Algorithmic Meta-Theorems

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ALGORITHMS AND COMPLEXITY GROUP

TU WIEN Informatics

Minor Characterization

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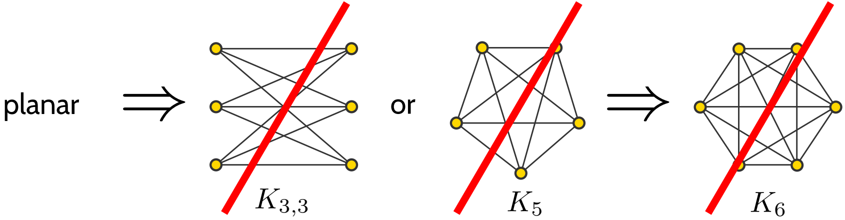

Shallow Minors

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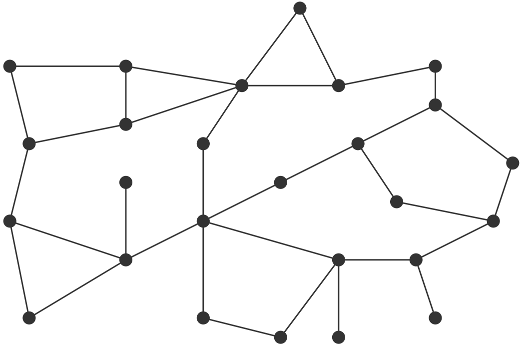

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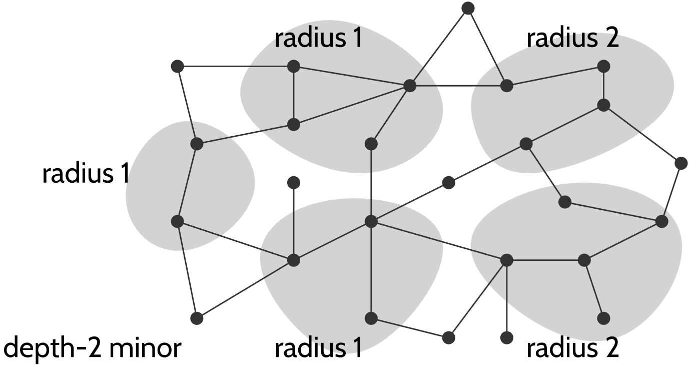

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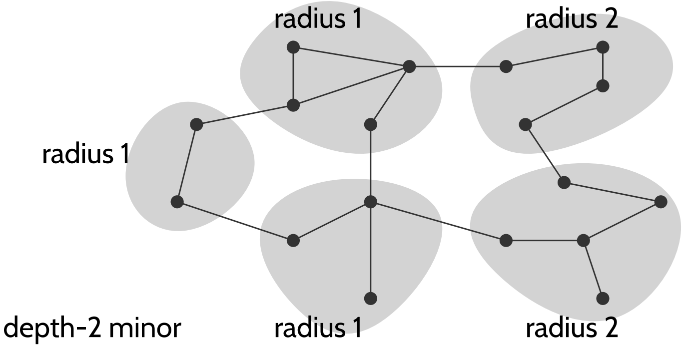

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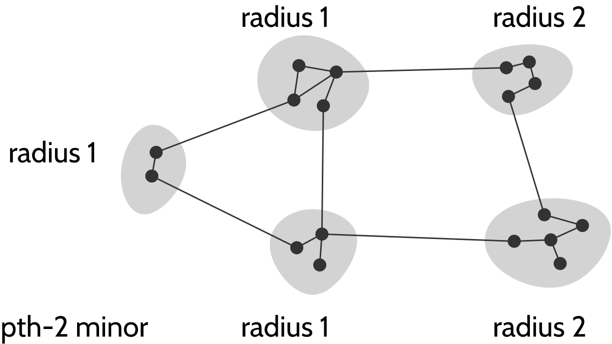

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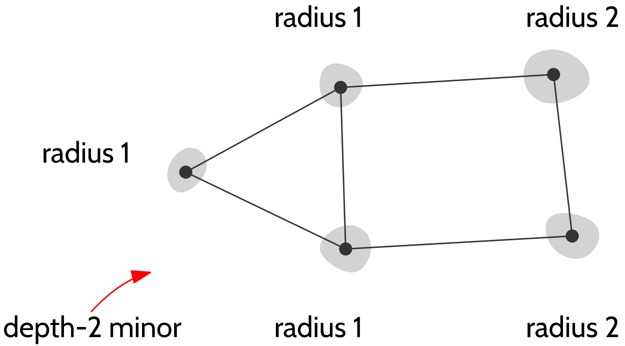

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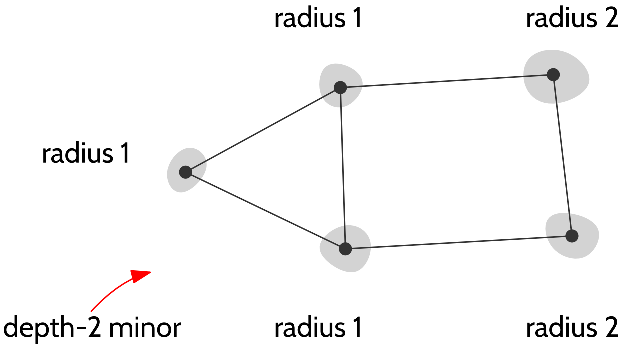

The Right Notion of Sparsity

We measure sparsity at depth r by measuring the depth-r minors of a graph G. This notion of sparsity was introduced by Nešetřil and Ossona de Mendez. We can think of two ways to do so

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Bounded Expansion

A graph class $\mathcal{C}$ has bounded expansion if there exists a function $f(r)$ such that for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$ we have $\nabla_r(G) \leq f(r)$.

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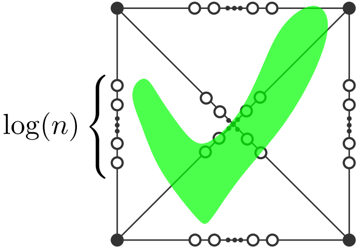
Nowhere Dense

A graph class $\mathcal{C}$ is nowhere dense if there exists a function $f(r)$ such that for all $r \in \mathbb{N}$ and all $G \in \mathcal{C}$ we have $\omega_r(G) \leq f(r)$.



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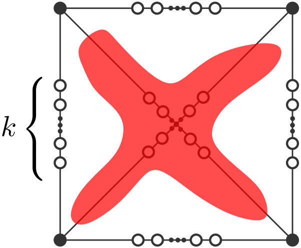

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Many Sparse Graph Classes

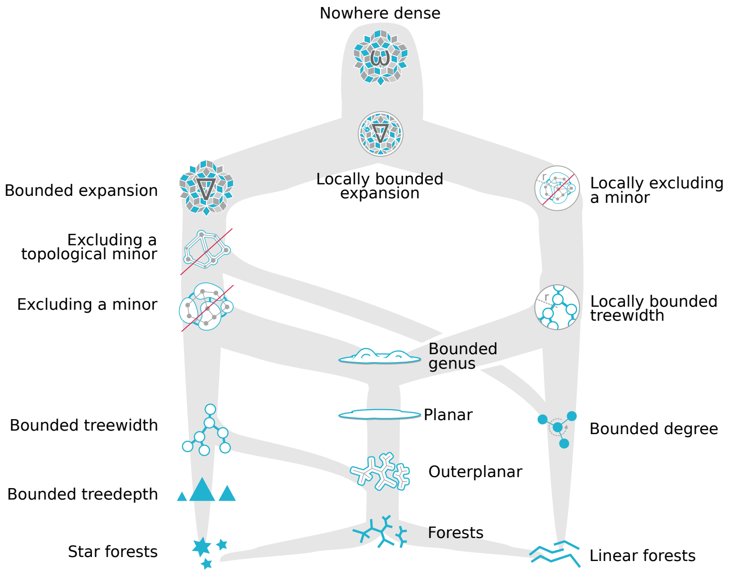

Figure by Felix Reidl

Main Results for Sparse Graphs

Theorem (Grohe, Kreuzer, Siebertz 2017)

For graph class $\mathcal{C}$ that is closed under subgraphs holds $\mathcal{C}$ is nowhere dense iff the first-order model-checking problem on $\mathcal{C}$ is FPT (assuming FPT $\neq$ AW[$*$]).

