Algorithmic Meta-Theorems

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Summary

Courcelle's theorem is a very powerful tool to solve problems on bounded treewidth. It comes in various flavours.

- MSO_1: base variant,
- MSO_2: edge quantifiers,
- CMSO: parity/modulo counting,
- LinEMSOL: optimization,
- and any combination thereof.

We now want to prove the following.

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For a MSO_1 formula φ and graph G one can decide whether $G \models \varphi$ in time $f(\mathrm{tw}(G), |\varphi|)n$ for some function f.

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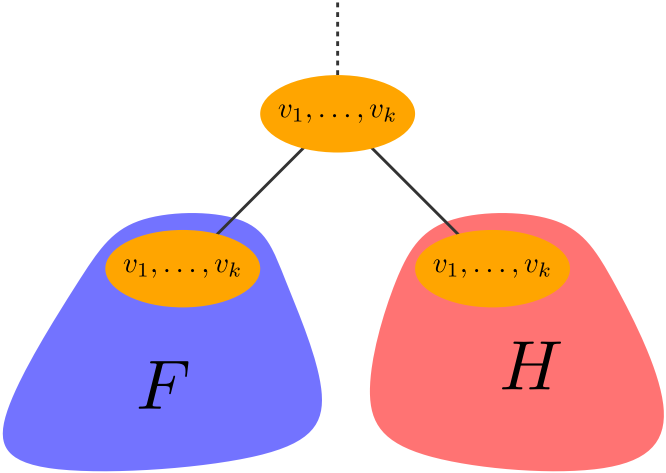
- Historically proven by converting MSO_1-formulas into tree-automata. Use this automaton to traverse the tree-decomposition.
- We prove it using a powerful logic-theorem by Fefermann and Vaught as a blackbox.

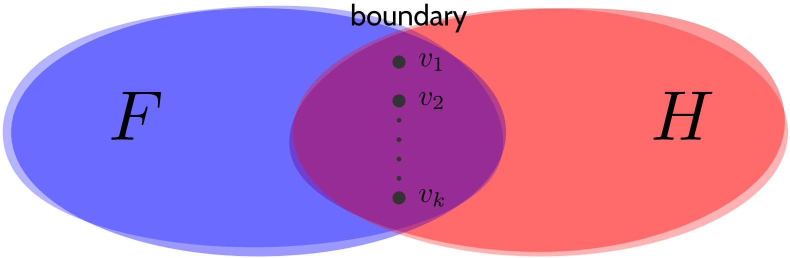
Join Nodes

We can assume we are given a nice tree decomposition. If we manage the *join* operation, *introduce* and *forget* are easy.



Fefermann–Vaught

Let H be a graph with boundary $v_1, \ldots, v_k$. We define q-type$(H; v_1, \ldots, v_k)$ to be the set of all MSO_1-formulas $\xi(x_1, \ldots, x_k)$ of quantifier-rank $\leq q$ with $H \models \xi(v_1, \ldots, v_k)$.



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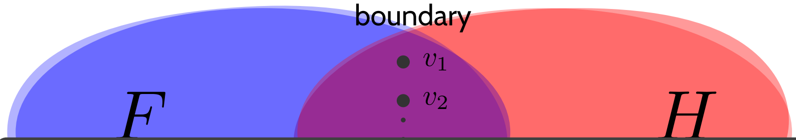

Theorem (Fefermann–Vaught)

Let F and H be graphs with $V(F) \cap V(H) = \{v_1, \ldots, v_k\}$. If we know

- q-type$(F; v_1, \ldots, v_k)$
- q-type$(H; v_1, \ldots, v_k)$

then we can compute q-type$(F \cup H; v_1, \ldots, v_k)$.

Dynamic Programming

We have a nice tree decomposition of a graph G and want to know whether $G \models \varphi$ for a formula with quantifier-rank q.

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Let r be the root-node. Then $G \models \varphi$ iff $G[V_r] \models \varphi$ iff $M_r(\varphi) = 1$.

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This bound cannot be improved much.

Dynamic Programming

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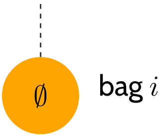
$$M_i(\xi) = \begin{cases} 1 & G[V_i] \models \xi(v_1, \ldots, v_k) \\ 0 & \text{otherwise.} \end{cases}$$

We now compute these values bottom-up.

Leaf Nodes

$G[V_i]$ is the empty graph.



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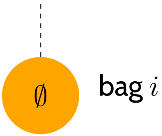

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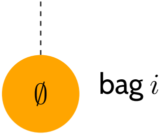
Simply evaluate them.



