

Respecting priorities versus respecting preferences in school choice: When is there a trade-off? *

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Abstract

A classic trade-off that school districts face when deciding which algorithm to use to match students to schools is that it is not possible to always respect both priorities and preferences. The student-proposing Deferred Acceptance algorithm respects priorities (fairness) but can lead to inefficient allocations. The Top Trading Cycle algorithm respects preferences (efficiency) but may violate priorities. We identify a new condition on school choice markets under which both algorithms yield the same allocation. Our condition differs from earlier conditions in the literature in that it places restrictions on how preferences and priorities relate to one another, rather than placing restrictions on either side separately. Such condition is natural as priorities are often the reflection of what school districts view as legitimate preferences. We quantify through simulations the extent to which our condition extends the known range of environments where fairness and efficiency do not conflict.

Keywords: Matching, stability, efficiency, uniqueness, priorities, preference alignment

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1 Introduction

Many countries promote parental choice for schools. How choice is actually implemented greatly varies across countries and school districts and is often hotly debated by stakeholders. A cornerstone of this debate is the choice of the algorithm to match students to schools, when capacity is limited. Indeed, different algorithms will typically result in different final allocations, even when submitted preferences and priorities are the same.

In their classic paper, [Abdulkadiroğlu and Sönmez \(2003\)](#) identify desirable properties of matching algorithms in the school choice context. These include efficiency, which can be interpreted as a measure of the extent to which the algorithm respects students’ preferences, and envyfreeness, a measure of the extent to which priorities are respected. Unfortunately, no algorithm exists that always produces an envyfree and efficient final allocation ([Balinski and Sönmez, 1999](#)): there is a trade-off between respecting students’ preferences and respecting school priorities.

How big the trade-off is in practice is an empirical question and existing evidence is mixed. For example, [Abdulkadiroğlu et al. \(2009\)](#) finds that the trade-off is significant in the NYC High School market but minor in Boston public schools (see also [Pathak \(2017\)](#)). [Calsamiglia et al. \(2020\)](#) finds significant differences across algorithms. These examples raise the question of when the choice of algorithms actually matters: when should a school district spend time weighting the choice among algorithms? When is the choice of algorithm a second order issue?

We identify a new condition on preferences and priorities under which there is no trade-off between the two goals: there is a unique envyfree allocation of students to schools and it is efficient. Our condition, Hierarchical Top Top (HTT), generalizes existing conditions identified in the literature. Like some of the existing conditions, it seeks to capture the degree to which priorities and preferences are “aligned” but it restricts attention to the part of the preferences and priorities that actually matters for the allocation (hence the qualifier “hierarchical”). Some schools that students have listed in their preferences may not be attainable anyways and likewise some students who have priority in a school may never need to consider it because they can get access to a school they prefer.

We show through simulations that our condition significantly expands the set of known environments for which there is no trade-off between efficiency and envyfreeness. We also discuss how this condition naturally arises in existing school choice environments. An expanding empirical literature has shown that parents typically value school quality and proximity, albeit potentially differentially according to their socio-economic status or academic inclination.

Positioning within the literature. A number of papers have explored the trade-off between envyfreeness and efficiency in school choice markets, starting from the seminal

papers by [Balinski and Sönmez \(1999\)](#) and [Abdulkadiroğlu and Sönmez \(2003\)](#). The student-proposing deferred acceptance (DA) algorithm and the top-trading cycle (TTC) algorithms are natural starting points here: the student-proposing deferred acceptance algorithm maximizes efficiency among algorithms that produce stable outcomes ([Balinski and Sönmez, 1999](#)), and TTC performs well (and under some circumstances best) on envyn-freenes, within the class of efficient and strategyproof mechanisms ([Abdulkadiroğlu et al., 2020](#)). When TTC does not involve any violation of priorities, it yields the same outcome as DA.