Main Results for Sparse Graphs

Theorem (Dvořák, Král, Thomas 2013)

Let $\mathcal{C}$ be a graph class with bounded expansion. There exists a function f such that for every FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n$.

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General rule: Things that work for bounded expansion also work for nowhere dense, but in an uglier way. This is why we focus on bounded expansion only in this course.

Existential Model-Checking

We will first prove a weaker result that is a building block in many other algorithms.

Let $\mathcal{C}$ be a class with bounded expansion. There exists a function f such that for every *existential* FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n$.

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This is (more or less) equivalent to deciding in time $f(|H|)n$ whether a pattern graph H occurs as induced subgraph.

Existential Model-Checking

Proof of equivalence:

- Assume we want to know whether $G \models \varphi$ for some existential formula with q quantifiers. For example $\varphi = \exists x \exists y \exists z x{\sim}y \wedge y{\not\sim}z$.

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- Now $G \models \varphi$ iff G contains some graph from $\mathcal{H}$ as induced subgraph.
  - Assume $G \models \varphi$. Then the satisfying assignment describes induced subgraph H of G with $H \models \varphi$.
  - Assume $H \in \mathcal{H}$ is induced subgraph of G. Then $H \models \varphi$. This does not change while adding the remaining vertices of G.

Alternative Characterizations

How can we prove these results?

- Gaifman does not help much because neighborhoods can be the whole graph.
- So far, all we know that certain shallow minors are not present.
- If we have a better understanding of the structure of sparse graphs, this will help us.

Alternative Characterizations

There are many alternative definitions of bounded expansion and nowhere dense classes.

- shallow minors
- generalized coloring numbers
- low treedepth colorings
- transitive fraternal augmentations
- quasi-wideness
- connector-splitter games

Which one is best depends on the task.

Low Treedepth Colorings

To prove the result, we will use the powerful notion of *low treedepth colorings*.

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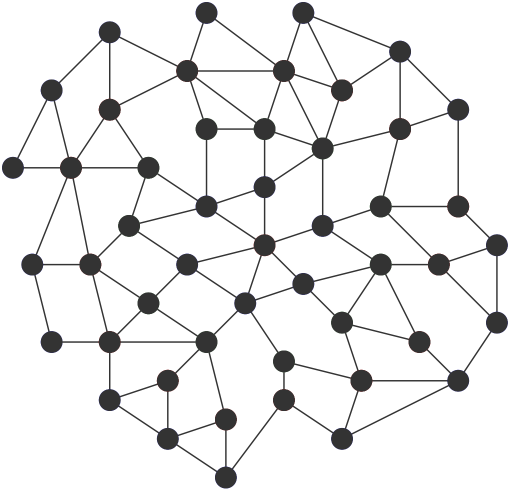
As a warmup, we solve the problem on planar graphs and then generalize the approach to bounded expansion.

Baker's Technique

On any planar graph you can do the following.



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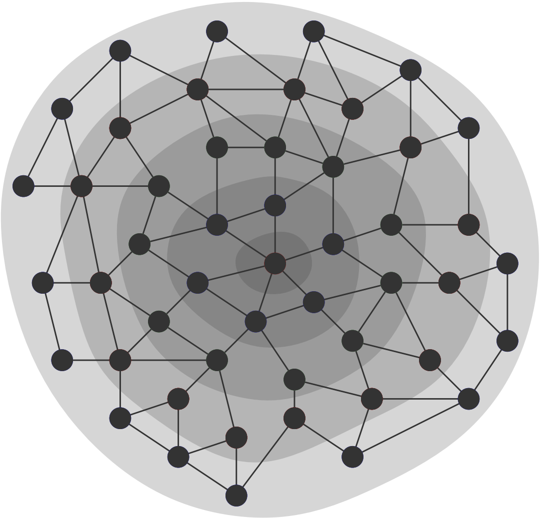

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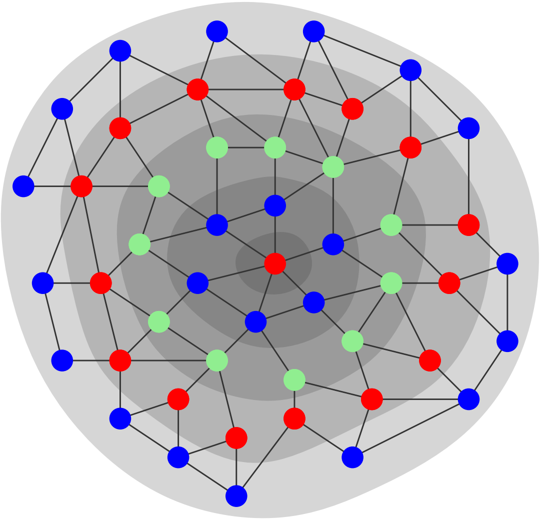

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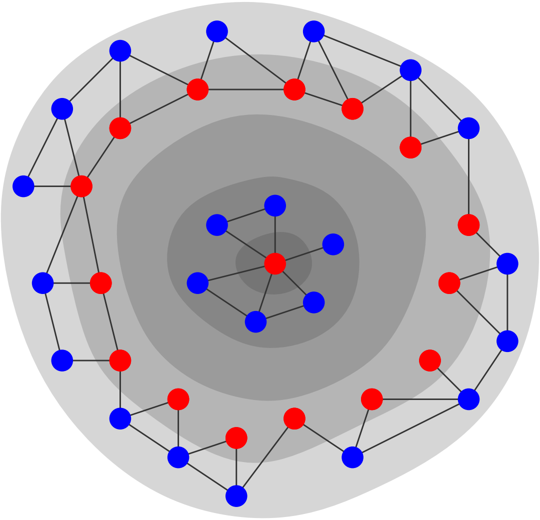

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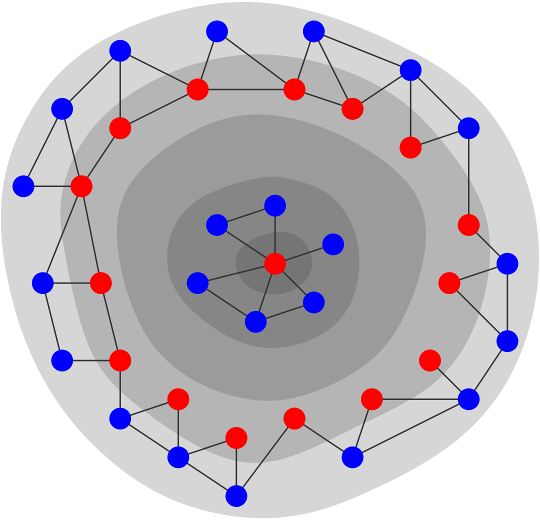
The resulting graph has treewidth at most $3p + 1$.