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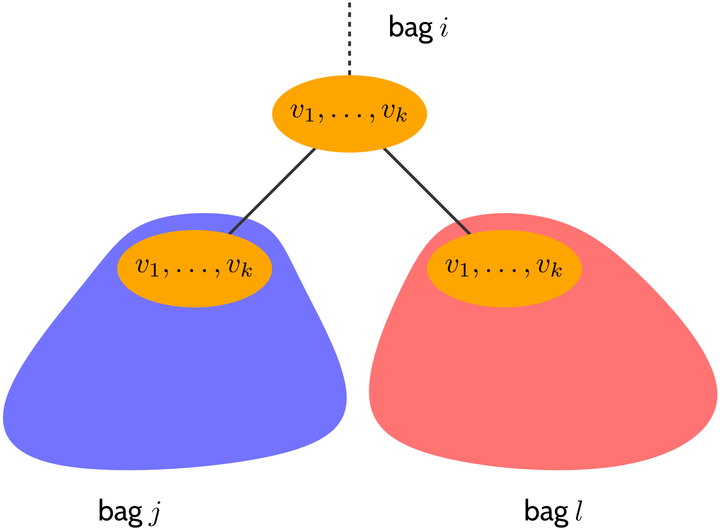

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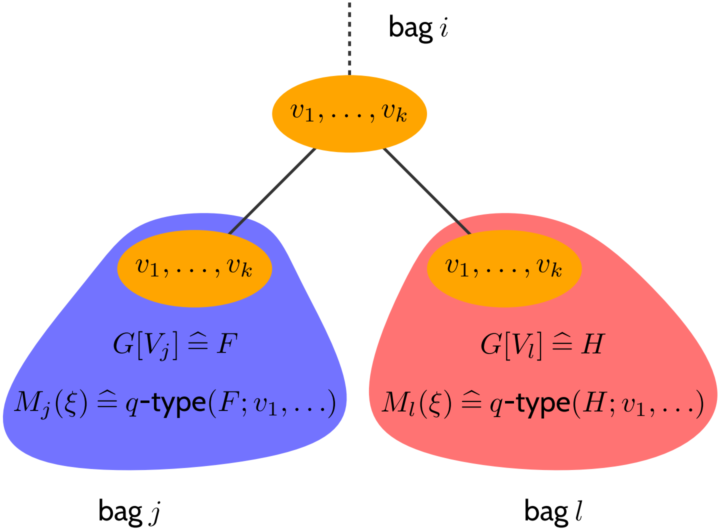

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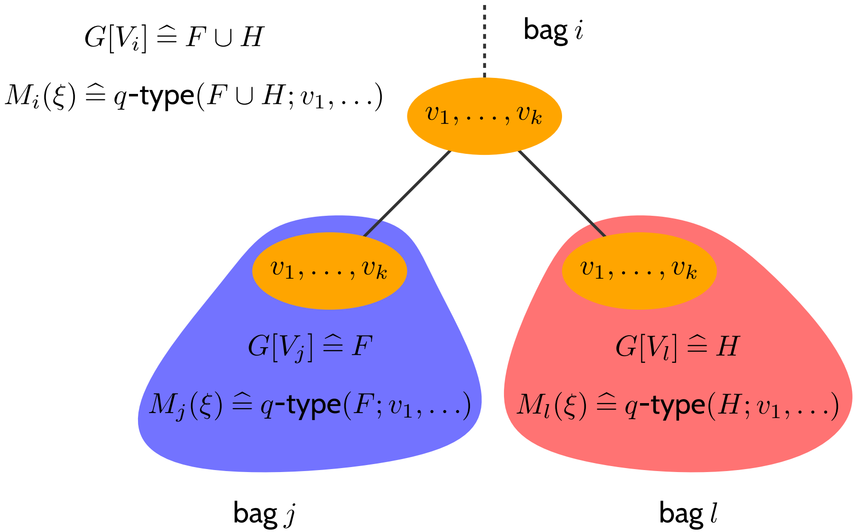

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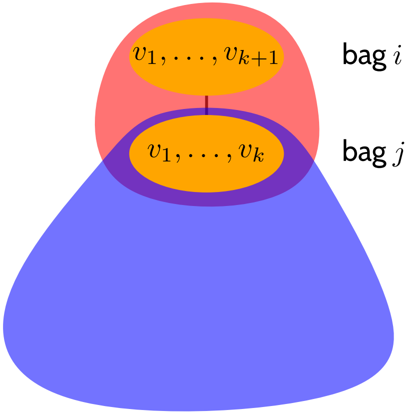
Introduce Nodes



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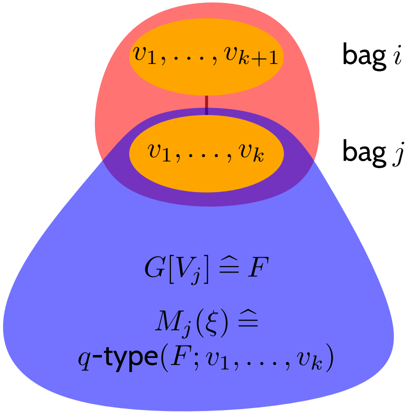

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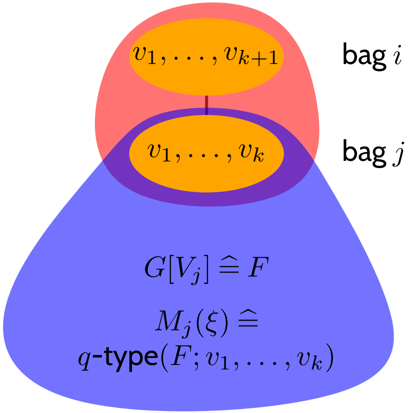

