Algorithmic Meta-Theorems

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ALGORITHMS AND COMPLEXITY GROUP
TU WIEN Informatics

Ehrenfeucht–Fraïssé Games

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- Show $G \equiv_q G'$ using *Ehrenfeucht–Fraïssé games*.

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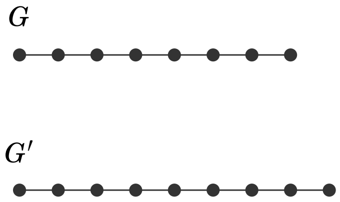

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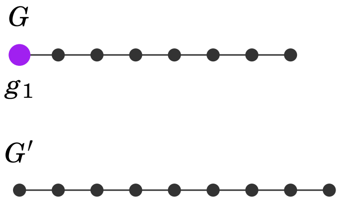

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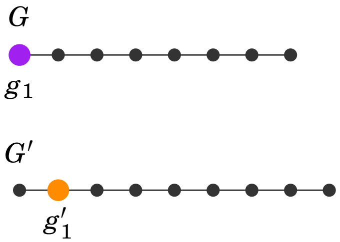

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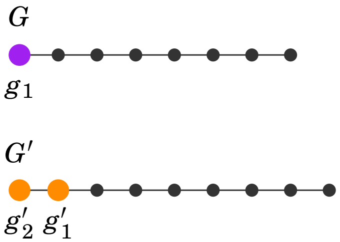

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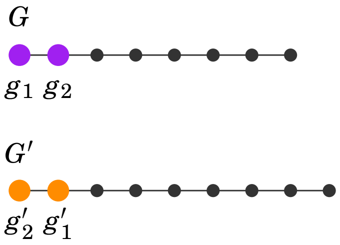

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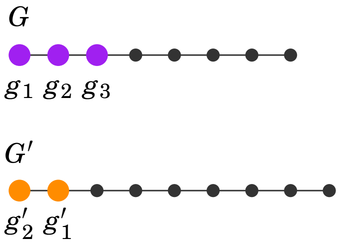

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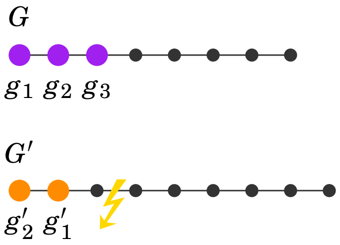

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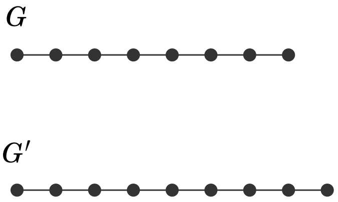

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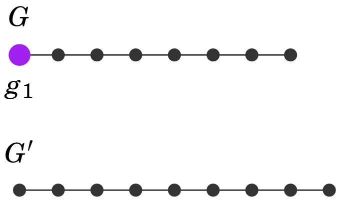

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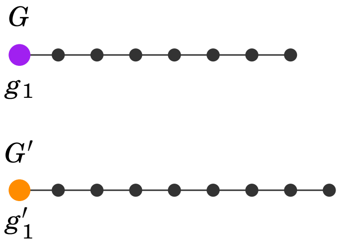

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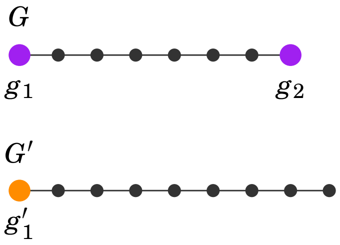

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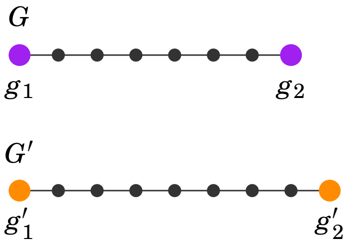

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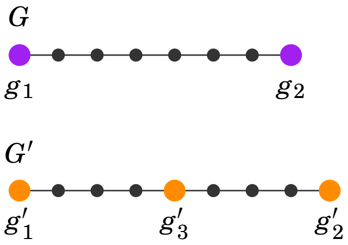

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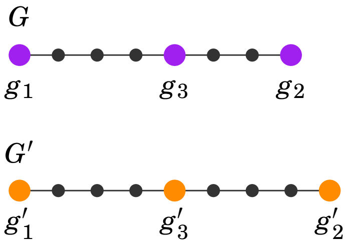

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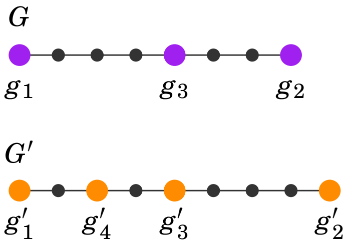

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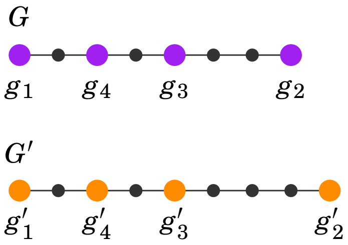

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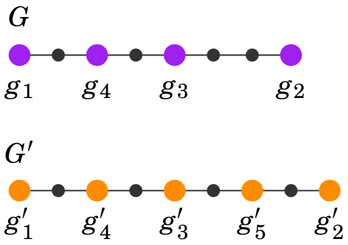

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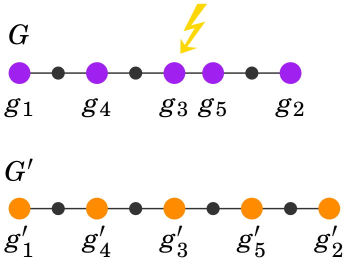

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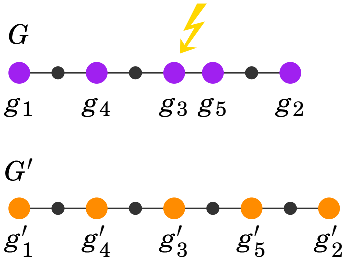


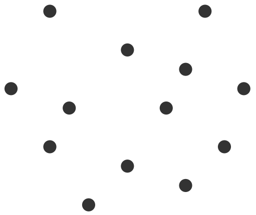
Theorem

$G \equiv_q G'$ iff the Duplicator wins the q-round Ehrenfeucht–Fraïssé game.

Parity

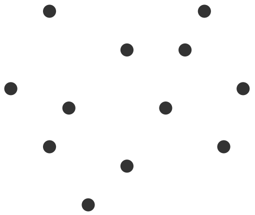
For every q there is G with an even number of vertices and G' with an odd number of vertices and $G \equiv_q G'$.

G



$q+1$ vertices

G'



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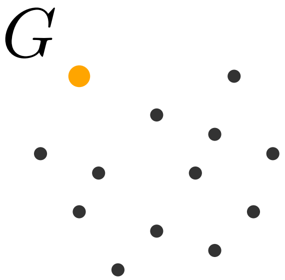

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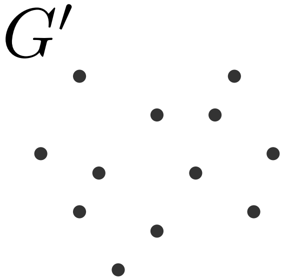

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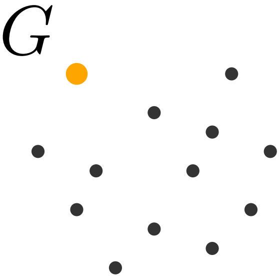

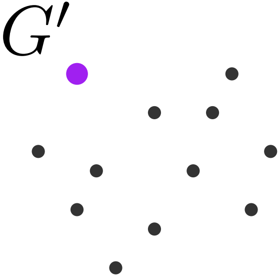

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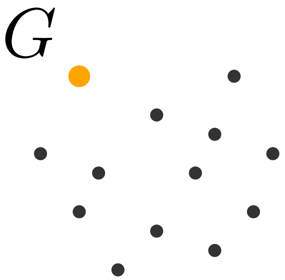