One approach has been to identify domain restrictions on the set of priorities such that DA is efficient ([Ergin, 2002](#); [Ehlers and Erdil, 2010](#); [Erdil and Kumano, 2019](#); [Ishida, 2019](#)) or TTC yields the same outcome as DA ([Kesten, 2006](#); [Ishida, 2019](#)), for any preference profile. [Heo \(2019\)](#) explores the flip-side question of domain restrictions on preferences such that DA is efficient or TTC yields the same outcome as DA, for any priority profiles.

Of course, whether efficiency and envyn-freeness conflict depends on the combination of preferences and priorities. Moreover, in practice, priorities are not entirely independent of preferences and instead a partial reflection of what school districts view as legitimate preferences. For example, many school districts give priorities to siblings or to students living close to schools and parents often prefer to send their children to nearby schools or to the school where a sibling is already educated, everything else equal.

Our Hierarchy Top Top condition captures this insight and generalizes existing conditions placed on how preferences and priorities relate and known to guarantee that DA is efficient, such as [Salonen and Salonen \(2018\)](#)’s single peakedness, [Clark \(2006\)](#)’s no crossing condition or [Reny \(2021\)](#)’s student-oriented preferences.

The trade-off between efficiency and envyn-freeness is conceptually connected to the issue of uniqueness of stable matchings which has been explored in two-sided markets ([Alcalde, 1994](#); [Eeckhout, 2000](#); [Clark, 2006](#); [Niederle and Yariv, 2009](#); [Legros and Newman, 2010](#); [Romero-Medina and Triossi, 2013](#); [Lee and Yariv, 2014](#); [Reny, 2021](#); [Gutin et al., 2021](#)). Intuitively, given the lattice structure of stable matchings, uniqueness of stable matchings suggests that preferences on both sides (priorities and preferences in the school choice context) are “sufficiently compatible” that starting from one side or the other when using deferred acceptance does not impact the final outcome.

Though uniqueness of stable matchings does not in itself guarantee efficiency (in a school choice context) nor the equivalence between DA and TTC, we show in Section 4 that most existing conditions that have been identified to imply uniqueness, also happen to imply that DA is efficient and yields the same outcome as TTC, thereby connecting these two strands of the literature. Our Hierarchy Top Top condition extends [Eeckhout \(2000\)](#)’s uniqueness condition to a setting where one side (schools) can be matched to several agents on the other side (students) and generalizes all other uniqueness conditions that lead to a student-efficient

outcome.

2 Model

There is a wide variety of algorithm and procedures used in practice to match students to schools. We focus on *direct* and *priority-based* mechanisms which elicit preference rankings from students (this is the “direct” part) and use priorities to assign students to schools when demand exceeds capacity (the “priority” part).

Let I denote the set of students ($|I| = n$), and S the set of schools ($|S| = m$). Let school s_{m+1} represent the outside option for the students. A market is defined by students’ preferences over schools, schools’ priorities over students and schools’ capacities, and is denoted by $\mathcal{E} = (\succ, P, q)$, where $\succ$ are students’ preferences over $S \cup \{s_{m+1}\}$, P are schools’ priorities, and $q = (q_1, \dots, q_m) \in \mathbb{N}^m$ are school capacities (without loss of generality, we let the capacity of s_{m+1} to be equal to n). We assume that preferences and priorities are strict linear orders (no indifference). Moreover, we assume that the priority order of each school only contains those students that consider the school as acceptable.¹

An allocation is a mapping $\mu : I \rightarrow S \cup \{s_{m+1}\}$ that describes to which schools students are assigned, with the understanding that $\mu(i) = s_{m+1}$ means that student i is unassigned. An allocation is feasible if it does not allocate more students to a school than its capacity, $|\mu^{-1}(s)| \leq q_s$, for every s . A feasible allocation μ is *Pareto efficient* if it is not Pareto dominated by any other feasible allocation μ' , such that $\mu'(i) \succeq \mu(i)$ for all $i \in I$ and $\mu'(i) \succ \mu(i)$ for some $i \in I$. A feasible allocation μ is blocked by a student i if it is not individually rational, that is, there is $i \in I$ who prefers remaining unassigned than to $\mu(i)$. A feasible allocation is blocked by a pair (i, s) where i prefers s to her assignment $\mu(i)$, and either s has some empty seats under μ , or there is a lower priority student j who is assigned under μ , that is, $s \succ_i \mu(i)$, and $|\mu^{-1}(s)| < q_s$ or $i P_s j$ for some $j \in \mu^{-1}(s)$. An allocation is said to be *envyfree* or *stable* if it is not blocked by any student or by any student school pair. A direct priority-based mechanism is a function that maps student preferences $\succ$ and school priorities P into a feasible allocation.