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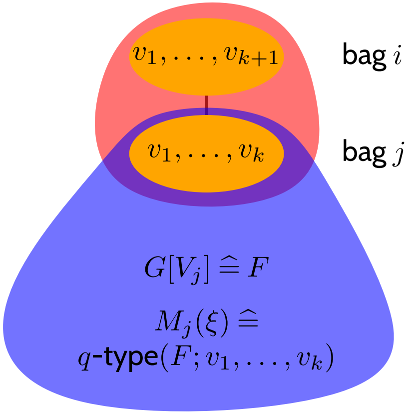

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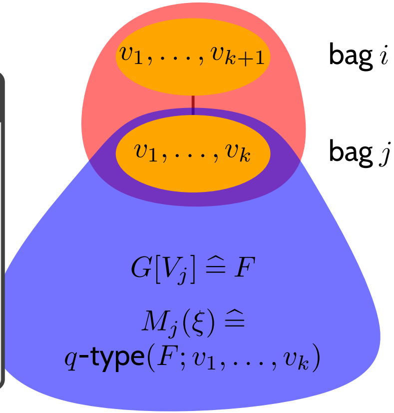

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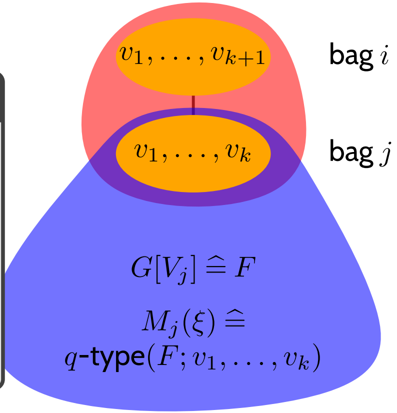

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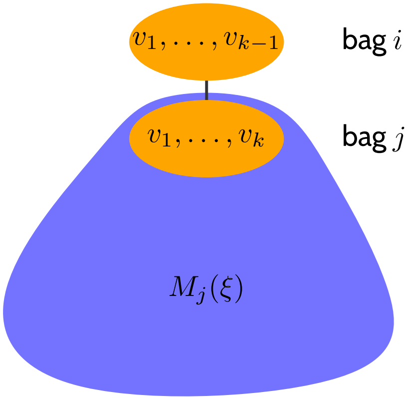

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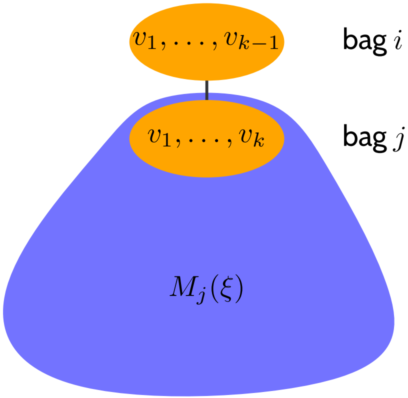

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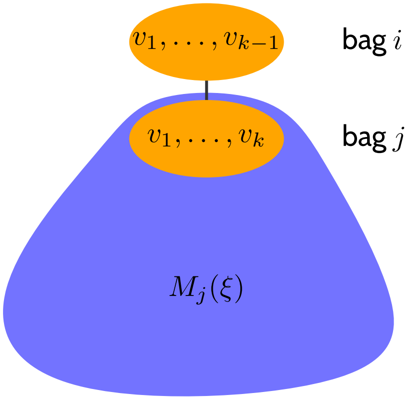

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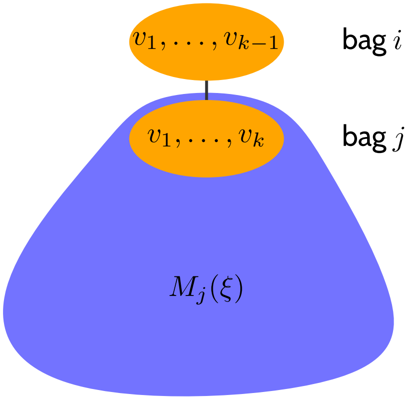

Forget Nodes

$G[V_i] = G[V_j]$

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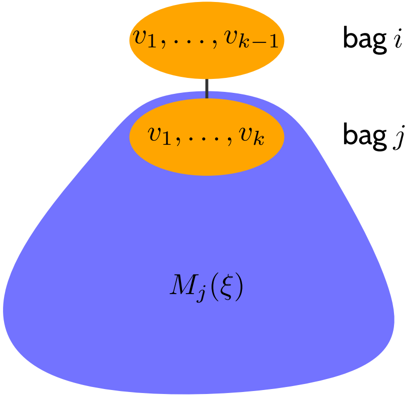
Remove all formulas that mention x_k.



Final Result

For a MSO_1 formula φ and graph G one can decide whether $G \models \varphi$ in time

$$\underbrace{2^{\cdot^{\cdot^{\cdot^{2^{O(\mathsf{tw}(G)^2|\varphi|)}}}}}}_{2|\varphi|} n.$$

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Can this be improved?

No. It cannot be done without such a tower of powers (Frick, Grohe 2004).

Implementations

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Usually, Courcelle's theorem is considered a theoretical classification tool. But it has been implemented by Kneis, Langer, and Rossmanith.

- Doing it naively has horrible, horrible run time ...
- By storing "game trees" instead, it becomes feasible `https://github.com/sequoia-mso`.