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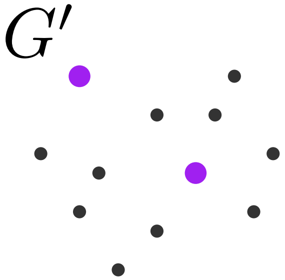

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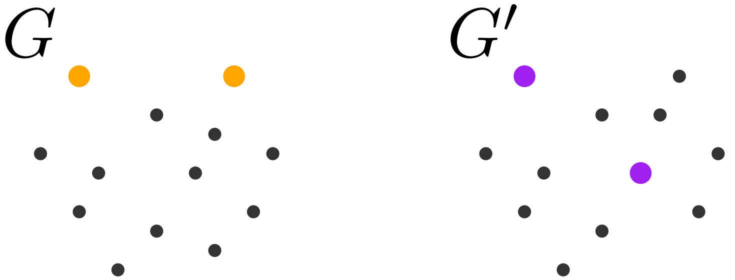

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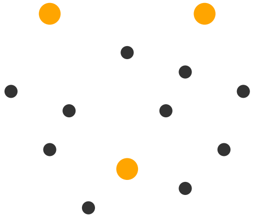

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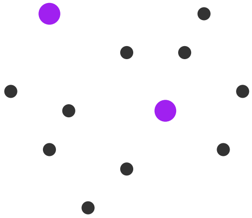

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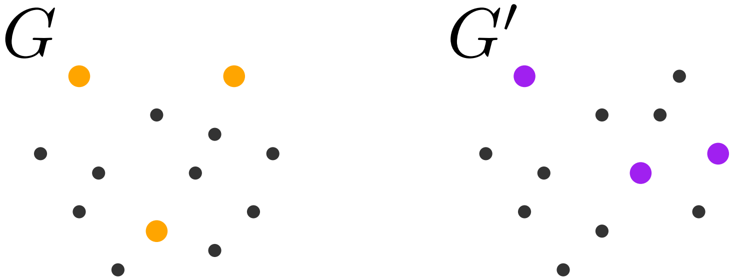

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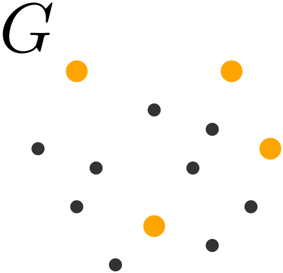

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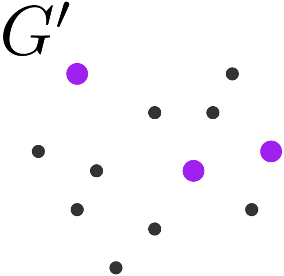

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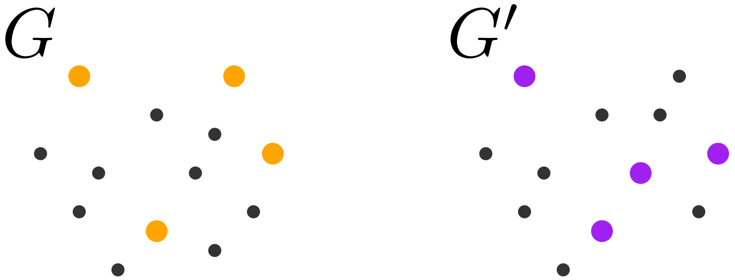

$q + 1$ vertices

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Connectivity

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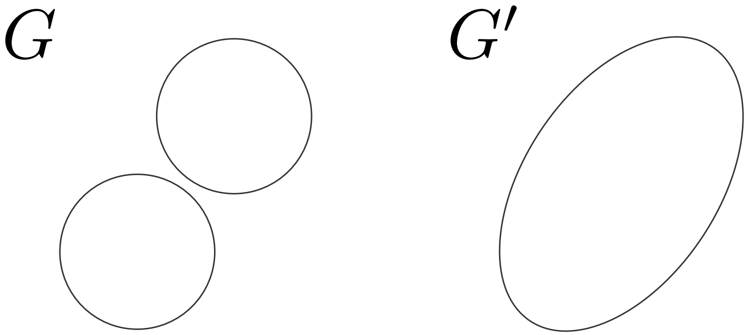

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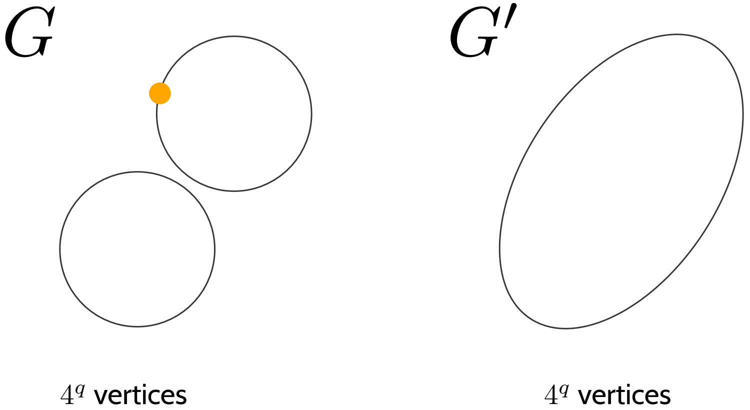

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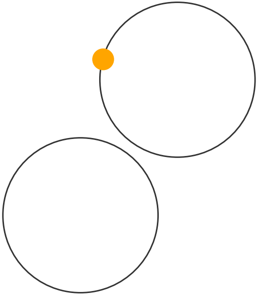
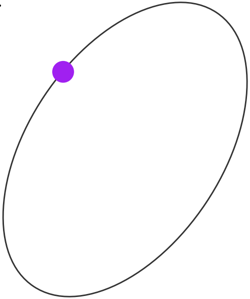
G



4^q vertices

G'



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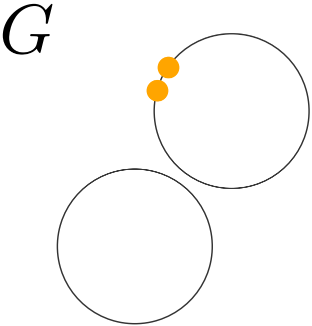

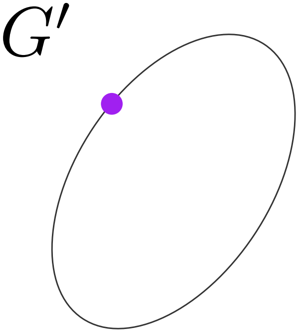

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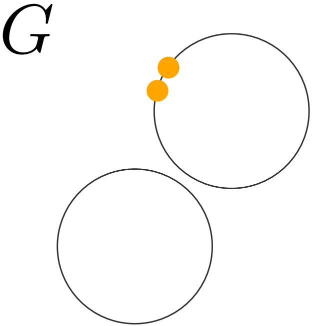

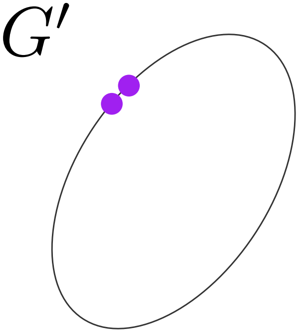

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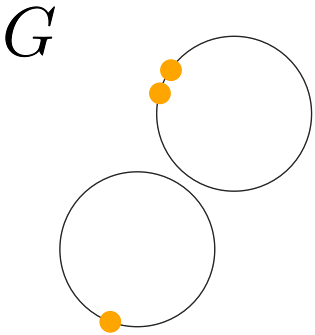