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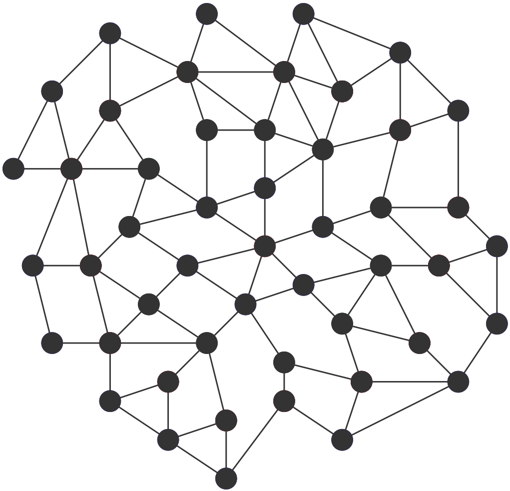

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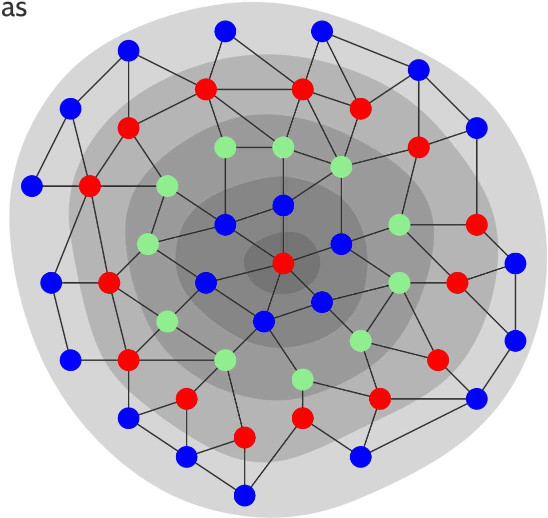

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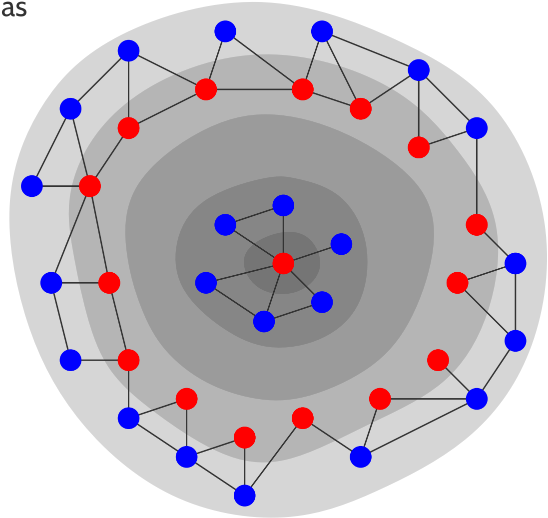

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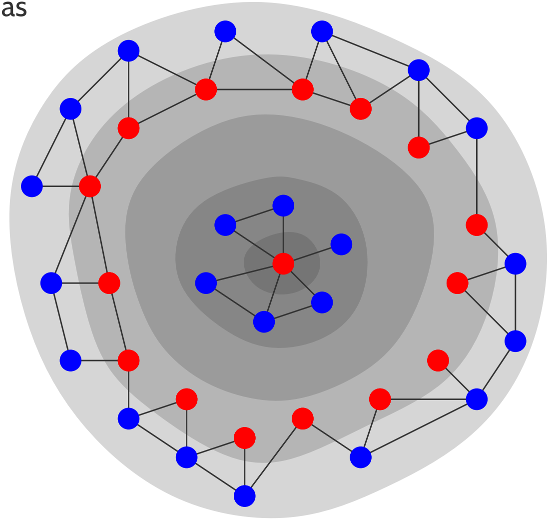

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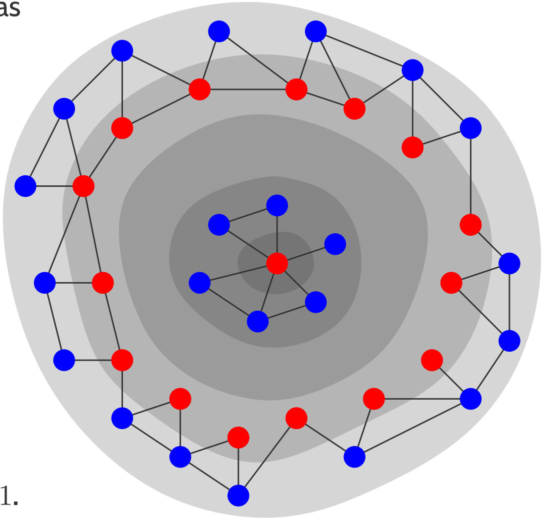

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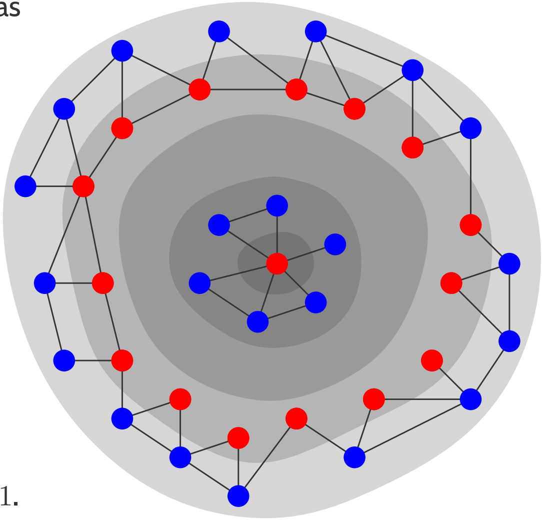
Run time $\binom{p}{p-1} \cdot f(3p+1, |H|) \cdot n$.



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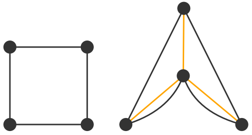
We can get something similar for bounded expansion.

- For every p one can color the graph with $f(p)$ colors such that every set of p colors induces a graph with treedepth at most p.

Treedepth

Definition

The *treedepth* of a graph G is the minimum height of a rooted forest on $V(G)$ such that all edges of G go between ancestors and descendants.



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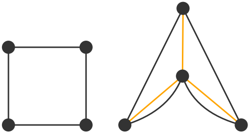

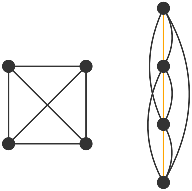

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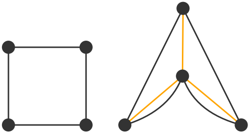

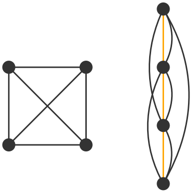

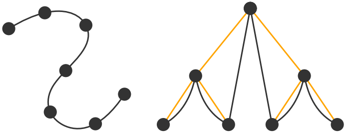

Treedepth vs. Treewidth

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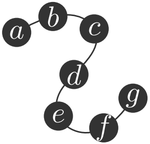

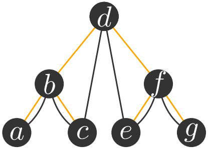

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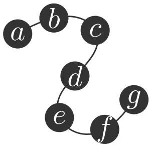

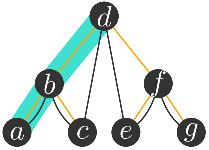

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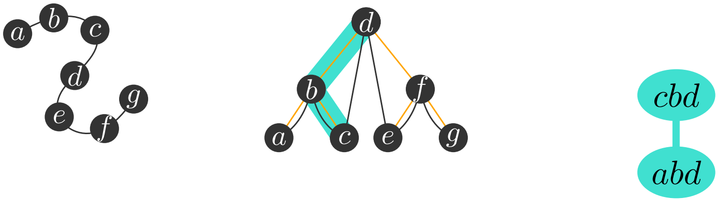

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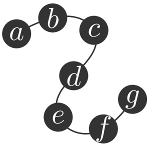

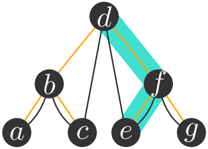

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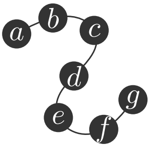

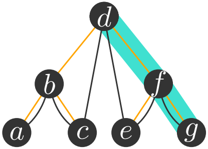

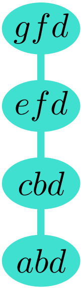

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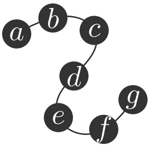
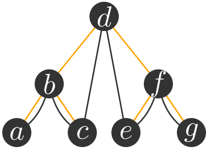
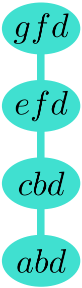
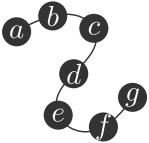


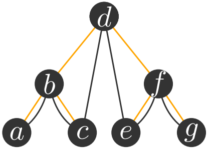




Treedepth of Paths

A treedepth of a path with n vertices is exactly $\lceil \log(n+1) \rceil$.