We consider two mechanisms that are in the center of the debate between stability and efficiency. The first one is the student-proposing DA algorithm, which was first proposed by [Gale and Shapley \(1962\)](#). The mechanism always produces an envyfree allocation based on the submitted preferences and priorities. Moreover, its outcome Pareto dominates any other envyfree allocation. Finally, the mechanism is strategyproof: it is a weakly dominant strategy for students to submit their true preferences. The second algorithm, Top Trading

¹This is without loss of generality, and it is useful in order to define the conditions over the “relevant” part of the priorities.

Cycle (TTC), was first described in [Shapley and Scarf \(1974\)](#) and adapted to the school choice problem by [Abdulkadiroğlu and Sönmez \(2003\)](#). The mechanism always produces an efficient allocation based on submitted preferences but may violate priorities. In what follows we will assume that students play the weakly dominant strategy of truth telling both under DA and TTC. Given the properties of the two mechanisms, the student-proposing DA and TTC serve as natural benchmarks to measure the tension between envyfreeness and efficiency. There is no tension between the two when DA is efficient, or when TTC does not involve any violation of priorities.

Additionally, a common mechanism used in practice is the Immediate Acceptance (IA) mechanism, also known as the Boston mechanism. This mechanism is efficient based on submitted preferences. Importantly, and in contrast to the two previous mechanisms, the IA mechanism is not strategyproof. Some students may have incentives to submit preferences different from their true preferences in order to get a preferred assignment. However, the set of Nash equilibrium outcomes is equal to the set of feasible envyfree allocation ([Ergin and Sönmez, 2006](#)). In the rest of the paper, we assume that students play Nash equilibrium strategies under the IA mechanism. All mechanisms are described in the Appendix.

3 Hierarchical Top Top condition

We are interested in analyzing the set of markets – conditions on priorities and preferences – for which there is no conflict between envyfreeness and efficiency. Since the DA allocation dominates all other envyfree allocations, the most direct formal translation of this question is to ask when DA is efficient. A slightly more demanding requirement is to ask when DA yields the same allocation as TTC. This requirement is more demanding because DA can be efficient, and yet yield a different allocation than TTC. In that case, TTC is efficient (as always) but not envyfree.

As we will see, the condition we identify implies both that DA is efficient and that DA yields the same allocation as TTC, but additionally that the set of envyfree allocations is a singleton. This implies that all major algorithms (the student-proposing DA, the school-proposing DA, the Immediate Acceptance algorithm, and TTC) yield the same allocation in equilibrium and are therefore of second-order importance for discussions about how to match students to schools.

Our condition extends [Eeckhout \(2000\)](#)'s sufficient condition for uniqueness in one-to-one matching environments to environments where one side (schools) can be matched with several agents of the other side (students). To do so it is convenient to convert schools into mini-schools of unit capacity, and complete preferences as follows: Consider a school $s \in S$ with capacity q . School s is replaced by q schools of unit capacity, $s_1, \dots, s_q$, such

that: 1) whenever s appears in student i preferences, s is replaced by $s_1, \dots, s_q$, and 2) each mini-school inherits the priority list of school s .²