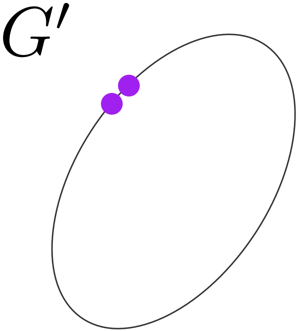

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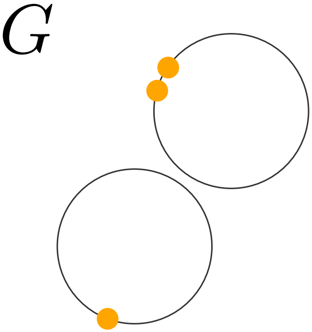

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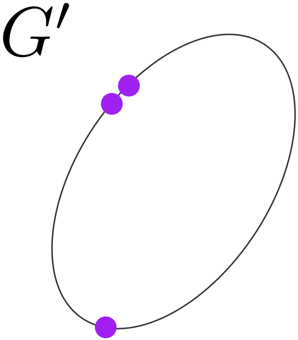

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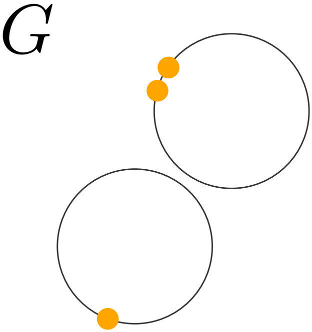

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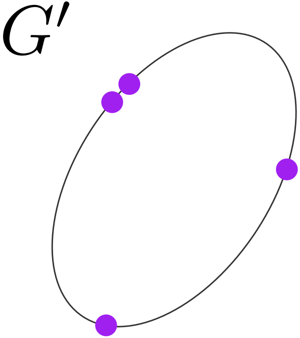

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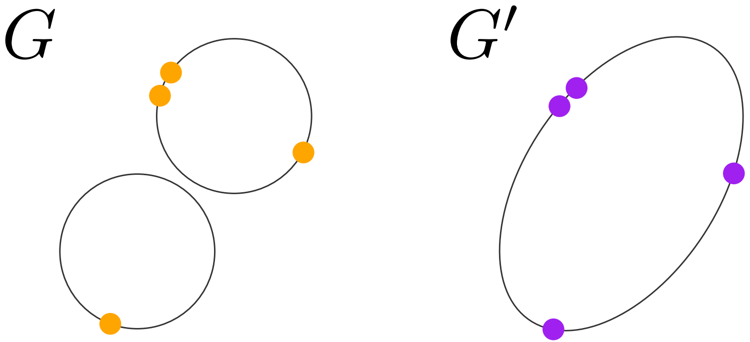

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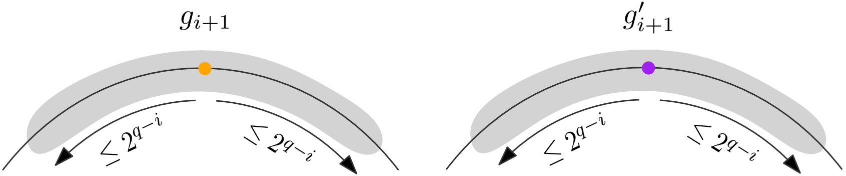
Proof by induction: The Duplicator can play such that for elements $g_1, \ldots, g_i$ and $g'_1, \ldots, g'_i$ choosen after i rounds $\text{dist}(g_j, g_k) =_{2^{q-i+1}} \text{dist}(g'_j, g'_k)$. Here $a =_c b$ means that a and b are equal or both larger than c. Show that this also holds for $i + 1$.



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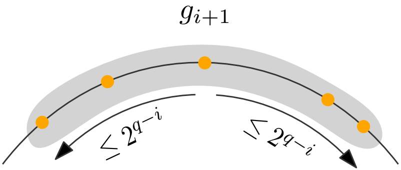

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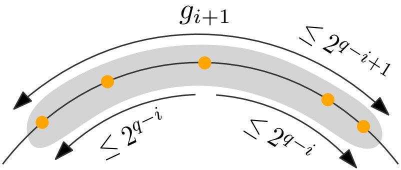

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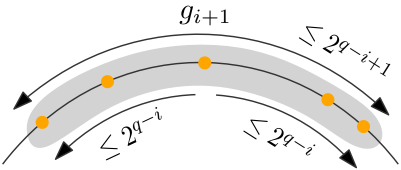

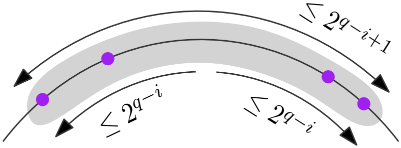

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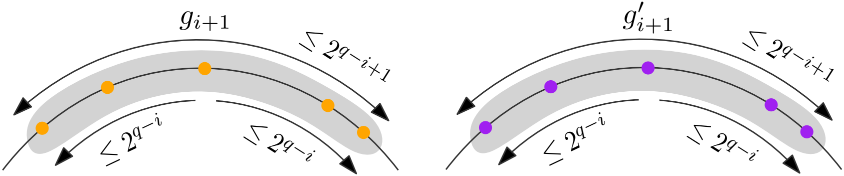

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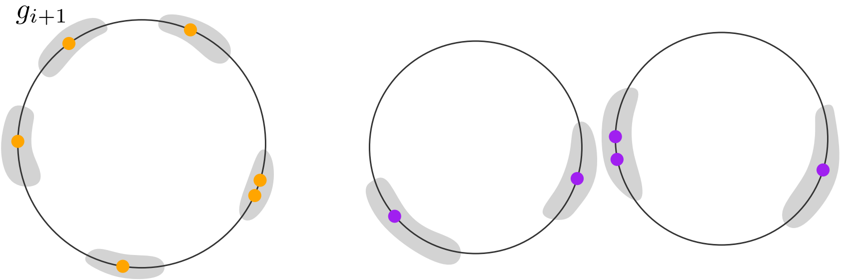

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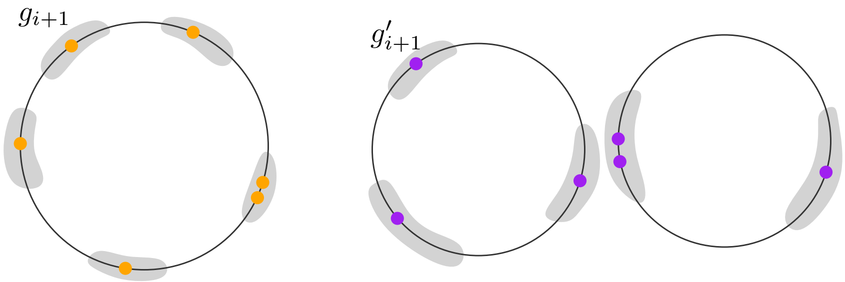
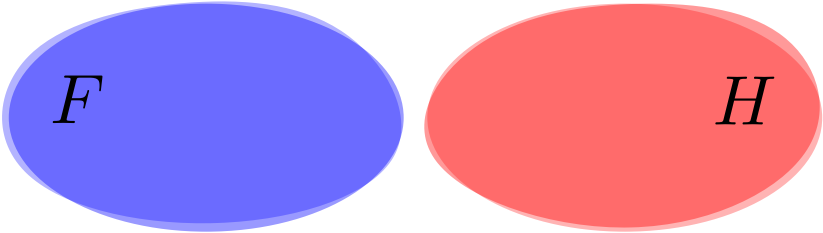


q-type(H) is the set of all FO-sentences ξ of quantifier-rank $\leq q$ with $H \models \xi$. Note that q-type$(F) = q$-type(H) iff $F \equiv_q H$.