Low Treedepth Colorings

Nešetřil, Ossona de Mendez

A graph class $\mathcal{C}$ has bounded expansion iff there exists a function f such that for every $G \in \mathcal{C}$ and $p \in \mathbb{N}$ one can color G with $f(p)$ colors and every set of p colors induces a graph with treedepth $\leq p$.

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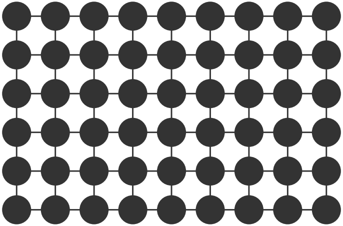
What is $f(2)$ for this graph?



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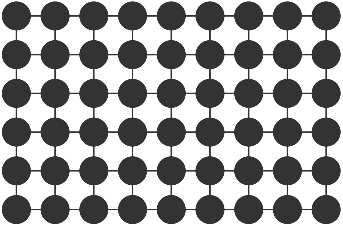
What is $f(2)$ for this graph? How many colors do we need here such that every set of 2 colors has treedepth ≤ 2?



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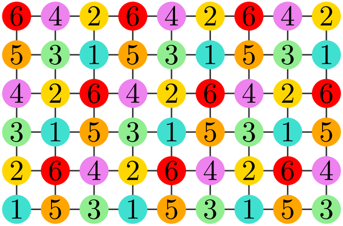

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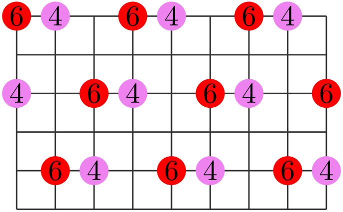

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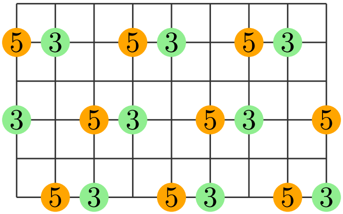

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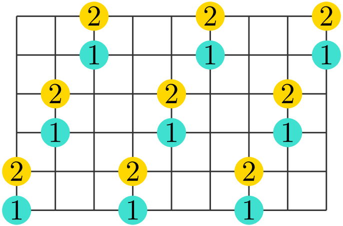

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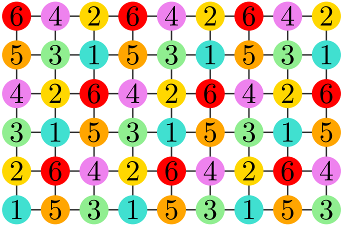


Low Treedepth Colorings

How many colors do we need to color a tree such that every set of p colors induces a graph with treedepth at most p?

Low Treedepth Colorings

How many colors do we need to color a tree such that every set of p colors induces a graph with treedepth at most p?

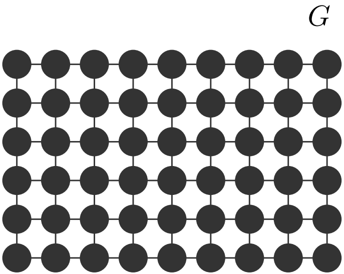
Color it with $p+1$ colors slicewise.

Existential Model-Checking

We can now use low-treedepth colorings to prove fpt existential model-checking.

Existential Model-Checking

Let $\mathcal{C}$ be a class with bounded expansion, having low treedepth colorings with function $f(p)$. We want to know in time $h(|H|) \cdot n$ whether a graph $G \in \mathcal{C}$ has H as induced subgraph.



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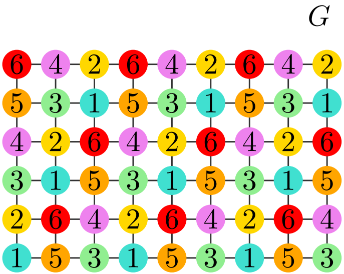

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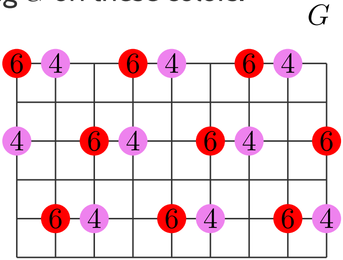

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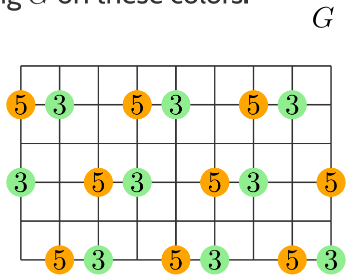

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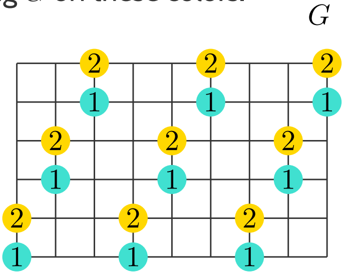

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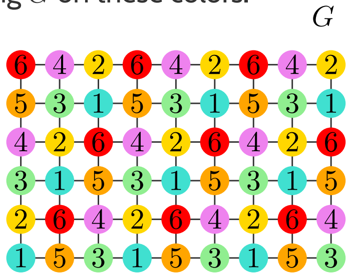

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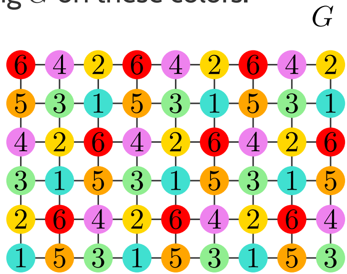
- We consider $\binom{f(|H|)}{|H|}$ color sets and for each we search for H in time $g(|H|) \cdot n$ using Courcelle.



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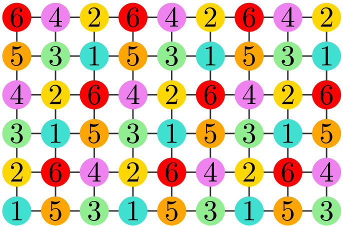
- Total run time $\binom{f(|H|)}{|H|} g(|H|) \cdot n$.

G



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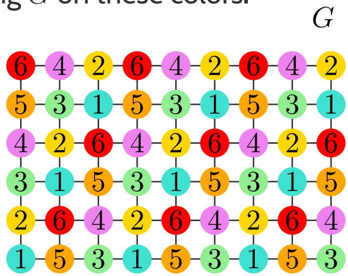
- Total run time $\binom{f(|H|)}{|H|} g(|H|) \cdot n$.
- Plus time needed to compute coloring!



Main Result for Sparse Graphs

After we previously proved the result for existential model-checking, we now prove the full version.

Theorem (Dvořák, Král, Thomas 2013)

Let $\mathcal{C}$ be a graph class with bounded expansion. There exists a function f such that for every FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n$.

This is the most involved proof presented in this course. We will again use low-treedepth colorings.

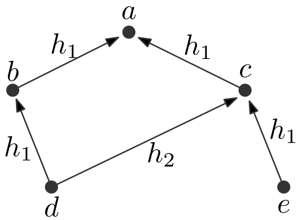
Functional Graphs

A *functional graph* $\vec{G}$ is a structure with signature $\tau = \{h_1, h_2, \ldots, R_1, R_2, \ldots, Q_1, Q_2, \ldots\}$ where

- $h_i : V \to V$ are unary functions
- $R_i \subseteq V$ are unary relations
- $Q_i \in \{0, 1\}$ are nullary relations

$h_1(d) = b$

$h_2(d) = c$

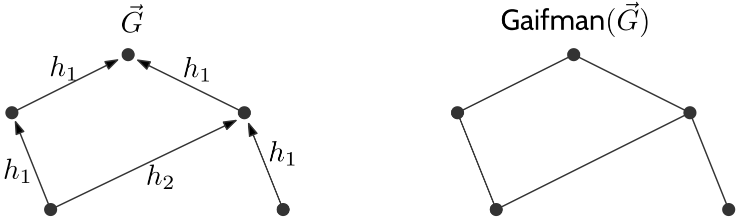


You can think of it as follows

- $h_i(u) = v$ equals directed edge from u to v of the ith type,
- $R_i(u)$ equals labeling u with ith label.
- Q_i equals a globally acessible truth value.