How to formulate Problems for Sequoia

```
-- More efficient formula for three-coloring; tests whether
-- (R, G, V\setminus (R\cup G)) is a proper three-coloring of the graph

ThreeCol(R, B) :=
        All x (
                (x notin R or x notin B)
            and
                All y (
                    ~adj(x,y) or (
                        (x notin R or y notin R) and
                        (x notin B or y notin B) and
                        ((x in R) or (x in B)
                                                      or
                         (y in R) or (y in B))
                    )
                )
        )
```

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Proof for Extensions

- We saw the proof for the base variant MSO_1.
- By swapping in FV theorem for CMSO, we get the proof for the parity/modulo variant CMSO.
- Also we already know how to reduce MSO_2 to MSO_1.
- The proof for the optimization variant LinEMSOL works similarly by also keeping track of the largest satisfying assignment.

How about Dense Graphs?

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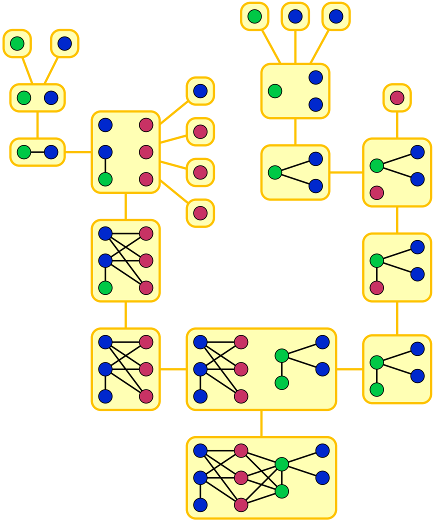
- There is a width measure *cliquewidth* similar to treewidth for which cliques have width one.
- Courcelle's theorem for MSO_1 also holds for cliquewidth.
- On the other hand MSO_2 only holds for treewidth.

Cliquewidth

Cliquewidth cw(G): Minimum number of colors needed to construct G using these operations.

- Creation of new vertex with color i
- Disjoint union of two graphs
- Joining by an edge every vertex with color i to every vertex with color j
- Changing color i to color j



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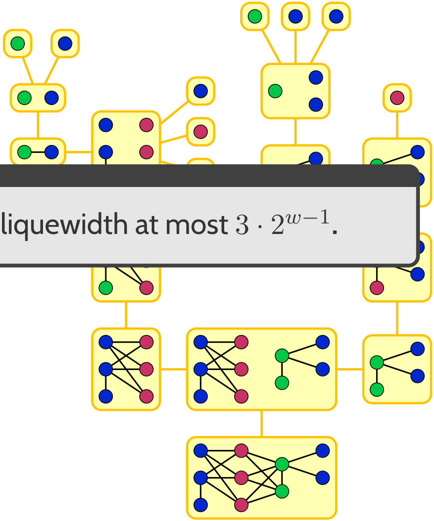

Graphs of treewidth w have cliquewidth at most $3 \cdot 2^{w-1}$.

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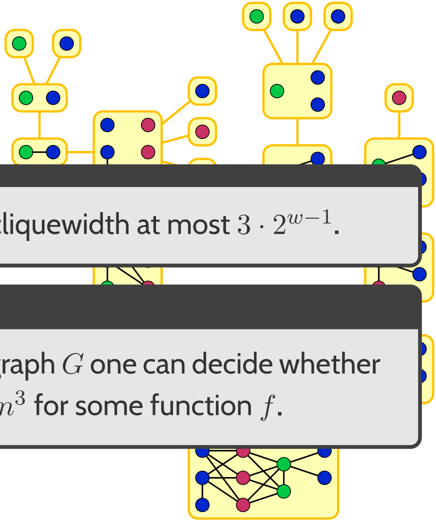


Graphs of treewidth w have cliquewidth at most $3 \cdot 2^{w-1}$.

Courcelle's Theorem

For a MSO_1 sentence φ and graph G one can decide whether $G \models \varphi$ in time $f(\text{cw}(G), |\varphi|)n^3$ for some function f.

- vertex with color j
- Changing color i to color j

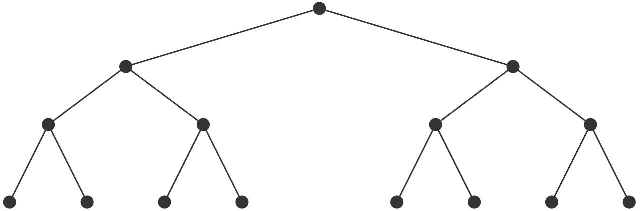
And Now For Something Completely Different...

Let us go back to the first lecture.

Independent Set on Trees

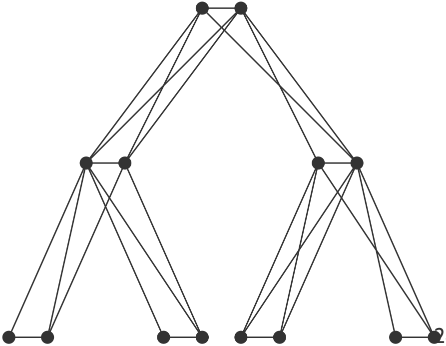
INDEPENDENTSET can be solved in linear time on trees.

