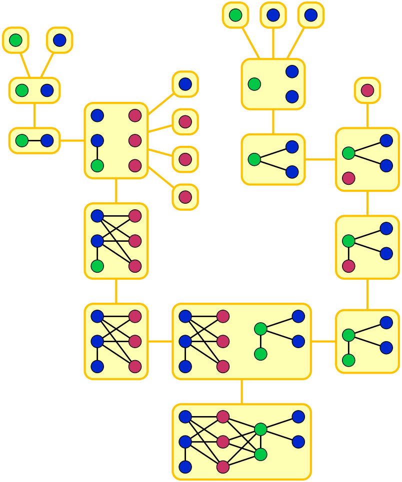


Cliquewidth



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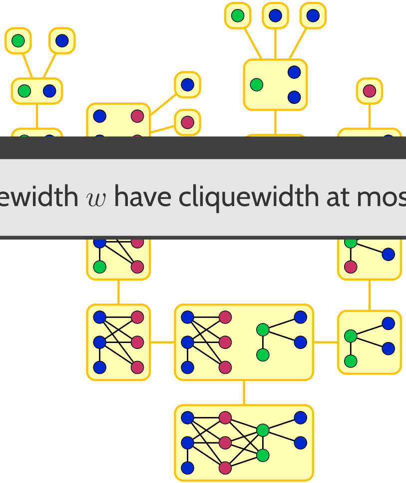

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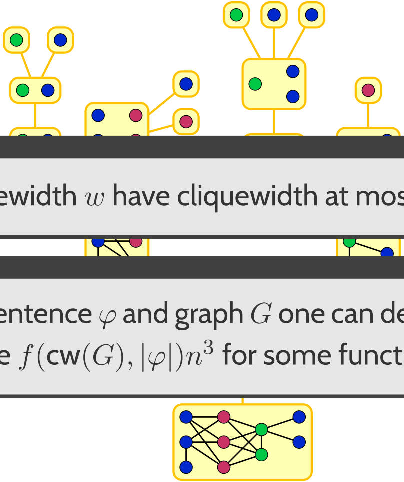

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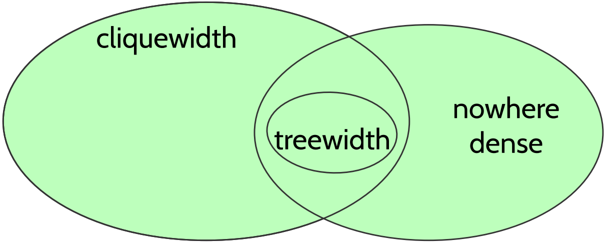
For a MSO_1 sentence φ and graph G one can decide whether $G \models \varphi$ in time $f(\mathsf{cw}(G), |\varphi|)n^3$ for some function f.

Classes with FPT first-order model-checking?



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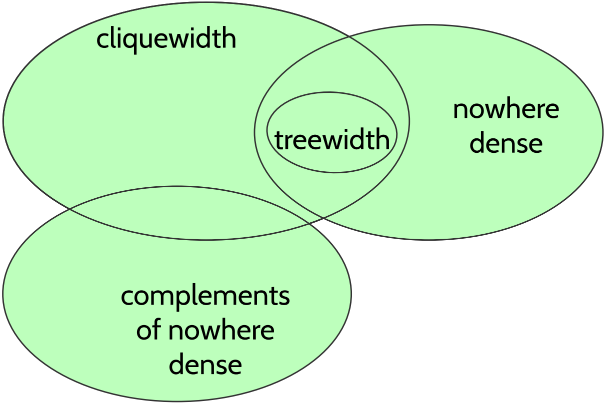
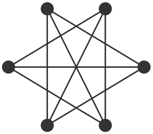


Complements

First-order model-checking is fpt on complements of nowhere dense classes by reduction.