Fefermann–Vaught (First-Order)

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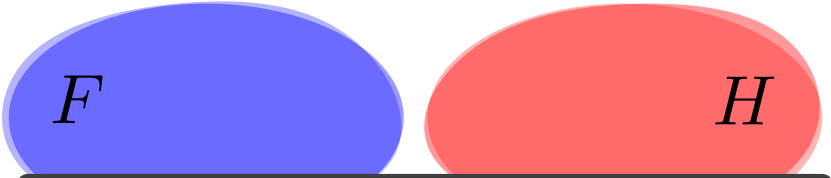
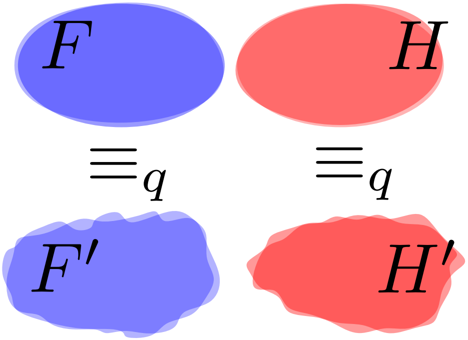


Theorem (Fefermann–Vaught)

Let F and H be graphs with $V(F) \cap V(H) = \emptyset$.
Then q-type$(F \cup H)$ is completely determined by

- q-type(F) and
- q-type(H).

Fefermann–Vaught (First-Order)



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Let F, F', H, H' be graphs. If

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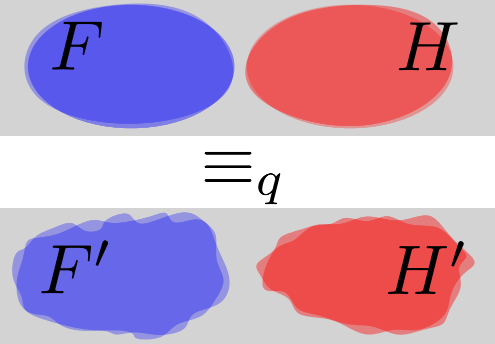

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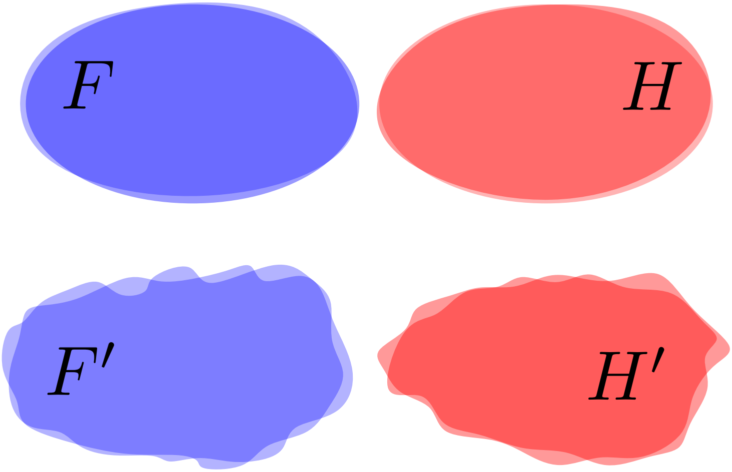
Proof: Combine the two winning strategies of the Duplicator.



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- The field of *descriptive complexity* dedicated to studying the expressiveness of logics.

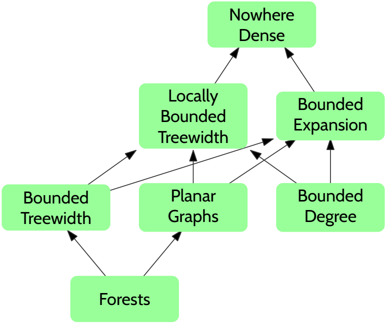
Our First First-Order Meta-Theorem

We will show:

Theorem (Seese 1996)

Let $\mathcal{C}$ be a graph class with bounded degree. There exists a function f such that for every FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n$.

Later we will generalize this to other sparse graph classes.



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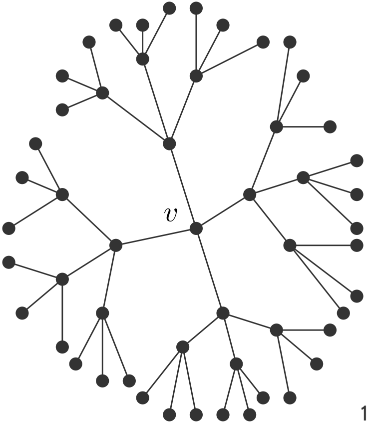
One can use the same technique for *locally bounded treewidth* graph classes. This includes: planar graphs, bounded treewidth, bounded degree, bounded genus, …

Main Tool: Bounded Neighborhood Size

Let v be a vertex in a graph with maximal degree d. Then $|N_r(v)| \leq d^{r+1}$.



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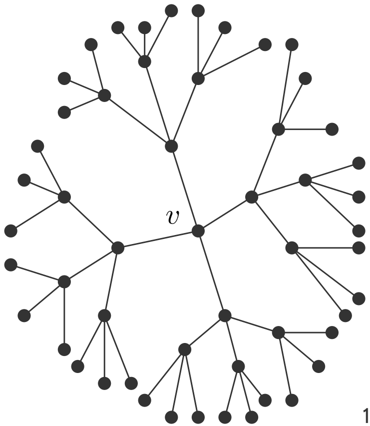

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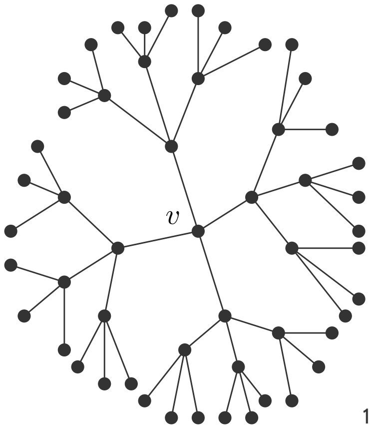

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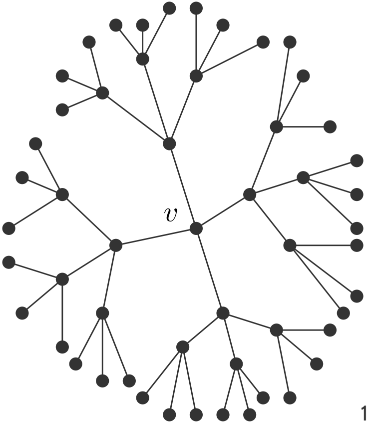


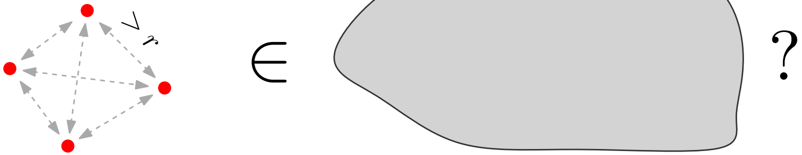
Finding Things That Are Far Appart

Let $\mathcal{C}$ be a graph class with bounded degree. One can decide in time $f(r,k)n$ whether a graph $G \in \mathcal{C}$ contains a red r-scattered set of size k.