Idea: Root the tree and do dynamic programming. Starting at the leafs, compute for each subtree the maximum size of a solution with and without its root.



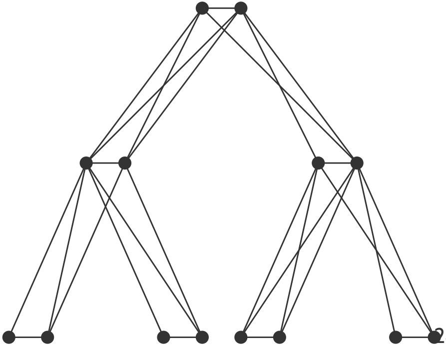
First Main Result

- This approach can be extended to tree-like graphs (bounded treewidth).



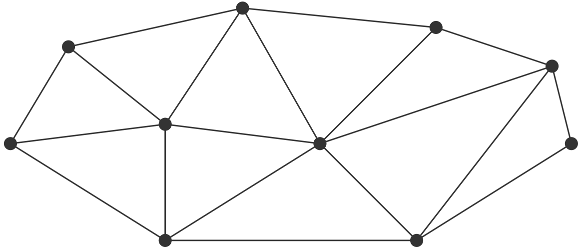
First Main Result

- This approach can be extended to tree-like graphs (bounded treewidth).
- First main result of the lecture (Courcelle's theorem): Every problem definable in monadic second-order logic can be solved in linear time on graphs of bounded treewidth.
- This includes
  - coloring
  - independent set
  - clique
  - dominating set
  - feedback vertex set
  - hamilton path
  - ...



Independent Set on Planar Graphs

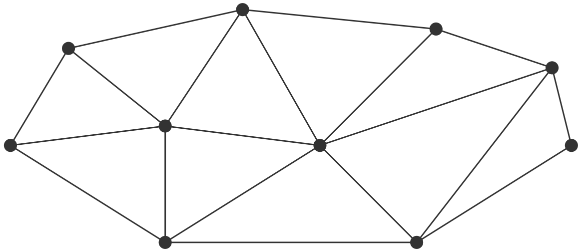
How about planar graphs?



Independent Set on Planar Graphs

How about planar graphs?

INDEPENDENTSET is NP-complete on planar graphs.



Algorithm

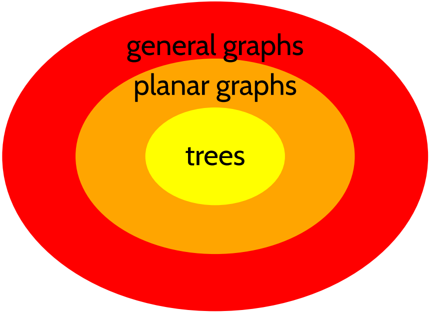
One can decide whether a planar graph has an independent set of size k in time $O(6^k n)$.

IS(G, k):
    if G is empty return $k == 0$
    find vertex v with degree ≤ 5 in G
    for all $w \in N(v)$:
        if IS($G \setminus N(w)$, $k - 1$) return True
    return False

Summary

INDEPENDENTSET is hard on general graphs. However,

- on trees, we can solve it in linear time
- on planar graphs, it is still fixed parameter tractable.

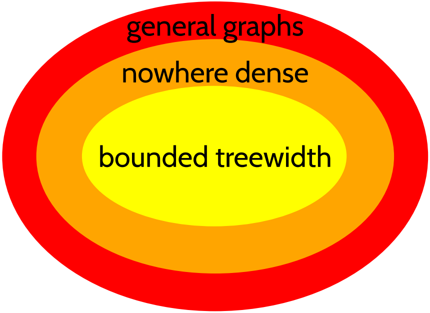


We will observe a similar behaviour for many other problems!

Summary

INDEPENDENTSET is hard on general graphs. However,

- on bounded treewidth, we can solve it in linear time
- on nowhere dense graphs, it is still fixed parameter tractable.



We will observe a similar behaviour for many other problems!

Graph Classes

- *Width measures* (treewidth, degree, . . .) capture the structure of a graph using *one number*. Sometimes, we may need more numbers to describe something.

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- *Width measures* (treewidth, degree, ...) capture the structure of a graph using *one number*. Sometimes, we may need more numbers to describe something.
- From now on, we work with (infinite) *graph classes*.

Graph Classes

- A class $\mathcal{C}$ has *bounded treewidth* if there exists a constant c such that for all $G \in \mathcal{C}$ holds $\text{tw}(G) \leq c$.

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- Generally, a class has bounded X if there is a constant c such that for all $G \in \mathcal{C}$ holds $X \leq c$.
- Assume we have a bounded treewidth class $\mathcal{C}$. On this class, Coucelle's theorem solves MSO_1 formulas in time $f(|\varphi|, \text{tw}(G))n \leq f(|\varphi|, c)n = f'(|\varphi|)n$.

Restating Courcelle

Courcelle's Theorem

Let $\mathcal{C}$ be a graph class with bounded treewidth. There exsists a function f (depending on $\mathcal{C}$!) such that for every MSO$_1$ sentence φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n$.