Definition 1 (Hierarchical Top Top (HTT)). A market $\mathcal{E} = (\succ, P, q)$ satisfies the Hierarchical Top Top property if, after transforming all schools into mini-schools of unit capacity, there exists a reordering of these mini-schools ($j = 1, \dots, n + \sum q_s$), and a reordering of students ($i = 1, \dots, n$), such that, for all i , and for all pairs of minischools s_i and s_k :

$$s_i \succ_i s_k \text{ for all } k > i \text{ and } i P_{s_i} k \text{ for all } k > i.$$

In words, in a market with the HTT property, there is always a student i and a school s , that the top choice of one another in the reduced market that results from sequentially removing pairs of mini-school and student that are the preferred choice of another. The

HTT condition is easily checked in any market:

Step 1: Find a pair of student-school (i_1, s_1) such that school s_1 is the first choice of student i_1 and student i_1 has the highest priority at school s_1 . Remove the school and the student from the market. We refer to the market after removals as the reduced market.

Step k: Consider the reduced market defined in the previous step. Find a new pair of student-school (i_k, s_k) , such that school s_k is the top choice of student i_k , and student i_k has the highest priority at school s_k . Remove the school and the student from the market.

The procedure terminates when all students are assigned or when there is no more pair of student and school (including school $s_m + 1$) such that they are each other's top choice. In the former case, the market satisfies HTT and the order in which pairs are removed produces the order in Definition 1, whereas in the latter case not.

Many school choice environments naturally satisfy the HTT condition. This is the case when priorities are common across schools e.g. because they are based students' academic performance (reorder students according to their grades and relabel school 1, student 1's preferred school, and so on, up to capacity). Alternatively, the HTT condition is also satisfied when priorities are based on distances and students value distance, schools' quality or a combination of both. For example, assume that student i 's preference over schools can be represented by the following utility function $u_{is} = \alpha_i q_s - d_{is}$ when q_s measures school s 's quality, d_{is} measures the distance between school s and student i 's home, and α_i measures the extent to which student i values quality. Such preferences naturally arise in school choice contexts, see e.g. [Abdulkadiroğlu et al. \(2017\)](#) or [Fack et al. \(2019\)](#). Combined with distance-based priorities ($\Phi_{is} = -d_{is}$), such school choice markets satisfies HTT. To see this, consider

²By convention, the outside option s_{m+1} is split into n unit capacity outside options, each of which being exclusively associated with a different student and that student benefiting from the highest level of priority in that school.

the school with the highest quality and the student who lives closest to that school. This school and student are the top choices of one another. Removing this student and one seat in that school, we can again look at the student living closest to the school of highest quality among those with remaining capacity. These student and school will again be the top choices of one another.

Proposition 1. *Suppose market $\mathcal{E} = (\succ, P, q)$ satisfies the Hierarchical Top Top property. Then there is no trade-off between envyfreeness and efficiency. The student-proposing DA, the school-proposing DA, TTC and IA yield the same allocation and this allocation is both efficient and envyfree.*

Proof. We first show that TTC, the student-proposing DA and the school-proposing DA produce the same allocation. Divide all schools into mini-schools of unit capacity and reorder students and priorities such that for all i , $s_i \succ_i s_k$ for all $k > i$ and $i P_{s_i} k$ for all $k > i$. The proof proceeds by iteration. Consider student 1 and mini-school 1. In TTC, mini-school 1 points to student 1 which points back to him. This is thus a cycle and student 1 is allocated a seat in mini-school 1. In the student-proposing DA, student 1 applies to mini-school 1 in which he benefits from the highest priority. He will therefore be admitted to mini-school 1 whatever happens in later rounds. Finally, in the school-proposing DA, school 1 offers a seat to student 1 and since school 1 is student i 's most preferred choice, he will decline any other offer afterwards. Remove student 1 and mini-school 1 from the market. By HTT, student 2 points to mini-school 2, which points back to him. This leads to a cycle in TTC which means that student 2 is assigned to mini-school 2. Likewise, student 2 (who may prefer mini-school 1 but would have been rejected in round 1 from mini-school 1) applies to mini-school 2 where he benefits from the highest priority (except possibly for student 1 but student 1 will never apply to mini-school 2). He is thus sure to get it, whatever happens in later rounds. Finally, in the school-proposing DA, school 2 offers a seat to student 2 who prefers school 2 among any alternative with remaining seat and will therefore hang on to that offer, whatever happens afterwards. This argument can be repeated for all i , establishing that all three algorithms yield the same allocation. This allocation is necessarily envyfree since it is produced by DA. It is also necessarily efficient since it is produced by TTC.