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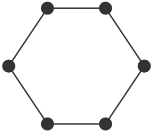

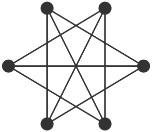

$$G \models \bar{\varphi} \iff \bar{G} \models \varphi$$

Complements

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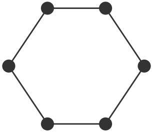
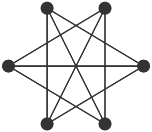


$G \models \bar{\varphi}$

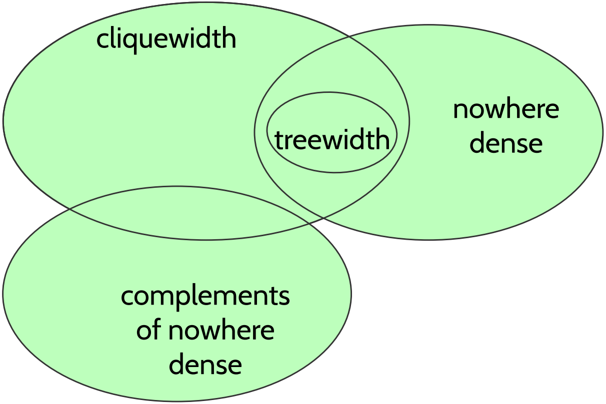
obtain $\bar{\varphi}$ from φ by swapping $\sim$ and $\not\sim$

$\Leftrightarrow$



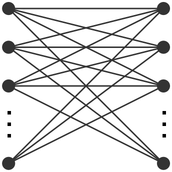
$\bar{G} \models \varphi$

Classes with FPT first-order model-checking?



Fully Bipartite Graphs

Can we do fpt model-checking on the class of fully bipartite graphs?



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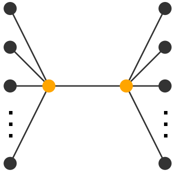
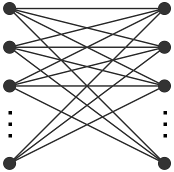


$G' \models \varphi'$



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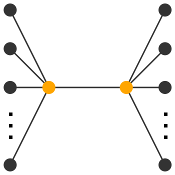

$G' \models \varphi'$

obtain φ' from φ by replacing $x{\sim}y$ with $\text{dist}(x,y)=3$

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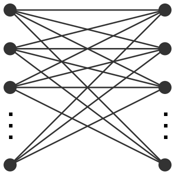

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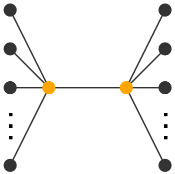

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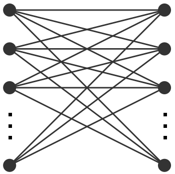
also restrict quantifiers to black vertices



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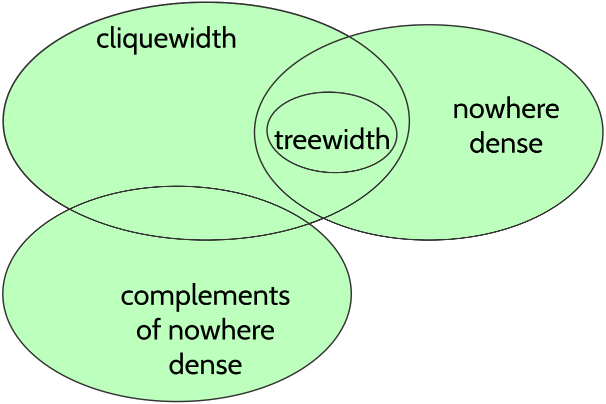

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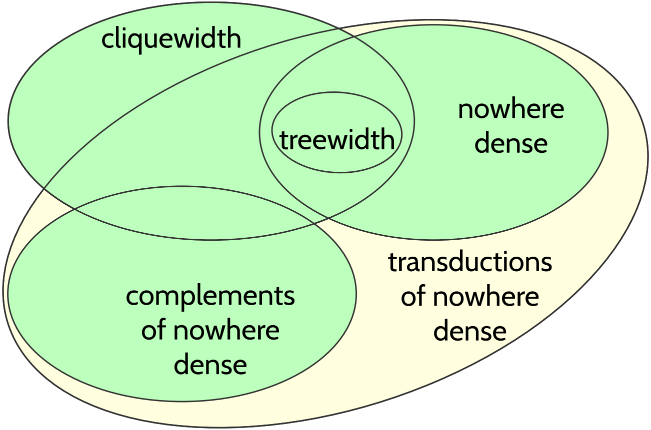
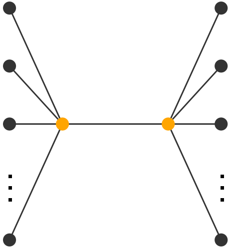


Interpretations

An interpretation $I = (\mu(x,y), \nu(x))$ maps G to $I(G)$.