Classes

For a functional graph $\vec{G}$, the graph Gaifman($\vec{G}$) has the same vertex set and edges uv iff $h_i(u) = v$ or $h_i(v) = u$ for some i.



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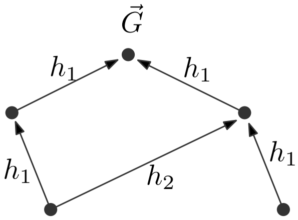

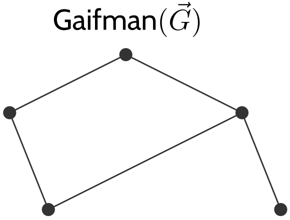

We say a class of functional graphs has *bounded expansion* if the class of their Gaifman graphs has.

Reduction

If we can solve the model-checking problem in time $f(|\varphi|)n$ on functional graph classes with bounded expansion then we can also do it in time $f(|\varphi|)n$ on normal graph classes with bounded expansion.

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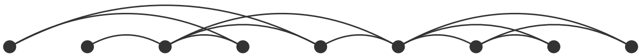
Proof: Let $\mathcal{C}$ be a normal class with bounded expansion. Then $\mathcal{C}$ has bounded degeneracy d.



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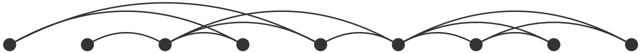

Compute degeneracy ordering. Let $h_i(v)$ point to ith left neighbor of v. Replace in φ every occurence of $x{\sim}y$ with

$$\bigwedge_{i=1}^{d} h_i(x) = y \vee h_i(y) = x.$$

Main Result for Functional Structures

We therefore want to prove the following.

Theorem (Dvořák, Král, Thomas 2013)

Let $\vec{\mathcal{C}}$ be a functional graph class with bounded expansion. For every graph $\vec{G} \in \vec{\mathcal{C}}$ and FO formula φ one can decide whether $\vec{G} \models \varphi$ in time $f(|\varphi|)n$.

Quantifier Elimination

Convert formula to prenex normal form.

$$\vec{G} \models \exists x_1 \left(\qquad\qquad\qquad\qquad \right)$$

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$$\vec{G} \models \exists x_1 \left(\forall x_2 \left(\exists x_3 \overbrace{\varphi(x_1 x_2 x_3)}^{\text{quantifer-free}} \right) \right)$$

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Convert formula to prenex normal form. Gradually simplify formula by removing quantifiers one by one from the inside.

$$\vec{G} \models \exists x_1 \left(\forall x_2 \left(\underbrace{\exists x_3 \; \overbrace{\varphi(x_1 x_2 x_3)}^{\text{quantifer-free}}}_{\text{replace with quantifier-free}} \right) \right)$$

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$$\vec{G}' \models \exists x_1 \left(\forall x_2 \ \overbrace{\varphi'(x_1 x_2)}^{\text{quantifier-free}} \right)$$

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Convert formula to prenex normal form. Gradually simplify formula by removing quantifiers one by one from the inside.

$$\vec{G}'' \models \exists x_1 \quad \overbrace{\varphi''(x_1)}^{\text{quantifier-free}}$$

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$$\vec{G}''' \models \overbrace{\varphi'''}^{\text{quantifier-free}}$$

We shift complexity from the formula to the graph. G', G'', G''' have same vertices and edges but additional unary and nullary relations.

$$\begin{array}{ccc} \vec{G} & & \vec{G}' \\ \exists y\, \varphi(y\bar{x}) & \longrightarrow & \varphi'(\bar{x}) \end{array}$$

Quantifier Elimination

$$\begin{array}{ccc} \vec{G} & & \vec{G}' \\ \exists y\, \varphi(y\bar{x}) & \longrightarrow & \varphi'(\bar{x}) \end{array}$$

Should still be short. Length depends only on φ.

Quantifier Elimination

Should still be sparse. Same Gaifman graph as $\vec{G}$.

$$\vec{G} \qquad \longrightarrow \qquad \vec{G}'$$

$$\exists y\, \varphi(y\bar{x}) \qquad \qquad \varphi'(\bar{x})$$

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$$\vec{G} \models \exists y\, \varphi(y\bar{v}) \quad \text{iff} \quad \vec{G}' \models \varphi'(\bar{v}) \qquad \text{for all } \bar{v} \in V(\vec{G})^{|\bar{x}|}.$$

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If there is a quantifier elimination procedure for classes with bounded expansion, this proves our main result.

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If there is a quantifier elimination procedure for classes with bounded expansion, this proves our main result. We use De Morgan to get rid of universal quantifiers.

$$\vec{G}' \models \exists x_1 \left(\forall x_2 \quad \overbrace{\varphi'(x_1 x_2)}^{\text{quantifier-free}} \right)$$

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$$\vec{G}' \models \exists x_1 \neg \Big(\underbrace{\exists x_2 \overbrace{\neg \varphi'(x_1 x_2)}^{\text{quantifier-free}}}_{\text{replace with quantifier-free}} \Big)$$

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$$\vec{G}''' \models \overbrace{\varphi'''}^{\text{quantifier-free}}$$

Roadmap

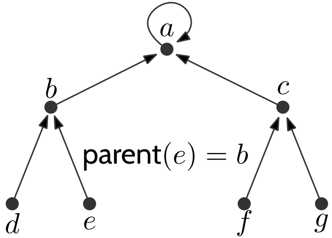
Roadmap: Construct quantifier elimination for graph classes of increasing complexity.

- forests of bounded depth
- bounded treedepth
- bounded expansion

Functional Forests

The simplest functional structures we work with are *functional forests*.

There are unary relations (labels) and exactly one unary function "parent" describing the parent relation of a rooted forest (roots point to themselves).



Functional Forests

Let $\mathcal{C}$ be a class of functional forests with bounded depth. Then $\mathcal{C}$ has a quantifier elimination procedure.

$$\vec{G} \qquad \qquad \vec{G}'$$

$$\exists y\, \varphi(y\bar{x}) \quad \longrightarrow \quad \varphi'(\bar{x})$$

Functional Forests

Since $\varphi(y\bar{x})$ is quantifier free, it is a boolean combination of atoms of the form

- R for some nullary relation R
- $R(\text{parent}^i(s))$ for unary R, $i \leq d$ and variable $s \in y\bar{x}$
- $\text{parent}^i(s) = \text{parent}^j(t)$ for $i, j \leq d$ and variables $s, t \in y\bar{x}$

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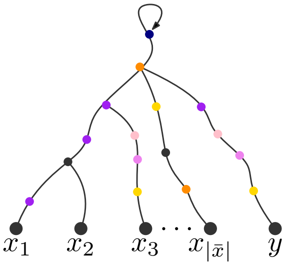
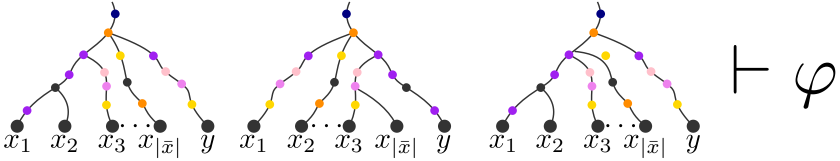
$\varphi(y\bar{x})$ can only talk about the connections between and labelings of ancestors of $y\bar{x}$.



Functional Forests

Let $\mathbb{T}$ be the set of all such trees that imply φ.