We only need to argue that IA also produce the same outcome. This follows directly from [Ergin and Sönmez \(2006\)](#)'s characterisation of the set of Nash equilibrium outcomes of IA and the lattice structure of the set of envyfree allocation. $\square$

HTT guarantees that a school market has an efficient envyfree allocation and that it can be equally produced by the student-proposing, school-proposing, TTC and IA. In order words, the choice of the algorithm is second order. A natural question is to what extent HTT

is also necessary. The answer is negative. There are school choice markets where efficiency and envyfreeness are compatible, yet do not satisfy HTT as example 2 illustrates.

Example 1. Consider the following market with preferences and priorities as follows:

$$\begin{array}{ll}
 i_1 : & s_1 \quad s_3 \quad s_2 \\
 i_2 : & s_2 \quad s_1 \\
 i_3 : & s_3 \\
 s_1 : & i_2 \quad i_1 \\
 s_2 : & i_1 \quad i_2 \\
 s_3 : & i_1 \quad i_3
 \end{array}$$

In this market, the matching found by student-proposing DA, $((i_1, s_1), (i_2, s_2), (i_3, s_3))$, is envyfree and efficient, and it coincides with the outcome found by TTC. In addition, it is the unique stable matching (envyfree allocation). However, this market does not have the HTT property. None of the students and the schools are the top choices of each other in the original market. $\square$

4 Relationship with Existing Conditions

As noted earlier, [Eeckhout \(2000\)](#) first identified the equivalent of HTT for one-to-one matching markets as a sufficient condition for the existence of a unique stable matching in two-sided matching markets. Our proposition 1 shows that, when extended to one-to-many one-sided matching markets, HTT also implies that DA is efficient and that DA and TTC yield the same allocation, neither of which are implied by the unicity of the stable matching. Because HTT also implies other uniqueness conditions, including [Clark \(2006\)](#)'s no-crossing condition, [Niederle and Yariv \(2009\)](#)'s aligned preferences, [Legros and Newman \(2010\)](#)'s co-ranking condition, [Salonen and Salonen \(2018\)](#)'s single-peakedness condition and [Reny \(2021\)](#)'s student-oriented preferences, these conditions also imply that DA is efficient and yield the same outcome as TTC, IA and the school-proposing DA. this is not entirely surprising since the two questions are closely related: there is a unique stable matching when preferences on both sides are sufficiently compatible, and likewise DA is efficient when priorities and preferences do not conflict.

Recently, [Gutin et al. \(2021\)](#) have proposed a sufficient and necessary condition for uniqueness. This condition, which is more general than all the other uniqueness conditions, is no longer sufficient for DA to be efficient or DA to yield the same outcome as TTC, thereby showing the limits of the connection between the two research questions.

5 Quantification and Empirical Relevance

We now quantify the extent to which our condition extends the known range of environments where envyfreeness and efficiency do not conflict. Schools priorities, denoted by Φ_{is} , are drawn independently and identically on $[0, 1]$ according to the uniform distribution. Φ_{is} can be interpreted as distance between a school and a student. Student preferences over schools contain two parts. The first part is based on priorities, Φ_{is} , and the second part includes a common utility component denoted by v_s , an idiosyncratic component denoted by ε_{is} . The preferences can be then represented by the following utility function:

$$(1 - \lambda)\Phi_{is} + \lambda(\delta\varepsilon_{is} + (1 - \delta)v_s),$$

where $\delta, \lambda \in [0, 1]$, and ε_{is}, v_s are both drawn independently and identically on $[0, 1]$ according to the uniform distribution. The parameter λ captures the sensitivity to priorities. When $\lambda = 0$, the function corresponds to the aligned preferences case, and when $\lambda = 1$, student preferences consist of a common utility part and an idiosyncratic utility part. For a given value of λ , δ measures the correlation between the common utility part and the idiosyncratic part. When $\delta = 0$, the second term of the utility function depends only on the common utility part, and when $\delta = 1$, it depends only on the idiosyncratic utility part.