G $\qquad\qquad$ $I(G)$



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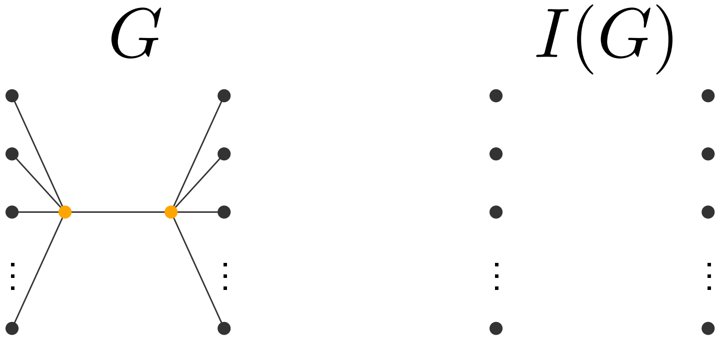

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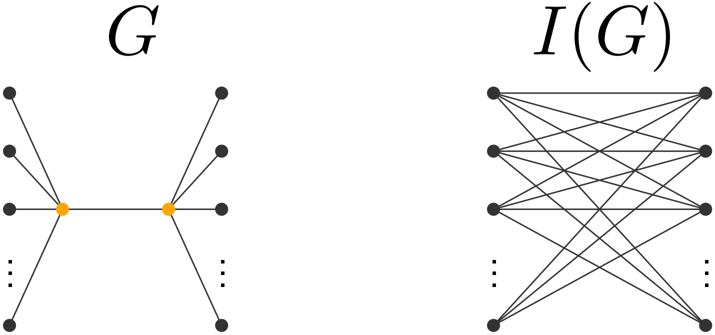

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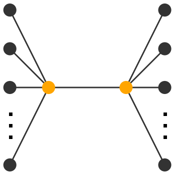

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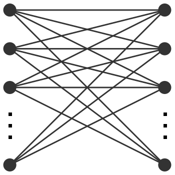

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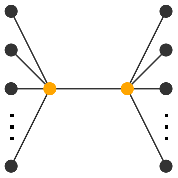

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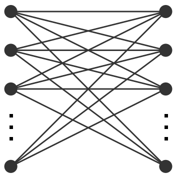

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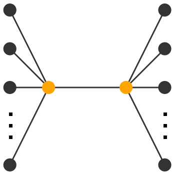
Instead of asking whether $x \sim y$ in $I(G)$, ask whether $\mu(x, y)$ in G.



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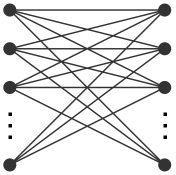

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Transductions

A class $\mathcal{C}$ is a *transduction* of a class $\mathcal{D}$ if there exists an interpretation I such that if for every $G \in \mathcal{C}$ there exists a graph $H \in \mathcal{D}$ and a coloring H' of H such that $G = I(H')$.

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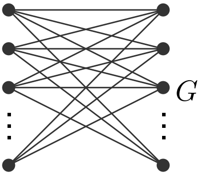

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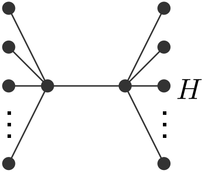

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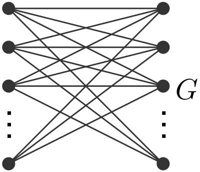

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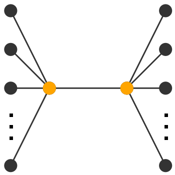

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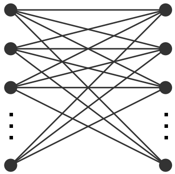