Finding Things That Are Far Appart

red r-scattered set:
k red vertices with
pairwise distance $> r$

degree $\leq d$



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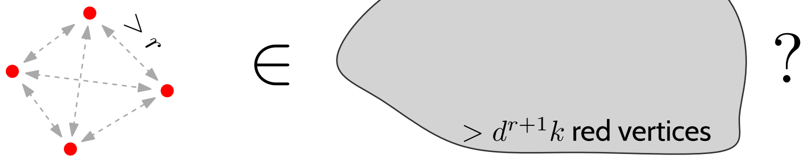

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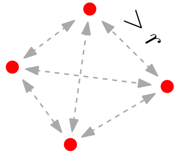

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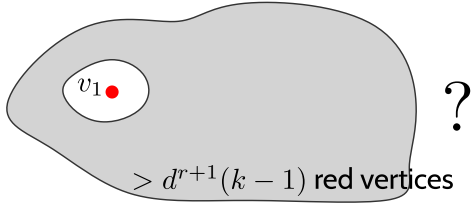

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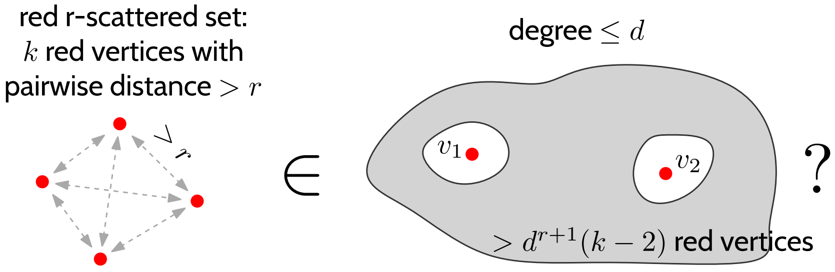

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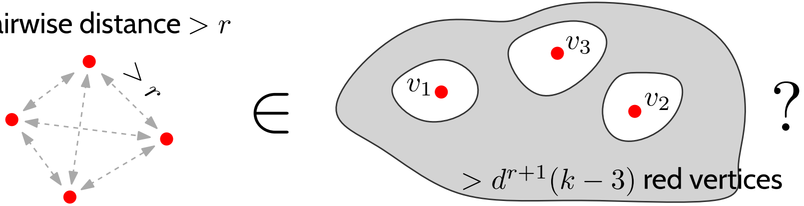

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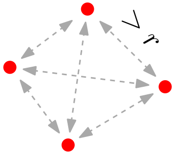

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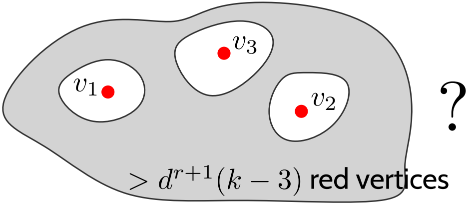

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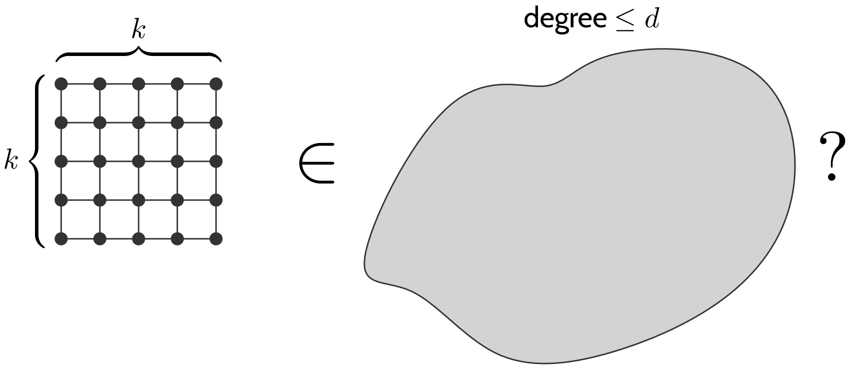
- Works because $\leq d^{r+1}$ vertices are removed each round.
- Run time: $(d^{r+1}k)^{O(k)}n$.

Finding Things That Are Local

Let $\mathcal{C}$ be a graph class with bounded degree. One can decide in time $f(k)n$ whether a graph $G \in \mathcal{C}$ contains a $k \times k$-grid as an induced subgraph.

Finding Things That Are Local



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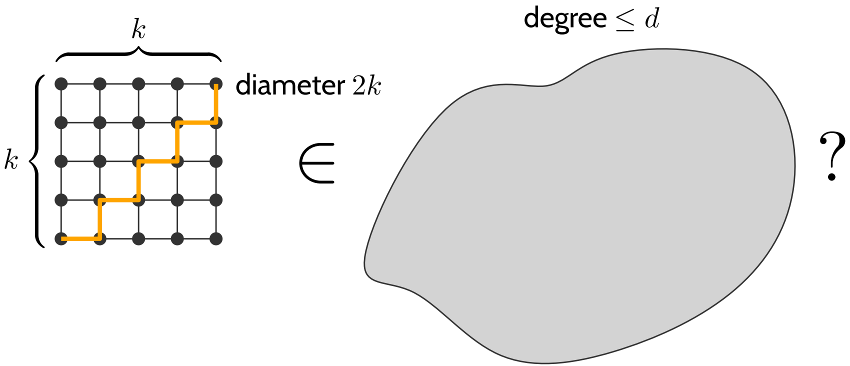

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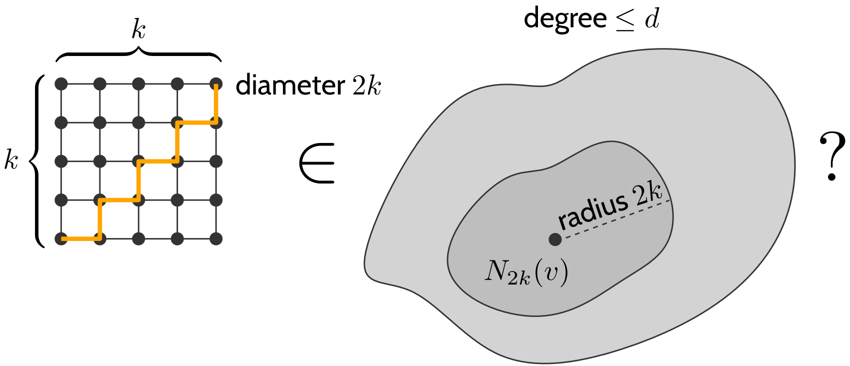

- Is there a $k \times k$ grid in a graph of degree $\leq d$?
- If yes, then it is contained in $N_{2r}(v)$ for some vertex v.

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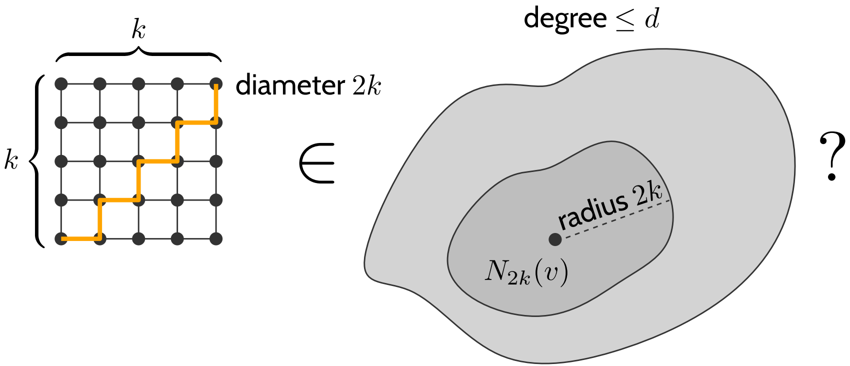

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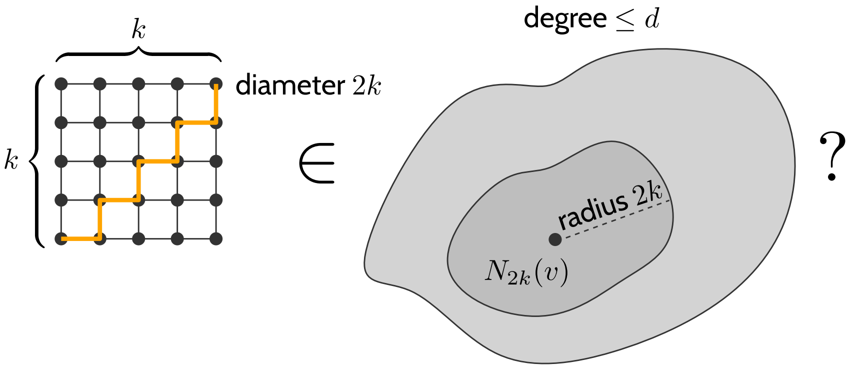