We conduct simulation experiments with 20 schools whose seats ranging from 10 to 100, and the number of students ranging from 200 to 2000. We present here the results for 2000 school seats and 2000 students, with 1000 repetitions. Figure 1 shows the proportion of markets where we know DA yields the same outcome as TTC versus the proportion of markets satisfying HTT. When λ is small, preferences depend more on priorities, therefore it is more likely to have markets where DA yields the same outcome as TTC and satisfies HTT. When λ increases, the dependence on priorities is less, and it is less likely to have markets where DA yields the same outcome as TTC and satisfies HTT. Even-though HTT is only a sufficient condition, it captures reasonably well the domains where DA yields the same outcome as TTC. Figure 2 shows that HTT expands the domains that are captured by aligned preferences. It expands the domain the most when we have a smaller δ , that is when preferences are more correlated. This approach assumes rather simple structure on priorities, vectors of priorities are equally likely whereas priorities are in practice shaped by the physical and social environment of the school district. To strengthen our results, we are conducting additional simulations to capture richer environments.

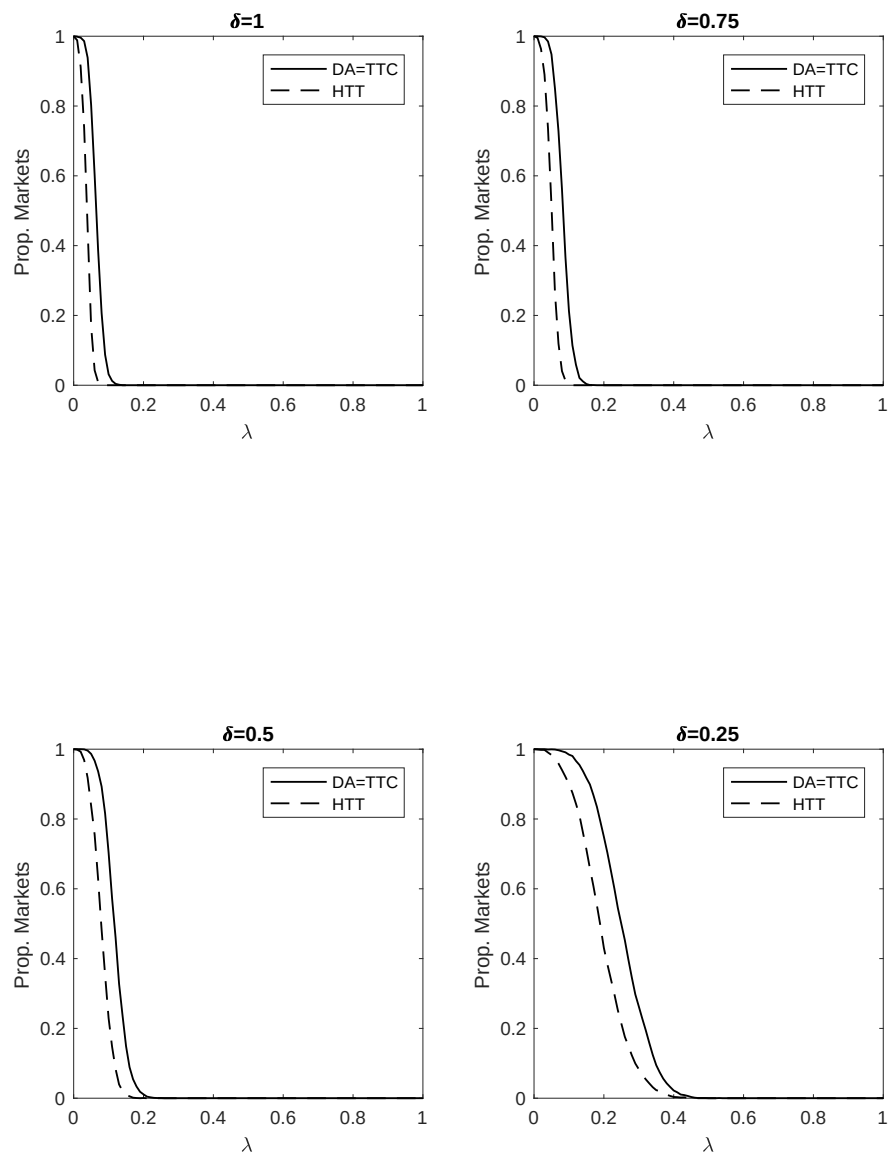


Figure 1: Proportion of markets where DA=TTC and proportion of markets with HTT

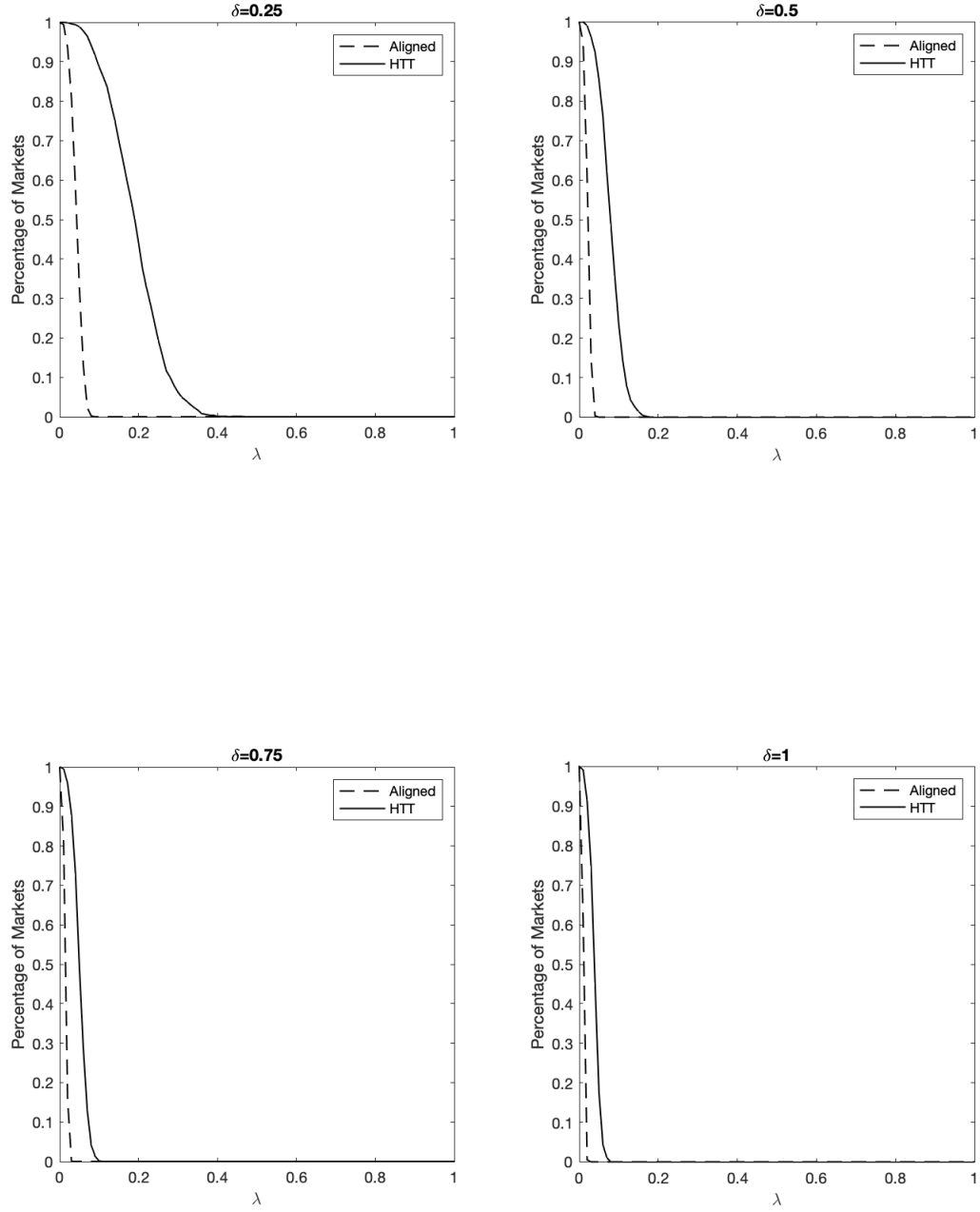


Figure 2: Proportion of markets with HTT and proportion of markets with aligned preferences

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A Appendix

A.1 Description of the mechanisms