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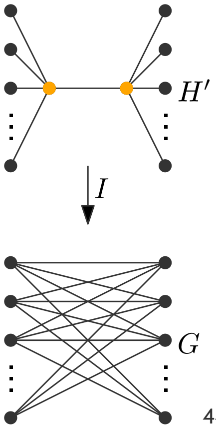
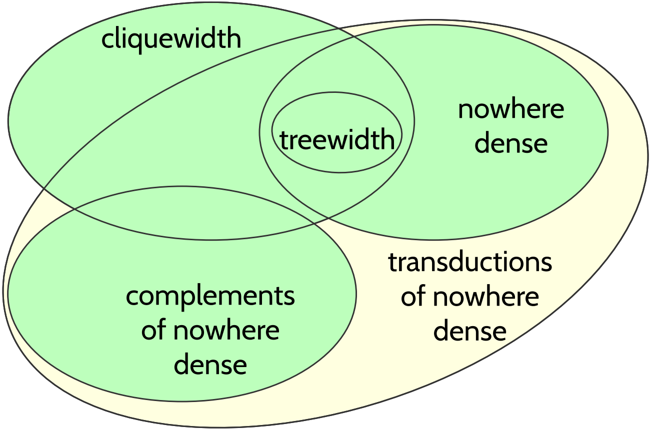


Classes with FPT first-order model-checking?



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For some sparse graph classes, this approach works.

Transductions

Gajarský, Hliněný, Lokshtanov, Obdržálek, Ramanujan 2018

Let $\mathcal{C}$ be a transduction of a class with bounded degree. First-order model-checking is fpt on $\mathcal{C}$.

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Bonnet, Dreier, Gajarský, Kreuzer, Mählmann, Toruńczyk 2022+

Let $\mathcal{C}$ be a transduction of a class with locally bounded treewidth. First-order model-checking is fpt on $\mathcal{C}$.

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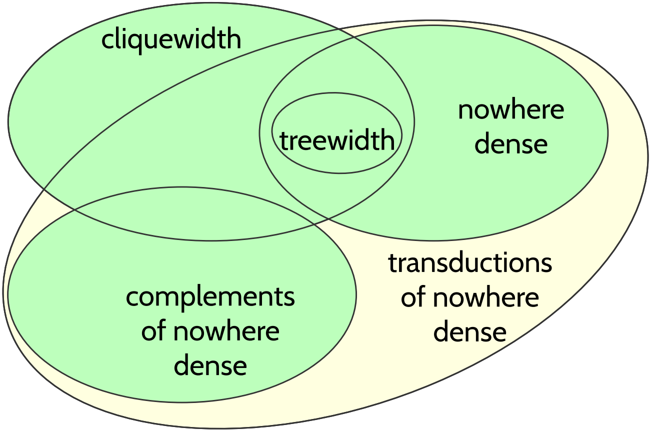

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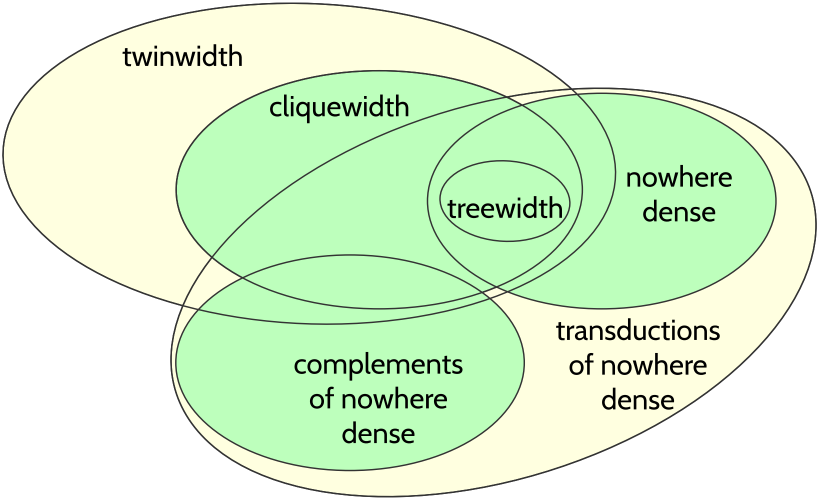
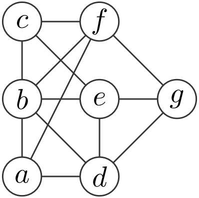


You already know normal graphs.