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If a fixed problem is expressible by some formula φ, then $f(|\varphi|) = O(1)$.

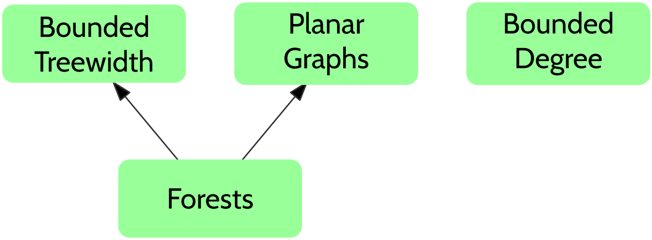
Courcelle's Theorem (most succinct formulation)

On graph classes with bounded treewidth, one can decide MSO_1-expressible problems in linear time.

Inclusion Diagrams

Each box represents a property of graph classes.

What do the arrows mean?



Sparsity

- What do these graphs have in common?
  - Graphs with treewidth w have at most wn edges.
  - Planar graphs have at most $3n$ edges.
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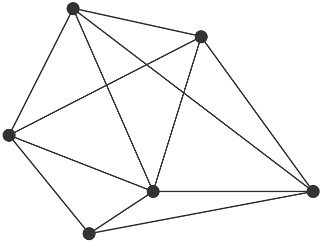
- What do these graphs have in common?
  - Graphs with treewidth w have at most wn edges.
  - Planar graphs have at most $3n$ edges.
  - Graphs with constant degree have $O(n)$ edges.
- Problems seem to be easier if the graphs are *sparse*!
- What does it really mean to be sparse?

Sparsity

- Every graph is "sparse" if you subdivide the edges.



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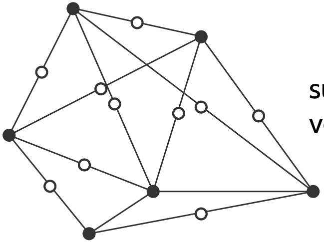

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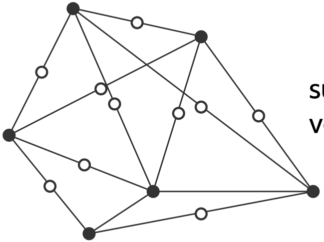

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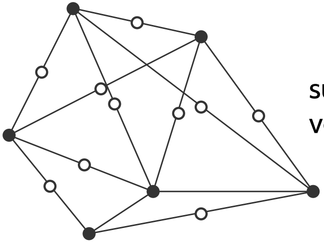

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  - No: Bounded expansion and nowhere dense graph classes
- We say “No” because it has nicer algorithmic theory.

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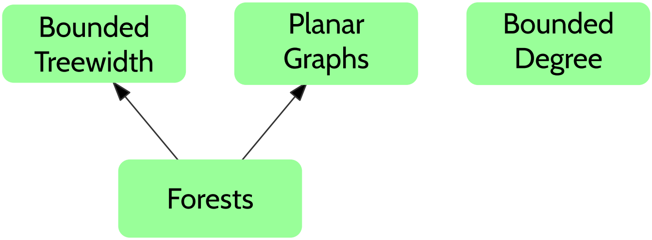

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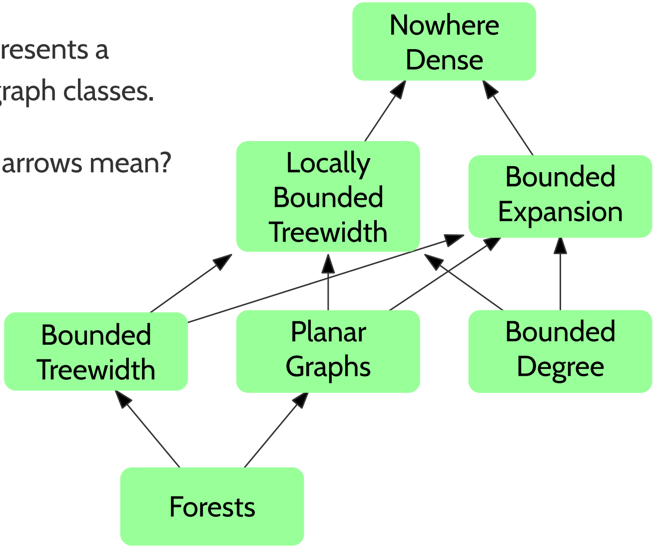