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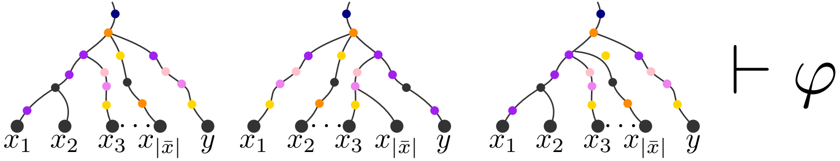

We see every $T \in \mathbb{T}$ as a formula $T(y\bar{x})$ checking if ancestor tree of $y\bar{x}$ equals T. Then

$$\exists y\, \varphi(y\bar{x}) = \exists y \bigvee_{T \in \mathbb{T}} T(y\bar{x})$$

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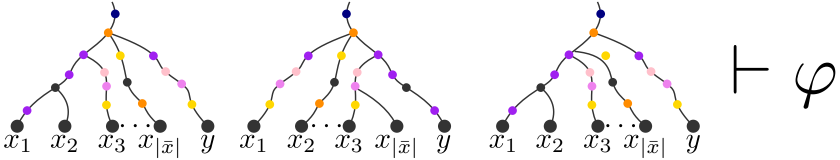

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$$\exists y\, \varphi(y\bar{x}) = \exists y \bigvee_{T \in \mathbb{T}} T(y\bar{x}) = \bigvee_{T \in \mathbb{T}} \exists y\, T(y\bar{x}).$$

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We focus on a single subformula $\exists y \, T(y\bar{x})$.

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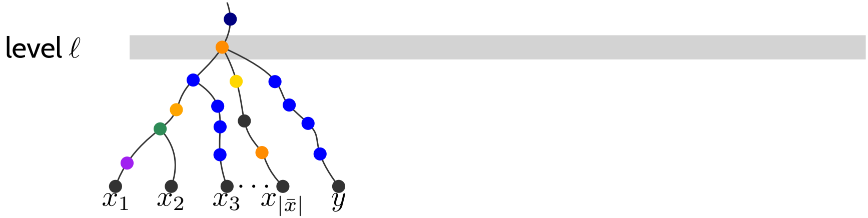
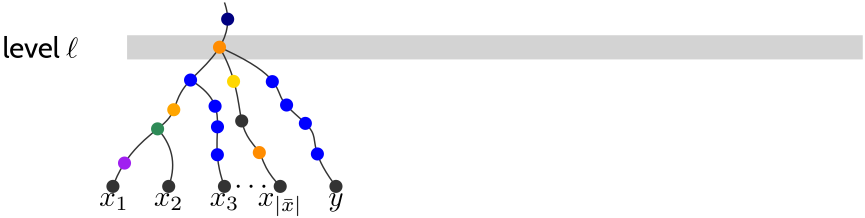


Functional Forests

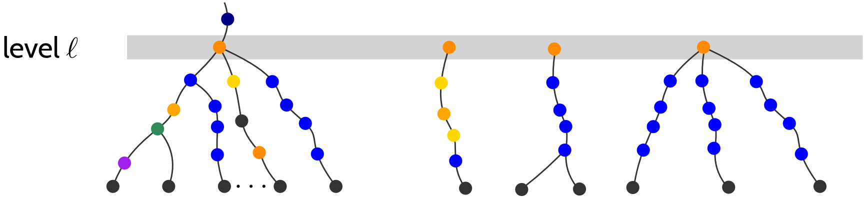
We focus on a single subformula $\exists y \, T(y\bar{x})$.

It enforces that the ancestors of $\bar{x}$ induce a certain tree and that there exists y that hits this tree via a "special path" with certain labels at a certain ancestor at height l.



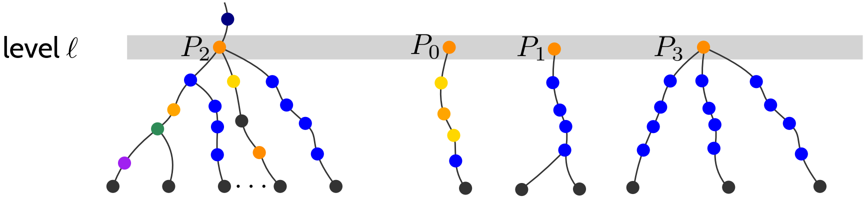
Functional Forests

We now augment the graph.



Functional Forests

We now augment the graph. For all v at level l use label $P_i(v)$ to store that i children of v can be completed to form a special path.



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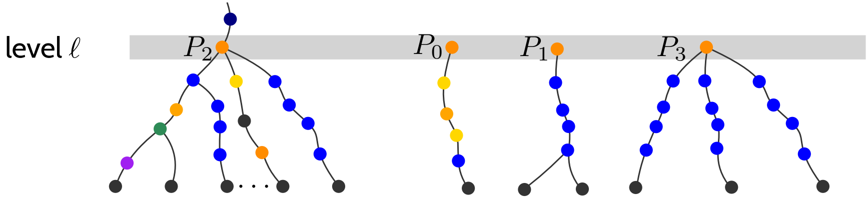
- round all numbers larger than $|\bar{x}|$ up to ∞.



Functional Forests

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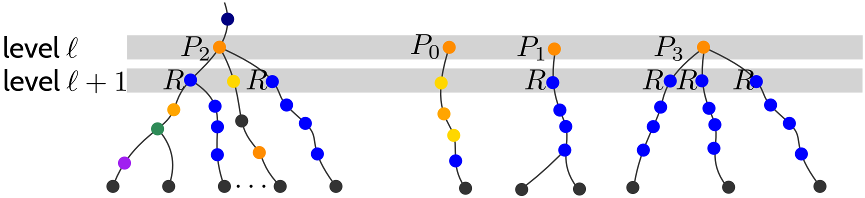
For all w at level $l+1$ add label $R(w)$ if w lies on a special path.



Functional Forests

We construct a quantifier-free formula $T'(\bar{x})$ equivalent to $\exists y\, T(y\bar{x})$. This formula can only check the ancestor tree of $\bar{x}$.

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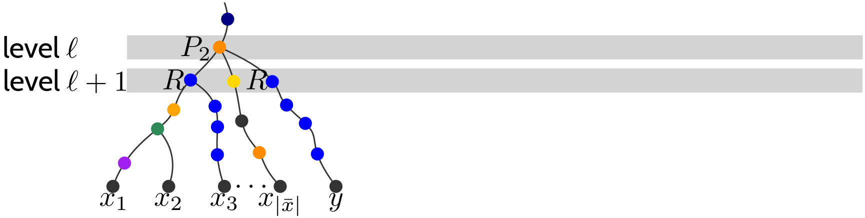
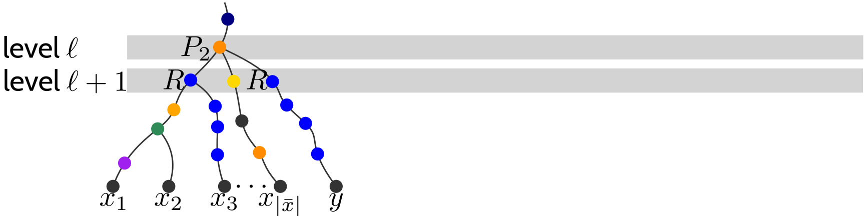


Functional Forests

We construct a quantifier-free formula $T'(\bar{x})$ equivalent to $\exists y\, T(y\bar{x})$. This formula can only check the ancestor tree of $\bar{x}$.

If we want to know if there is y reaching an ancestor s of $\bar{x}$, we can check whether s has more ancestors than are in the tree of $\bar{x}$.



Functional Forests

For every ancestor tree $T \in \mathbb{T}$ we can perform quantifier elimination.

$$\begin{array}{ccc} \vec{G} & & \vec{G}' \\ \exists y\, T(y\bar{x}) & \to & T'(\bar{x}) \end{array}$$

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$$\begin{array}{ccc} \vec{G} & & \vec{G}' \\ \exists y\, T(y\bar{x}) & \rightarrow & T'(\bar{x}) \end{array}$$

$\vec{G} \models \exists y\, T(y\bar{v})$ iff $\vec{G}' \models T'(\bar{v})$ for all $\bar{v} \in V(\vec{G})^{|\bar{x}|}$.