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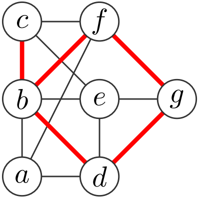
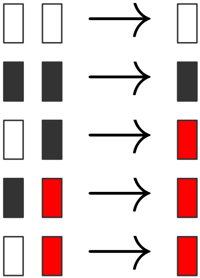
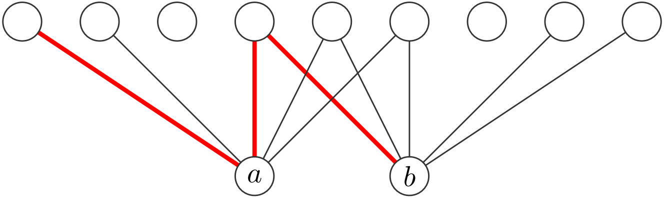


In *trigraphs* there are additional red error edges.

Contraction



We can contract two (not neccessarily adjacent) vertices a and b. The edges of the new vertex ab follow this table.



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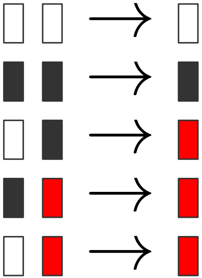
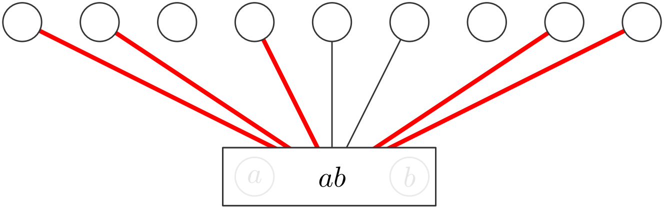
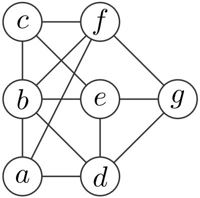
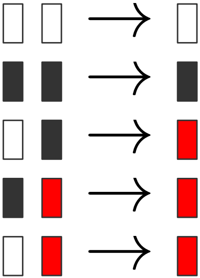




Contraction Sequences





A *contraction sequence* is a sequence of contractions until only a single vertex is left.

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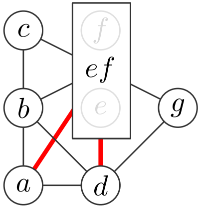

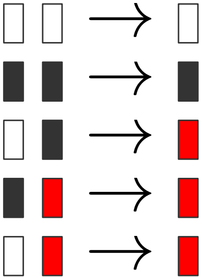

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Example by Édouard Bonnet

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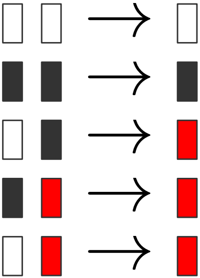

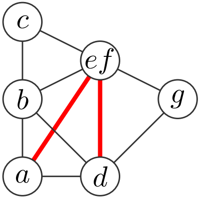

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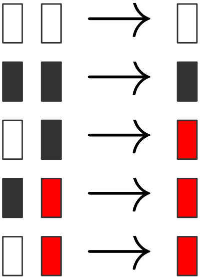

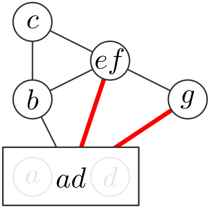

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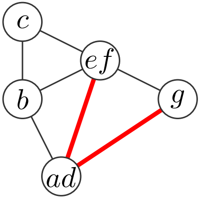

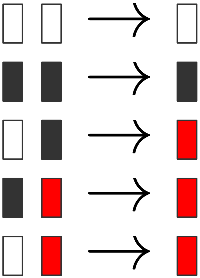

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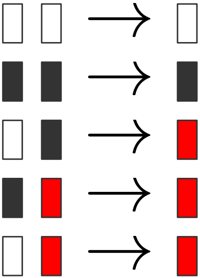