Many Sparse Graph Classes

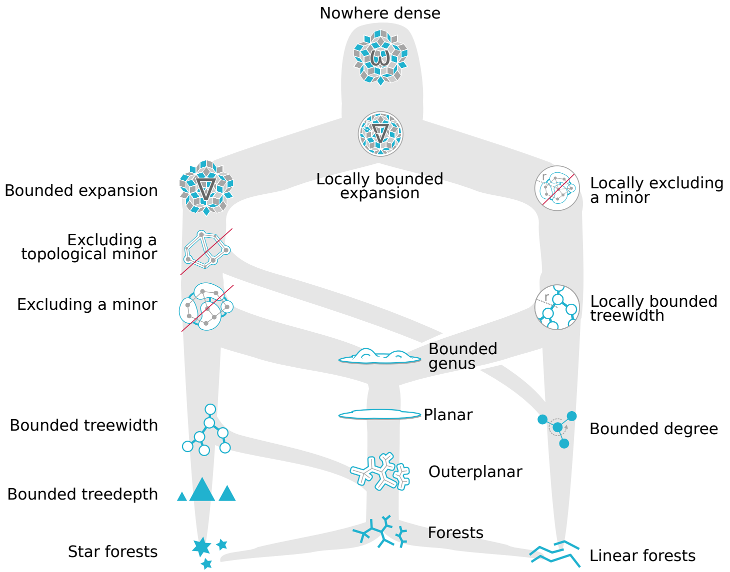

Figure by Felix Reidl

First-Order Logic

For sparse graphs, MSO_1 is too powerful. For example Independent Set, Coloring, Dominating Set are NP-complete on planar graphs or bounded degree graph classes. However, *first-order logic* fits just right.

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Main Result (roughly)

Let $\mathcal{C}$ be a sparse graph class. For an FO formula φ and graph $G \in \mathcal{C}$ one can decide whether $G \models \varphi$ in time $f(|\varphi|)n$ for some function f.

First-Order Logic

For a given signature τ, first-order logic has ...

- element-variables $(x, y, z, \dots)$
- the equality relation $=$ as well as the relations from τ.
- quantifiers $\exists$ and $\forall$, as well as operators $\wedge$, $\vee$ and $\neg$

We mostly work on colored undirected graphs with $\tau = \{\sim, c_1, c_2, \dots\}$. Here, we call the logic FO.

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- Boolean query $\Leftrightarrow$ Decide whether $G \models \varphi$.
- There exist extensions of first-order logic simulating SQL's COUNT operator.

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Input: Graph G and first-order sentence φ

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First-Order Query Counting

Input: Graph G and first-order formula $\varphi(x_1, \ldots, x_k)$

Question: Count number of tuples $v_1, \ldots, v_k$ with $G \models \varphi(v_1, \ldots, v_k)$.

Theorem (Vardi 1982)

The model-checking problem is PSPACE-complete.

Complexity

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FO model-checking on planar graphs in NP-complete.

Proof: Reduction from Independent Set.

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However, usually database queries are very small compared to the size of the database. Parameterize by $|\varphi|$.

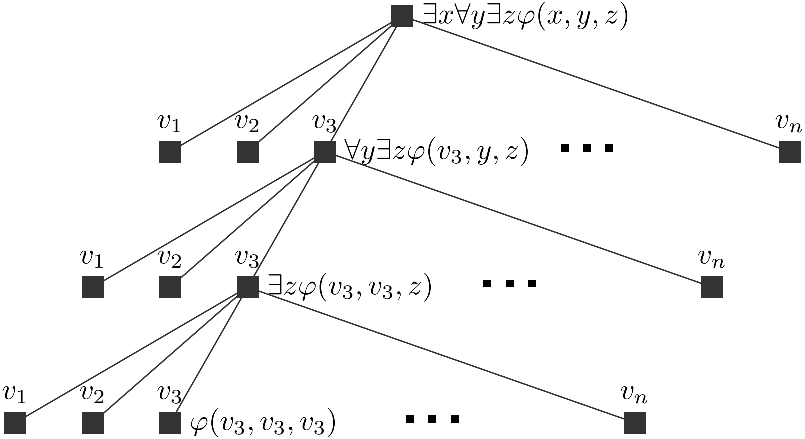
Parameterized Complexity (Upper Bound)

Theorem

One can decide whether $G \models \varphi$ in time $O(|G|^{|\varphi|})$.

Evaluation Trees

Proof: We can assume φ to be in prenex normal form. Construct an evaluation tree of size $O(|G|^{|\varphi|})$.



Parameterized Complexity (Lower Bound)

Conjecture (based on SETH)

It is believed one cannot decide whether $G \models \varphi$ in time $O(|G|^{q-1-\varepsilon})$ for any $\varepsilon > 0$ where q is the number of quantifiers of φ.

The previous algorithm is probably more or less optimal.

A faster model-checking algorithm would lead to a faster algorithm for many other problems.

On certain graph classes, we can do much better though.

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Certain properties cannot be decided in first-order logic. For example, there is no first-order formula φ such that $G \models \varphi$ iff G is connected. How do we prove that?

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- The *quantifier-rank* of a formula is the maximum number of nested quantifiers.
- We write $G \equiv_q H$ if for all first-order sentences φ of quantifier-rank $\leq q$ holds $G \models \varphi \iff H \models \varphi$.

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- Show $G \equiv_q H$ using *Ehrenfeucht–Fraïssé games*.