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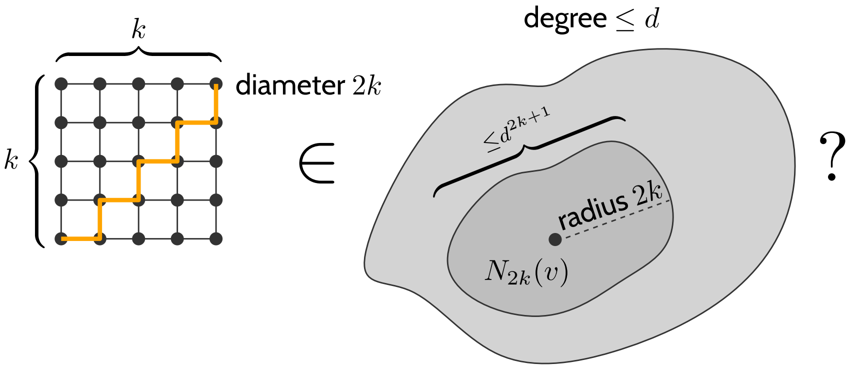

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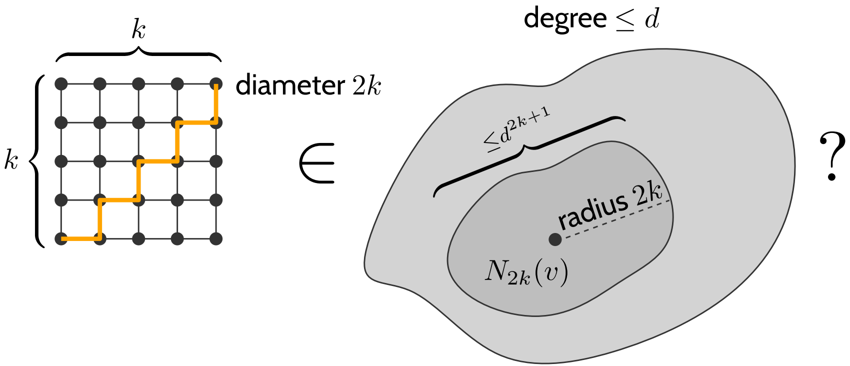

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- Total time: $d^{(2k+1)O(k^2)}n$

Combining Both Ideas

Homework

Let $\mathcal{C}$ be a graph class with bounded degree. For every graph H and $G \in \mathcal{C}$ one can decide in time $f(|H|)n$ whether G contains H as induced subgraph.

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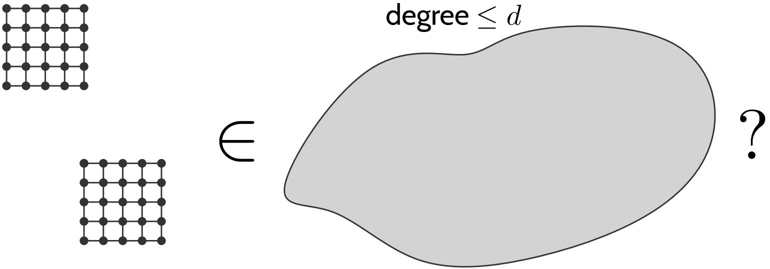
Let $\mathcal{C}$ be a graph class with bounded degree. One can decide in time $f(k)n$ whether a graph $G \in \mathcal{C}$ contains two disjoint $k \times k$-grids as induced subgraph.

Combining Both Ideas



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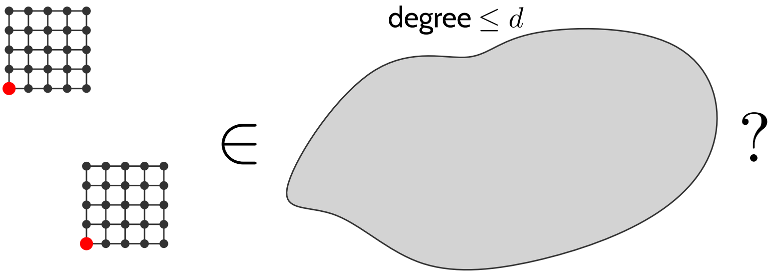

- Are there two disjoint $k \times k$ grids in a graph of degree $\leq d$?
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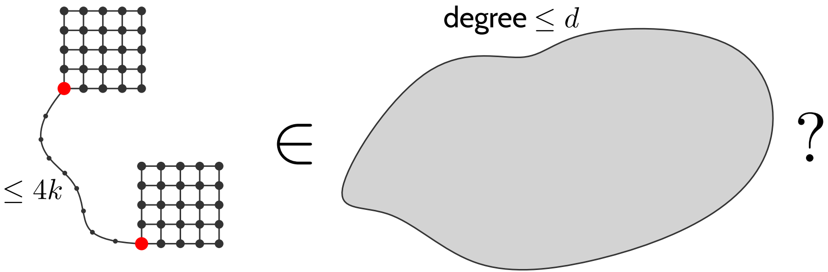

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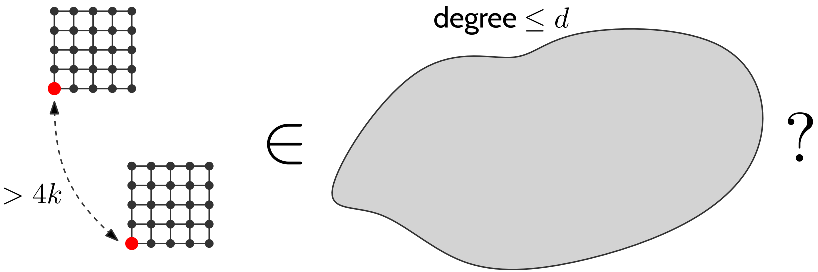

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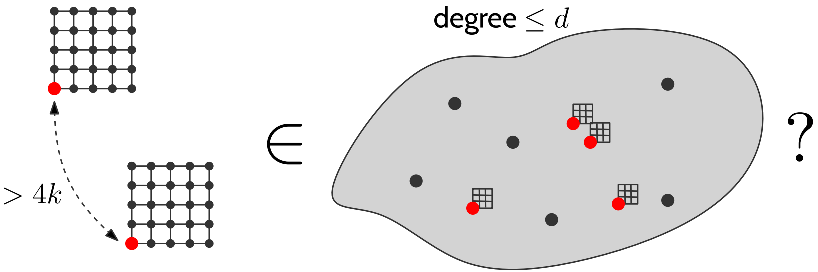

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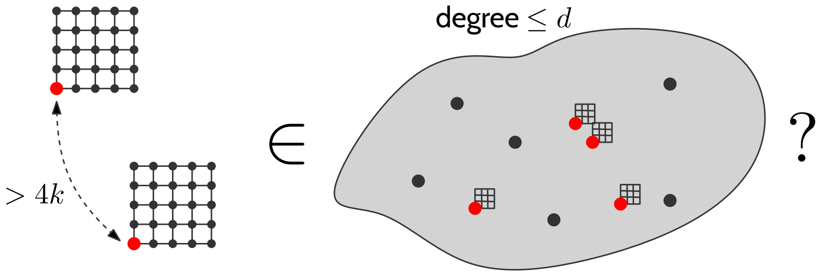

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  - Use existing approach to find two red vertices with distance $> 4k$. Since grids have radius $2k$, their grids are disjoint.