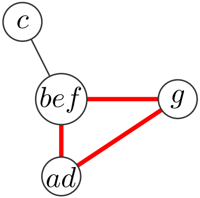

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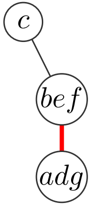

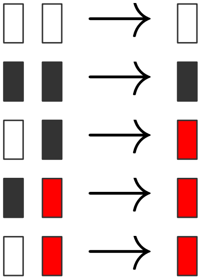

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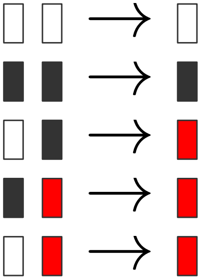
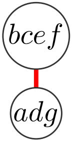
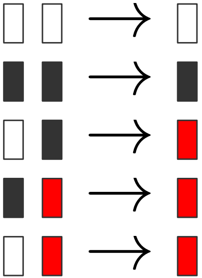




A *contraction sequence* is a sequence of contractions until only a single vertex is left.

Example by Édouard Bonnet

Contraction Sequences



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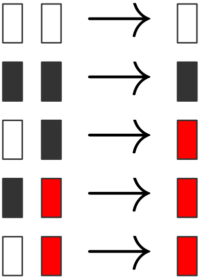

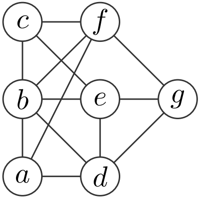

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Twinwidth: Smallest integer d such there is a contraction sequence where the red degree is *at all times* at most d.

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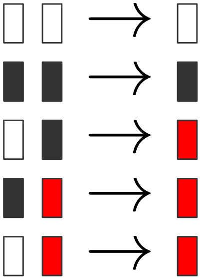

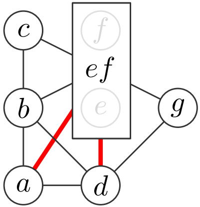

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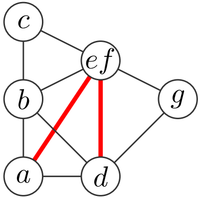

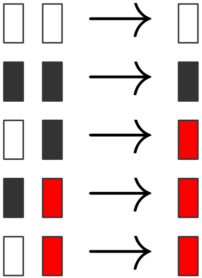

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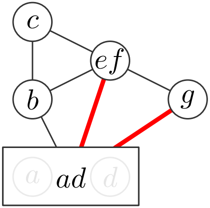

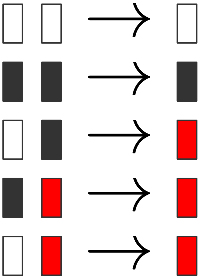

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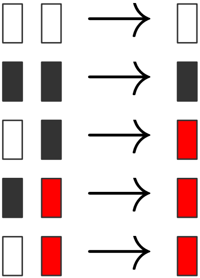

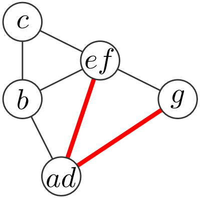

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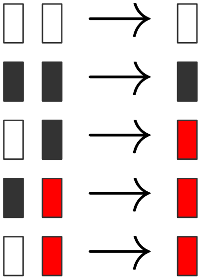

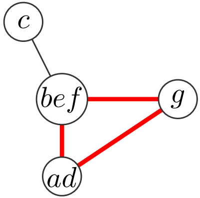

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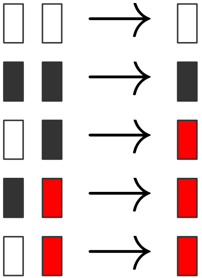

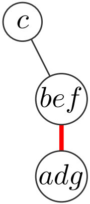

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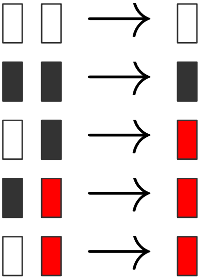

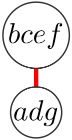

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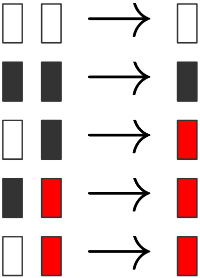