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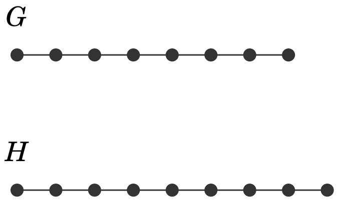

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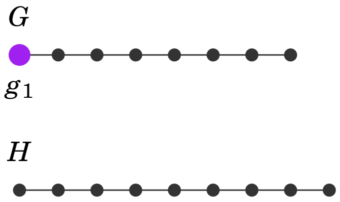

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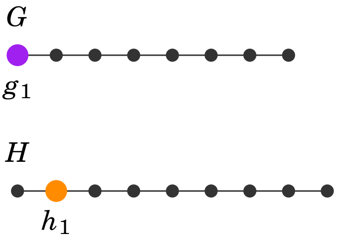

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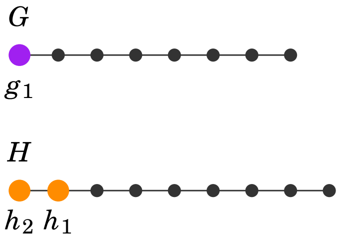

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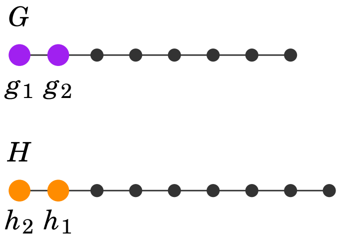

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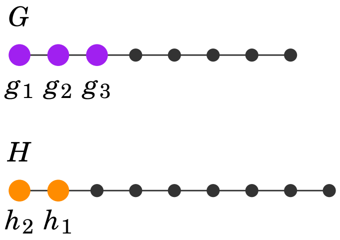

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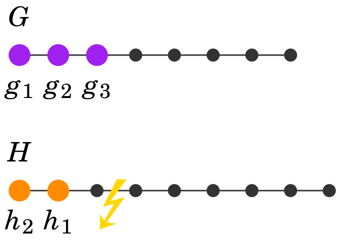

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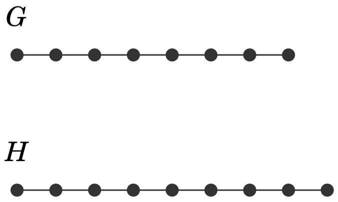

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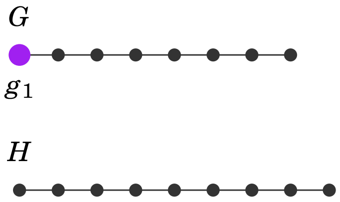

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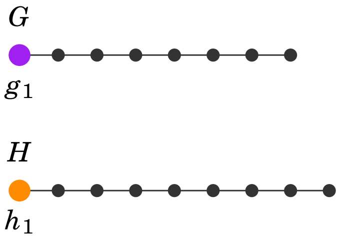

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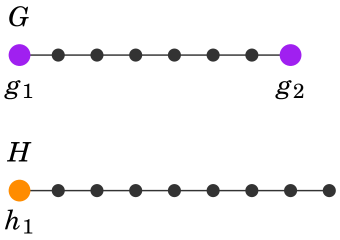

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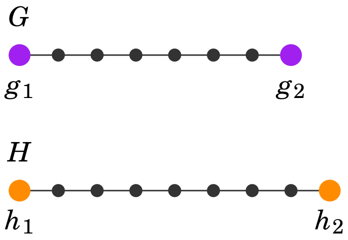

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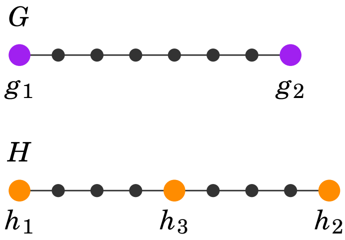

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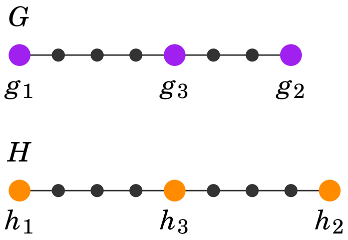

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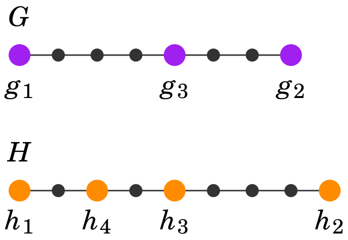

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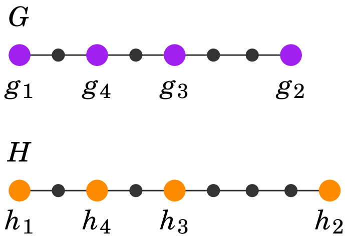

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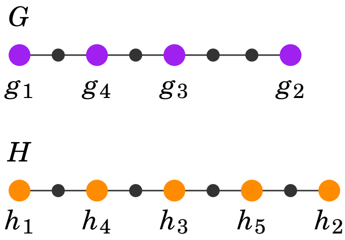

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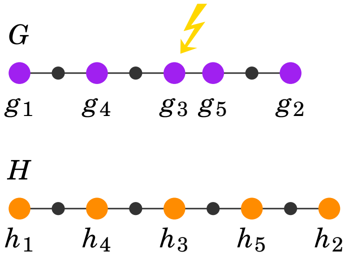

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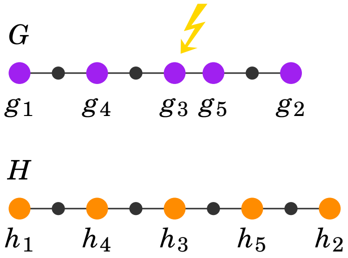

Theorem

$G \equiv_q H$ iff the Duplicator wins the q-round Ehrenfeucht–Fraïssé game.