However, we want quantifier elimination for our original formula

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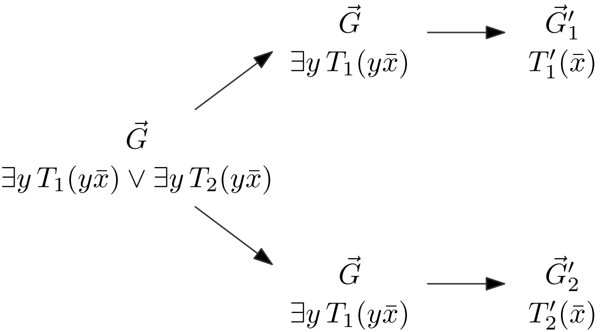

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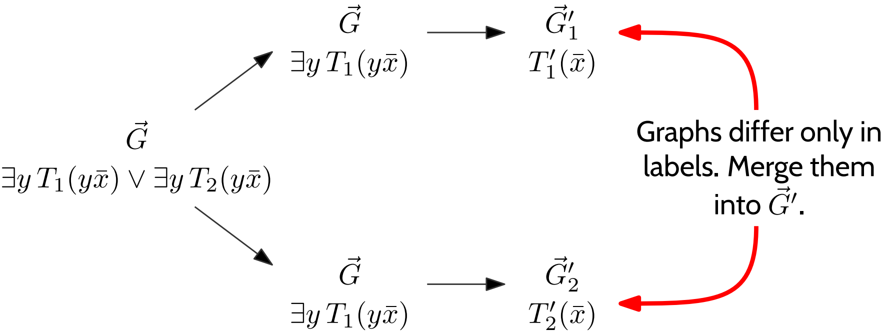

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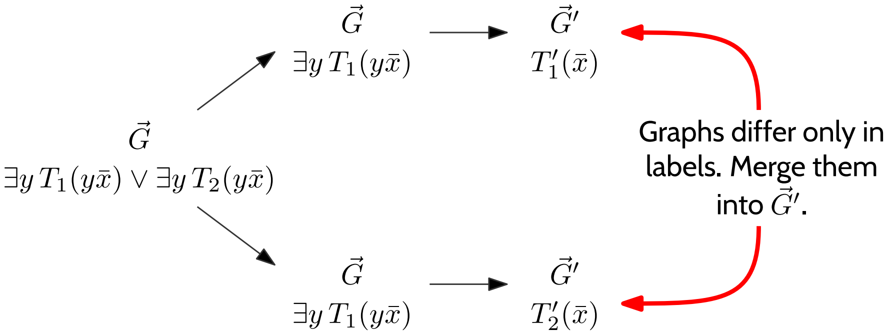

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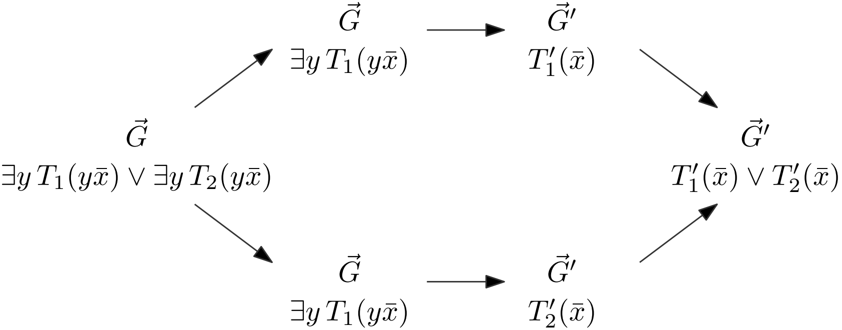
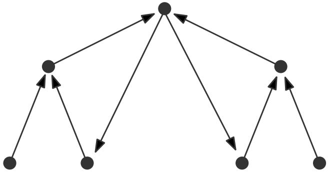


Bounded Treedepth

Next, we consider functional graph classes whose Gaifman graphs have *bounded treedepth*.



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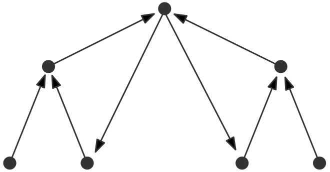
Let $\mathcal{C}$ be a class whose Gaifman graphs have *bounded treedepth*. Then $\mathcal{C}$ has a quantifier elimination procedure.

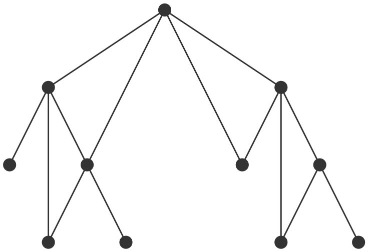


Bounded Treedepth

bounded
treedepth

$$\vec{G}$$

$$\exists y \, \varphi(y\bar{x})$$

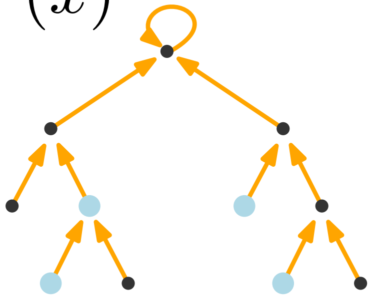


bounded treedepth

forest of bounded depth

$$\vec{G} \longrightarrow \vec{G}'$$

$$\exists y\, \varphi(y\bar{x}) \qquad \exists y\, \varphi'(\bar{x})$$



Bounded Treedepth

bounded treedepth

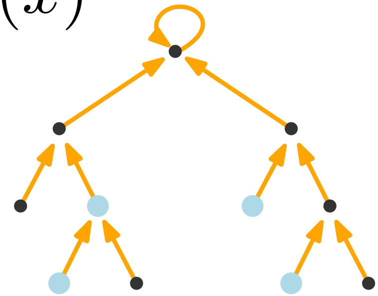
$$\vec{G}$$

forest of bounded depth

$$\vec{G}''$$

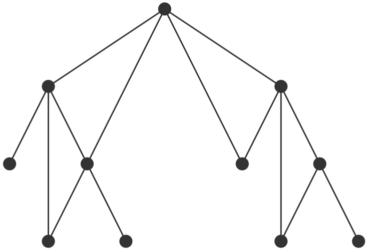
$$\exists y\, \varphi(y\bar{x}) \quad \longrightarrow \quad \varphi''(\bar{x})$$

Use previous theorem for quantifier elimination on forests.



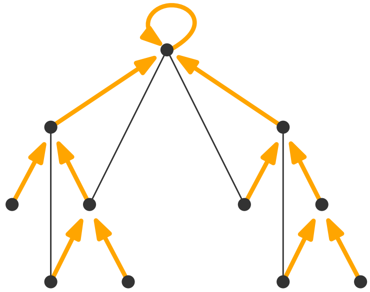
Bounded Treedepth

- Consider Gaifman graph of $\vec{G}$.



Bounded Treedepth

- Consider Gaifman graph of $\vec{G}$. Build depth-first search tree and make it functional.



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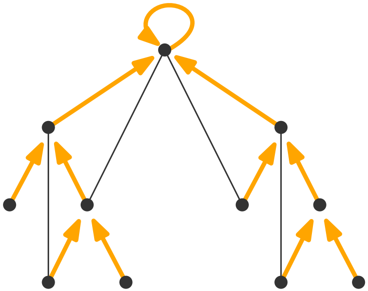
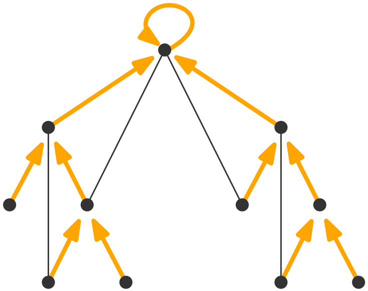
- Consider Gaifman graph of $\vec{G}$. Build depth-first search tree and make it functional. Graphs with treedepth d contain no paths longer than 2^d.



Bounded Treedepth

- Consider Gaifman graph of $\vec{G}$. Build depth-first search tree and make it functional. Graphs with treedepth d contain no paths longer than 2^d. We have a functional forest of depth at most 2^d.