STUDENT-PROPOSING DEFERRED ACCEPTANCE (DA):

Step 1: Each student i proposes to the best school according to $\succ_i$. Each school s provisionally accepts the q_s -highest ranked students, according to P_s , among those students that have proposed to s , and rejects the others.

Step k : Each student i , who has not been previously accepted, proposes to the best school according to $\succ_i$, among those schools that have not yet rejected i . Each school s provisionally accepts the q_s -highest ranked students, according to P_s , among those students that have proposed to s along steps 1 to $k + 1$, and rejects the others.

The algorithm terminates at the step where no rejections are made and provisional acceptances become definitive by matching each school s to the set of students provisionally accepted at this step.

TOP TRADING CYCLES (TTC).

Step 1: Each student i points to the best school according to $\succ_i$. If no school is acceptable, i points to s_{m+1} and is removed from the problem. Each school s points to the best student according to P_s , and s_{m+1} points to all students.

Since the sets of students and schools are finite, there exists at least one cycle which is of the form $i_1 \rightarrow s_1 \rightarrow \dots \rightarrow i_K \rightarrow s_K \rightarrow i_1$ or $i \longleftrightarrow s_{m+1}$, where $x \rightarrow y$ means “ x points to y ”.³ Each student i in a cycle is matched to the school s that she points to (if $i \rightarrow s$), in which case i and a seat in s are removed from the problem. If i points to s_{m+1} , she remains unmatched and i is removed from the problem.