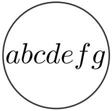

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Classes of Bounded Twinwidth

The following classes have bounded twinwidth

- planar graphs,
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The following classes do not have bounded twinwidth

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First-Order Model-Checking

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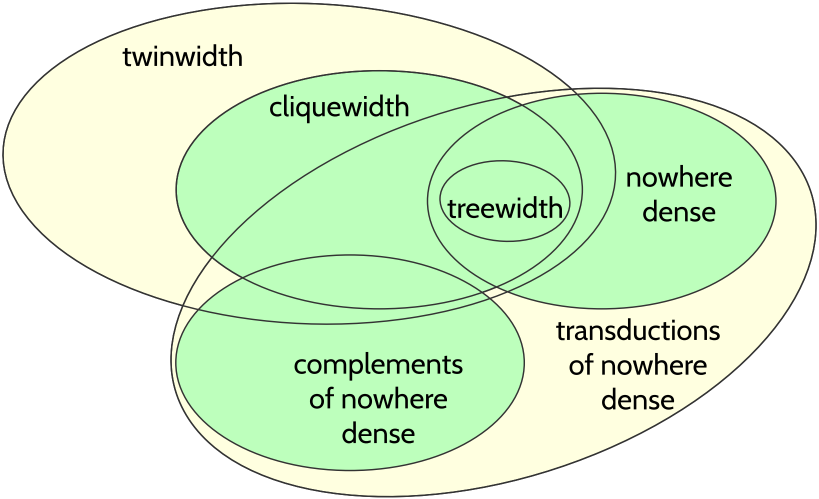
Let $\mathcal{C}$ be a class of bounded twinwidth. Then first-order model-checking is fpt on $\mathcal{C}$, if one is additionally provided a contraction sequence of bounded twinwidth.

Classes with FPT first-order model-checking?



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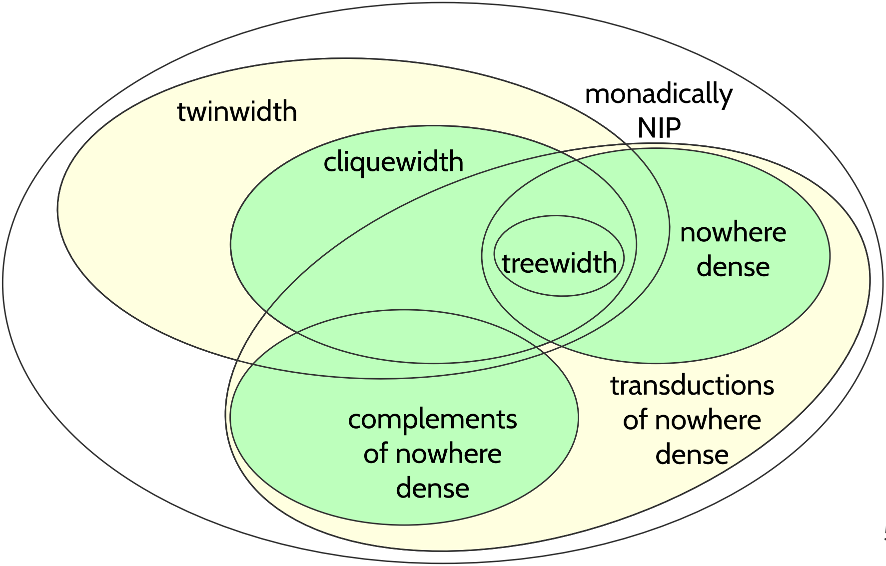
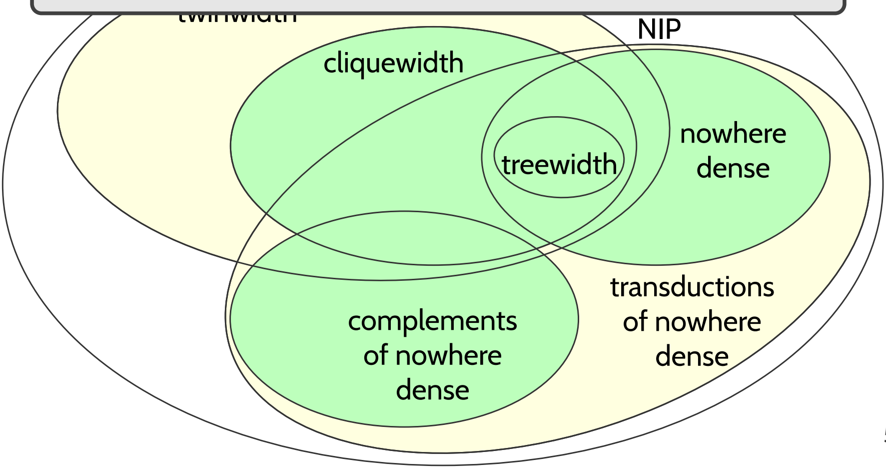