Bounded Treedepth

- Consider Gaifman graph of $\vec{G}$. Build depth-first search tree and make it functional. Graphs with treedepth d contain no paths longer than 2^d. We have a functional forest of depth at most 2^d.
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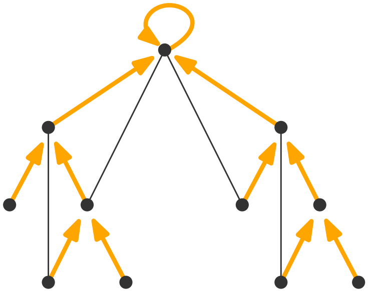

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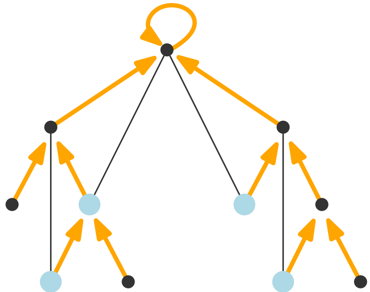

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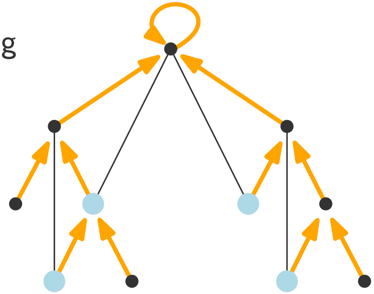
$P_{i,j,\updownarrow}(v)$: "There is edge from v to parent$^i(v)$ labeled with j going upwards/downwards".



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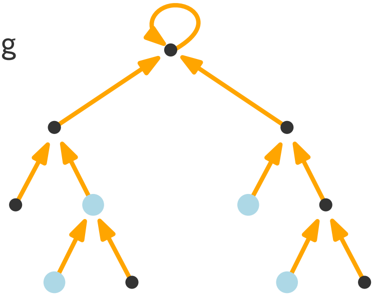
- This tree fully encodes $\vec{G}$.



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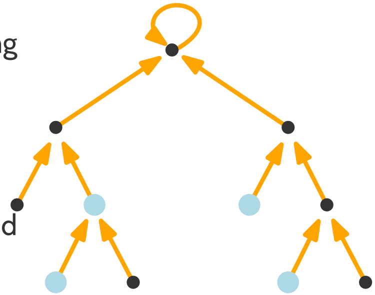
  $P_{i,j,\updownarrow}(v)$: "There is edge from v to parent$^i(v)$ labeled with j going upwards/downwards".

- This tree fully encodes $\vec{G}$.
- Replace atoms $f_j(x) = y$ by guessing tree-relationship and checking the new predicates.

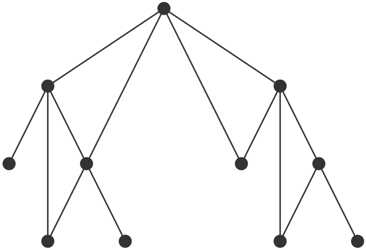


Bounded Treedepth

bounded
treedepth

$$\vec{G}$$

$$\exists y\, \varphi(y\bar{x})$$



Bounded Treedepth

bounded treedepth

$$\vec{G}$$

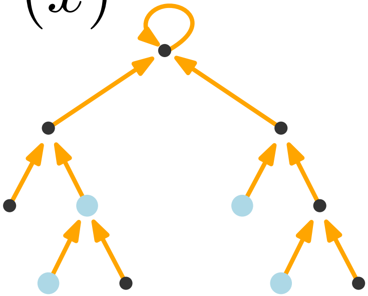
$$\exists y\, \varphi(y\bar{x})$$

$$\longrightarrow$$

forest of bounded depth

$$\vec{G}'$$

$$\exists y\, \varphi'(\bar{x})$$



Bounded Treedepth

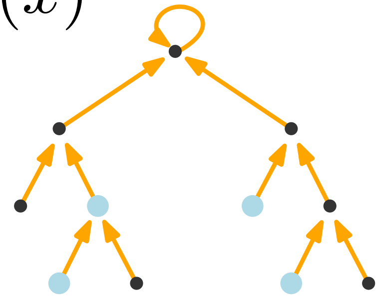
bounded treedepth → $\vec{G}$

forest of bounded depth → $\vec{G}''$

$$\vec{G} \quad \longrightarrow \quad \vec{G}''$$

$$\exists y\, \varphi(y\bar{x}) \qquad \varphi''(\bar{x})$$

Use previous theorem for quantifier elimination on forests.



Bounded Expansion

The last step.

Let $\mathcal{C}$ be a class with bounded expansion. Then $\mathcal{C}$ has a quantifier elimination procedure.

Bounded Expansion

We are given a formula $\exists y \varphi(\bar{x})$ where φ is quantifier-free.

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Replace all function applications such as $p(g(x_i)) = x_j$ with directed labeled edges such as $\exists y\, x_i \underset{g}{\rightsquigarrow} y \wedge y \underset{p}{\rightsquigarrow} x_j$.

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The result is a formula $\exists \bar{y} \, \psi(\bar{y}\bar{x})$ on a normal (non-functional) directed edge-labeled graph G.

Bounded Expansion

For every $\bar{v} \in V(G)^{|\bar{x}|}$

$$G \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

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Let Λ be the colors of a low-treedepth coloring of G where every subgraph on $|\bar{y}\bar{x}|$ colors has treedepth $\leq g(|\bar{y}\bar{x}|)$.

Bounded Expansion

For every $\bar{v} \in V(G)^{|\bar{x}|}$

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$$\bigvee_{\substack{S \subseteq \Lambda \\ |S| = |\bar{y}\bar{x}|}} \text{colors}(\bar{v}) \subseteq S \wedge$$

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For all graphs $G[S]$ (of bounded treedepth) we perform quantifier elimination.

Bounded Expansion

For every $\bar{v} \in V(G)^{|\bar{x}|}$

$$G \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

$$\bigvee_{\substack{S \subseteq \Lambda \\ |S| = |\bar{y}\bar{x}|}} \text{colors}(\bar{v}) \subseteq S \wedge G[S] \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

$$\bigvee_{\substack{S \subseteq \Lambda \\ |S| = |\bar{y}\bar{x}|}} \text{colors}(\bar{v}) \subseteq S \wedge \vec{G}'_S \models \psi_S(\bar{v}) \quad \text{iff}$$

For all graphs $G[S]$ (of bounded treedepth) we perform quantifier elimination.

Bounded Expansion

For every $\bar{v} \in V(G)^{|\bar{x}|}$

$$G \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

$$\bigvee_{\substack{S \subseteq \Lambda \\ |S| = |\bar{y}\bar{x}|}} \text{colors}(\bar{v}) \subseteq S \wedge G[S] \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

$$\bigvee_{\substack{S \subseteq \Lambda \\ |S| = |\bar{y}\bar{x}|}} \text{colors}(\bar{v}) \subseteq S \wedge \vec{G}' \models \psi_S(\bar{v}) \quad \text{iff}$$

For all graphs $G[S]$ (of bounded treedepth) we perform quantifier elimination. And stack the graphs on top of each other.

Bounded Expansion

For every $\bar{v} \in V(G)^{|\bar{x}|}$

$$G \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

$$\bigvee_{\substack{S \subseteq \Lambda \\ |S| = |\bar{y}\bar{x}|}} \text{colors}(\bar{v}) \subseteq S \wedge G[S] \models \exists \bar{y}\, \psi(\bar{y}\bar{v}) \quad \text{iff}$$

$$\bigvee_{\substack{S \subseteq \Lambda \\ |S| = |\bar{y}\bar{x}|}} \text{colors}(\bar{v}) \subseteq S \wedge \vec{G}' \models \psi_S(\bar{v}) \quad \text{iff}$$

$$\vec{G}' \models \bigvee_{\substack{S \subseteq \Lambda \\ |S| = |\bar{y}\bar{x}|}} \text{colors}(\bar{v}) \subseteq S \wedge \psi_S(\bar{v}).$$

Finally, pull out the graph.

Summary

This completes the proof. We have solved the model-checking problem on bounded expansion by performing quantifier elimination on trees and lifting it up using low treedepth colorings.