Step k . Each remaining student i points to the best school according to $\succ_s$, among the schools that still have empty seats. Each school s with an empty seat, points to the best student, according to P_s , among the remaining students, and s_{m+1} points to all of these students. There is at least one cycle. Each student i in a cycle of the form $i_1 \rightarrow s_1 \rightarrow \dots \rightarrow i_K \rightarrow s_K \rightarrow i_1$ is matched to the school s that she points to, and i and a seat in s are removed from the problem. If i points to s_{m+1} , she remains unmatched and i is removed from the problem.

The algorithm terminates when each student i is either matched to a school or to s_{m+1} . TTC is strategy-proof and Pareto efficient, and it admits reasonably small justified envy in many to one matching, and minimizes justified envy in one to one matching ([Abdulkadiroğlu et al., 2020](#)).

³There may be many cycles, although each student and school $s \neq s_{m+1}$ can be part of at most one cycle.

IMMEDIATE ACCEPTANCE (IA):

Step 1: Each student i proposes to the best school according to $\succ_i$. Each school s accepts the q_s -highest ranked students, according to P_s , among those students that have proposed to s , and rejects the others. Accepted students are definitive matched to the school.

Step k : Each student i , who has not been previously accepted, proposes to the best school according to $\succ_i$, among those schools that have not yet rejected i . Each school s accepts the q_s -highest ranked students, according to P_s , among those students that have proposed to s , and rejects the others. Accepted students are definitive assigned to the school.

The algorithm terminates at the step where no rejections are made.