Gaifman

The same kind of split into “local” and “far apart” parts can be done for first-order logic as well.

In a sense, all that first-order logic can do is say certain “local” properties are “far apart”.

This is formalized in Gaifman’s locality Theorem.

Local Formulas

A formula $\omega(x)$ is called r-*local* if for all graphs G and all $v \in V(G)$

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Evaluating Local Formulas

Let G be a graph with degree $\leq\ d$ and $\omega(x)$ be an r-local formula. One can compute in in time $O(d^{(r+1)|\varphi|}n)$ the set of all $v \in V(G)$ with $G \models \omega(v)$.

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Proof:

- Enumerate all vertices v.
- Instead of evaluating $\varphi(v)$ on G, we evaluate it on $G[N_r(v)]$.
- $G[N_r(v)]$ contains $\leq d^{r+1}$ vertices.
- We can evaluate $\varphi(v)$ on a graph with k vertices in time $O(k^{|\varphi|})$.

Basic Local Sentences

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Basic Local Sentences

Let $\omega(x)$ be an r-local formula. Let $\text{dist}_{>2r}(x_1, x_2)$ be the first-order formula denoting that the distance between x_1 and x_2 is greater than $2r$. A *basic local sentence* is a sentence of the form

$$\exists x_1 \ldots \exists x_s \Big(\bigwedge_{i \neq j} \text{dist}_{>2r}(x_i, x_j) \wedge \bigwedge_{i} \omega(x_i) \Big).$$

"There are s vertices that all satisfy some local property ω and are far apart."

Example: There exist s vertices with degree 4 and pairwise distance larger than 10.

Gaifman's Locality Theorem

Every first-order sentence is equivalent to a boolean combination of basic local sentences.
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- either there exists s_1 that is part of a subgraph □–□ ($6k$-local),
- or there exist s_1, s_2, both part of a subgraph □ ($2k$-local) with $\text{dist}_{>4k}(s_1, s_2)$.

Proof of First-Order Meta-Theorem

Theorem (Seese 1996)

Let $\mathcal{C}$ be a graph class with bounded degree. There exists a function f such that for every FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n$.

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Proof:

- By Gaifman's theorem, φ is equivalent of boolean combination of basic local sentences such as

$$\exists x_1 \ldots \exists x_s \Big(\bigwedge_{i \neq j} \mathrm{dist}_{>2r}(x_i, x_j) \wedge \bigwedge_i \omega(x_i) \Big).$$

  The size and number of these basic local sentences depends only on φ. We show how to evaluate such sentences.

Proof

Consider a basic local sentence

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- Neighborhoods $G[N_r(v)]$ have size $\leq d^{r+1}$.

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- Doing it for all $v \in V(G)$ takes time $O(d^{(r+1)|\varphi|} n)$.

We have to find a red $2r$-scattered set of size s in G. This can be done in time $f(r, s)n$ in bounded degree classes.

Locally Bounded Treewidth

Our approach for bounded degree easily generalizes to *locally bounded treewidth*.

Locally Bounded Treewidth

A graph class $\mathcal{C}$ has *locally bounded treewidth* if there exists a function $g(r)$ such that for every graph $G \in \mathcal{C}$, every $v \in V(G)$ and every $r \in \mathbb{N}$ the neighborhood $G[N_r(v)]$ has treewidth at most $g(r)$.

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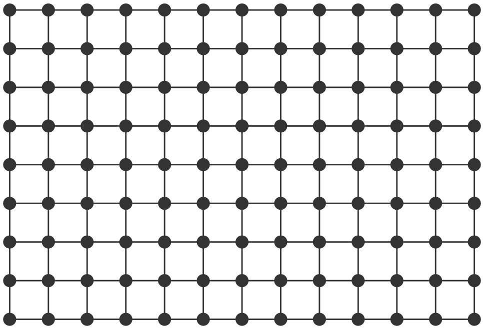
Which of these classes have locally bounded treewidth?



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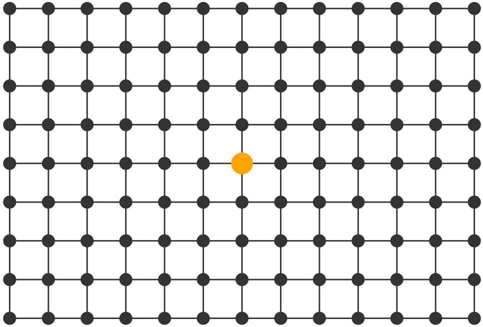

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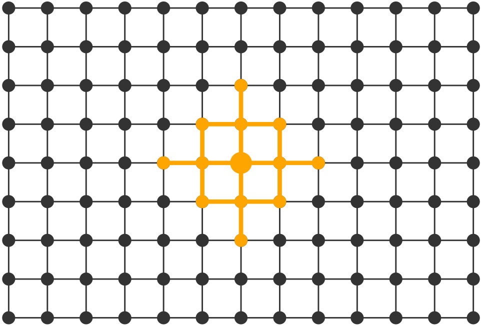

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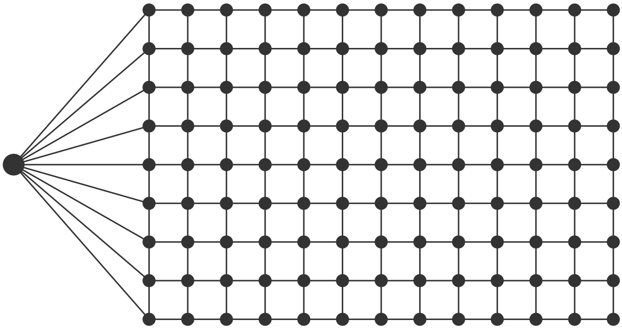

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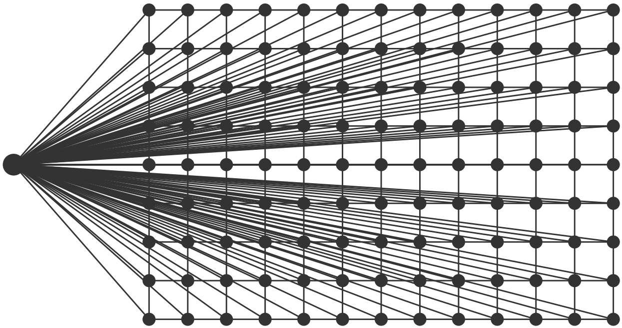
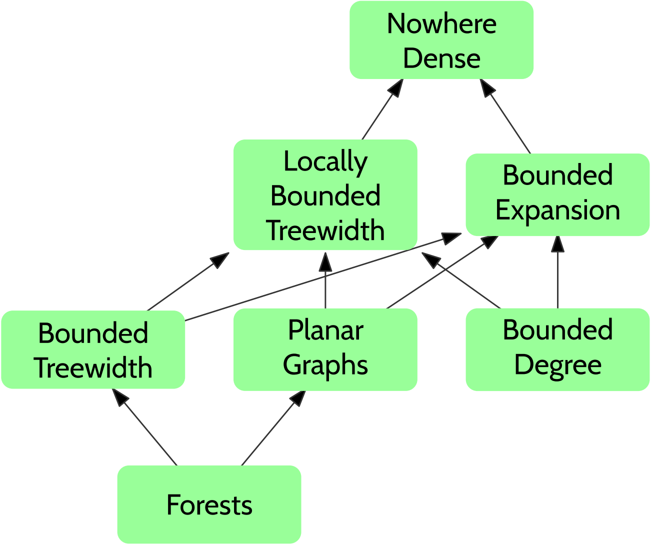


Relationship to Other Classes



Relationship to Other Classes

Locally bounded treewidth generalizes

- planar graphs,
- bounded degree,
- bounded treewidth.