Classes with FPT first-order model-checking?

A class $\mathcal{C}$ is a *transduction* of a class $\mathcal{D}$ if there exists an interpretation I such that if for every $G \in \mathcal{C}$ there exists a graph $H \in \mathcal{D}$ and a coloring H' of H such that $G = I(H')$.



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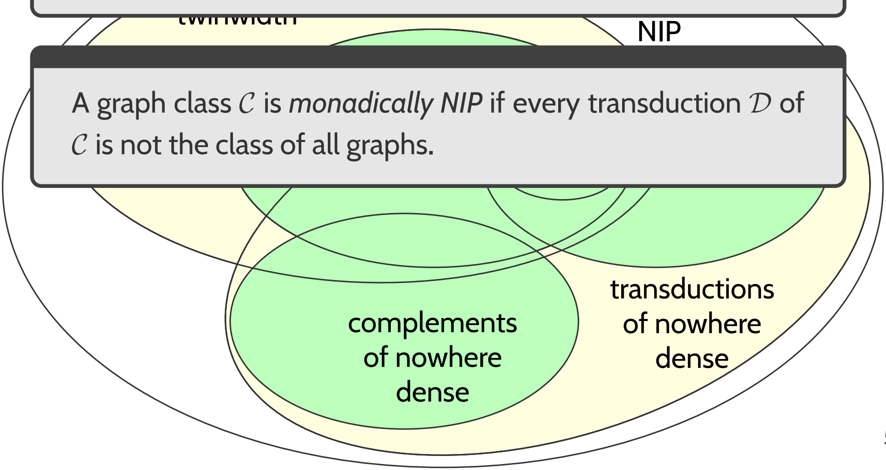

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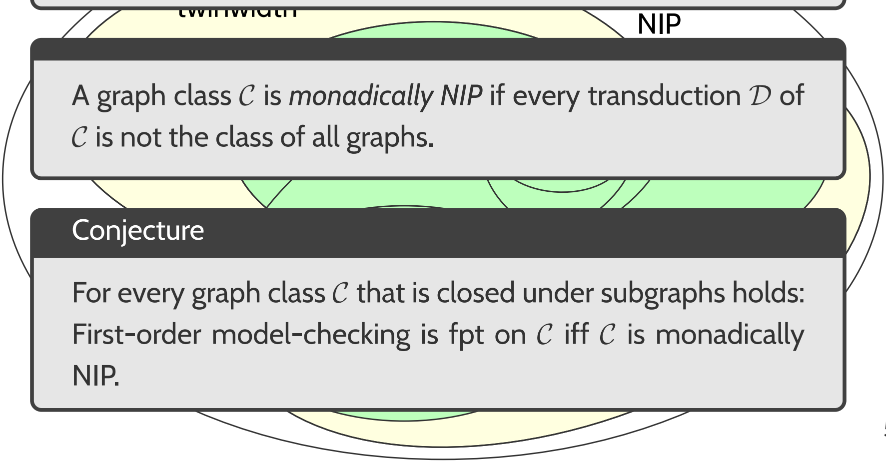

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Conjecture

For every graph class $\mathcal{C}$ that is closed under subgraphs holds: First-order model-checking is fpt on $\mathcal{C}$ iff $\mathcal{C}$ is monadically NIP.

The End