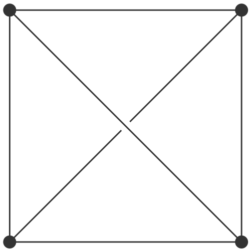
Theorem (Bodlaender)

The treewidth of a planar graph with radius r is at most $3r - 1$.

The size of a graph with with radius r and maximal degree d is at most d^{r+1}.

Subdivisions

Does the class of $\log(n)$-subdivisions of n-cliques have locally bounded treewidth?



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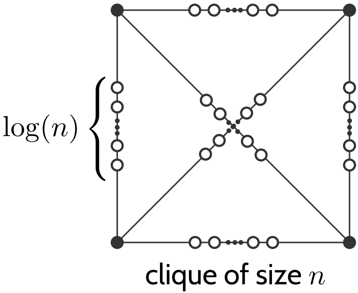

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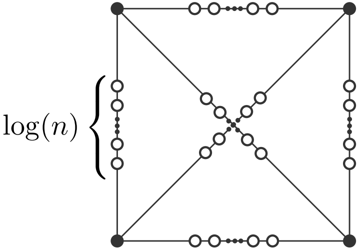

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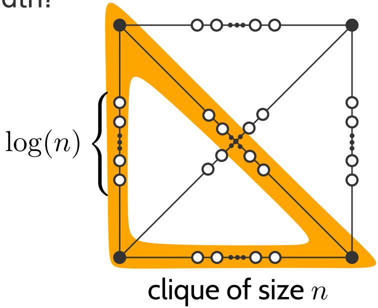

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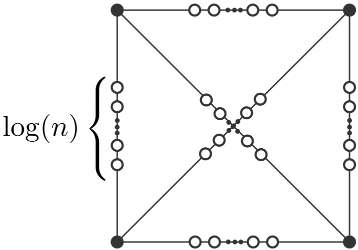

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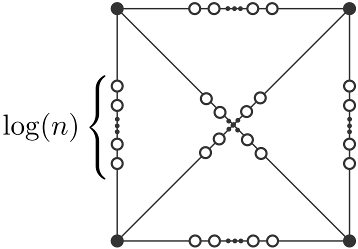

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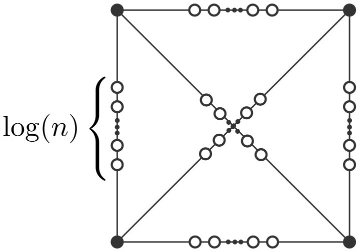
- If $r > \log(n)$ then $G[N_r(v)]$ has treewidth at most $n \leq 2^r$.

Subdivisions

Does the class of $\log(n)$-subdivisions of n-cliques have locally bounded treewidth?

Yes! Proof:



clique of size n

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- If $r > \log(n)$ then $G[N_r(v)]$ has treewidth at most $n \leq 2^r$.
- For every r, $G[N_r(v)]$ has treewidth at most 2^r.

A More General First-Order Meta-Theorem

Theorem (Frick, Grohe 2001)

Let $\mathcal{C}$ be a graph class with locally bounded treewidth. There exists a function f such that for every FO formula φ, graph $G \in \mathcal{C}$ and $\varepsilon > 0$ one can decide whether $G \models \varphi$ in time $f(\varepsilon, |\varphi|)n^{1+\varepsilon}$.

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We prove the following slightly weaker statement.

Theorem (Frick, Grohe 2001)

Let $\mathcal{C}$ be a graph class with locally bounded treewidth. There exists a function f such that for every FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n^2$.

Proof

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Proof:

- By Gaifman's theorem, φ is equivalent of boolean combination of basic local sentences such as

$$\exists x_1 \dots \exists x_s \big(\bigwedge_{i \neq j} \mathrm{dist}_{>2r}(x_i, x_j) \wedge \bigwedge_i \omega(x_i) \big).$$

  We show how to evaluate such sentences.

Proof

Consider a basic local sentence

$$\exists x_1 \ldots \exists x_s \Big(\bigwedge_{i \neq j} \mathsf{dist}_{>2r}(x_i, x_j) \wedge \bigwedge_i \omega(x_i) \Big).$$

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We compute the set R of all vertices v with $G[N_r(v)] \models \omega(v)$ and color them red.

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- Neighborhoods $G[N_r(v)]$ have treewidth at most $g(r)$.

Proof

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- Neighborhoods $G[N_r(v)]$ have treewidth at most $g(r)$.
- For $v \in V(G)$, check whether $G[N_r(v)] \models \omega(v)$ in time $f'(|\omega|, g(r))n$ using Courcelle's theorem.

Proof

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We have to find a red $2r$-scattered set of size s in G. This is equivalent to finding a red $2r$-scattered set of size s in $G[N_r(R)]$.

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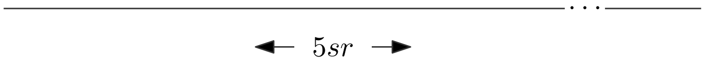
We have to find a red $2r$-scattered set of size s in G. This is equivalent to finding a red $2r$-scattered set of size s in $G[N_r(R)]$.

We do a case distinction on the diameter of $G[N_r(R)]$.

Proof

Assume $G[N_r(R)]$ has diameter at least $5sr$.

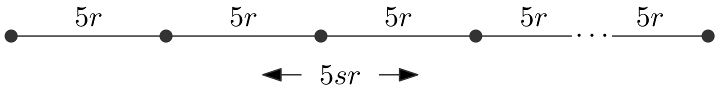
- Then there is a shortest path of length $5sr$.



Proof

Assume $G[N_r(R)]$ has diameter at least $5sr$.

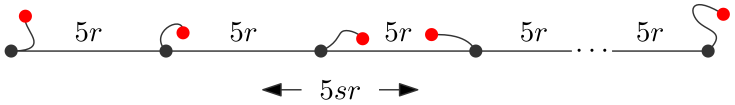
- Then there is a shortest path of length $5sr$.
- There are s vertices on it with distance $5r$ of each other.



Proof

Assume $G[N_r(R)]$ has diameter at least $5sr$.

- Then there is a shortest path of length $5sr$.
- There are s vertices on it with distance $5r$ of each other.
- Each vertex on the path has distance $\leq r$ from a red vertex.



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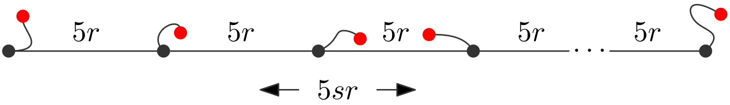
- There are s vertices on it with distance $5r$ of each other.
- Each vertex on the path has distance $\leq r$ from a red vertex.
- These red vertices have pairwise distance $> 2r$.



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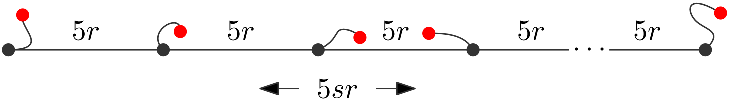
Assume $G[N_r(R)]$ has diameter at least $5sr$.

- Then there is a shortest path of length $5sr$.
- There are s vertices on it with distance $5r$ of each other.
- Each vertex on the path has distance $\leq r$ from a red vertex.
- These red vertices have pairwise distance $> 2r$.
- $2r$-scattered set of size s exists!



Proof

Assume $G[N_r(R)]$ has diameter at most $5sr$. Then it has treewidth at most $g(5sr)$, where g is the treewidth bound of $\mathcal{C}$. Use Courcelle's theorem to decide if $G[N_r(R)]$ has red $2r$-scattered set.

Summary

We now have a first-order meta-theorem for locally bounded treewidth.

This captures three natural classes of graphs.

- bounded treewidth
- planar graphs
- bounded degree

However, locally bounded treewidth is not very robust. It is not closed under adding apex-vertices.