

Efficient matching under general constraints*

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Abstract

We study indivisible goods allocation problems under constraints and provide algorithms to check whether a given matching is Pareto efficient. We first show that the serial dictatorship algorithm can be used to check Pareto efficiency by assuming a constraint to be a matroid. To prove this, we develop a generalized top trading cycles algorithm. Moreover, we show that the matroid structure is necessary for obtaining all Pareto efficient matchings by the serial dictatorship algorithm. Second, we provide an extension of the serial dictatorship algorithm to check Pareto efficiency under general constraints. Subsequently, we propose an efficiency-adjusted deferred acceptance algorithm for path-independent choice functions.

1 Introduction

We study indivisible goods allocation problems, including real-life applications such as student placement in public schools and refugee resettlement. These applications are often subject to constraints. A school district requires specific diversity of the student body at each school (*type-specific quotas* and *proportional constraints*). A school needs at least a certain number of students to operate (*minimal quotas*). In refugee resettlement, the central authority needs to consider heterogeneous family sizes and numerous additional requirements such as job training, language class, etc. (*multidimensional constraints*). In a school district, multiple school programs often share one building. There is a limit on the total number of students in these programs in addition to each program's capacity because the building also has a physical capacity (*regional quotas*). Framing the allocation problem as a matching between students and schools, we examine the Pareto efficiency (for students) in a general model of matching with constraints, including all these constraints. Pareto efficiency is desirable in an allocation problem and has received a substantial attention from both theoretical and practical perspectives (see Section 1.3 for details).

In this paper, we study how to check whether a given matching is Pareto efficient or not. Checking Pareto efficiency is one of the fundamental steps in developing a mechanism that satisfies desirable properties. For example, consider the deferred acceptance (DA) mechanism, which may produce a Pareto inefficient matching for students. Thus, we would like to examine whether a current matching produced by the DA is Pareto efficient. Moreover, if the current matching is Pareto inefficient, the subsequent interest would be to Pareto improve the current matching. We provide a mechanism that finds such a Pareto improvement.

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Checking Pareto efficiency under constraints is non-trivial. In a standard model with capacity constraints, we can check the Pareto efficiency of a given matching by repeatedly applying the following procedure: find a student who is assigned to her best school among the available ones and fix the assignment of the student.¹ If the given matching is Pareto efficient, the process can continue till the end; if the given matching is Pareto inefficient, the process cannot continue because such a student will not exist in some steps. The reason why the procedure works stems from the property that any Pareto efficient matching can be produced by the serial dictatorship (SD) with some order of the students. Here, the order of the students corresponds to the one obtained in the above procedure. This property does not hold under a general constraint, that is, some Pareto efficient matchings cannot be produced by the SD. The following example illustrates this fact.

1.1 Motivating example

Example 1. Suppose there are three students i_1, i_2, i_3 and two schools s_1, s_2 . The student preferences (e.g., $\succ_i$ denotes the preference of student i) are as follows:

$$s_2 \succ_{i_1} s_1, \quad s_1 \succ_{i_2} s_2, \quad \text{and} \quad s_2 \succ_{i_3} s_1.$$

The school constraints (e.g., $\mathcal{F}_s$ denotes the constraint of school s) are as follows:

$$\mathcal{F}_{s_1} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_1, i_3\}\} \text{ and } \mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}\}.$$

While $\mathcal{F}_{s_2}$ is a capacity constraint, $\mathcal{F}_{s_1}$ is not: $\{i_1, i_3\} \in \mathcal{F}_{s_1}$, but $\{i_1, i_2\}, \{i_2, i_3\} \notin \mathcal{F}_{s_1}$. Constraints like $\mathcal{F}_{s_1}$ appear as a budget constraint (e.g., i_2 has a disability and incurs more cost) or an anti-bullying constraint (e.g., i_2 has bullied i_1 and i_3).

Consider the matching μ such that $\mu(i_1) = s_1$, $\mu(i_2) = s_2$, and $\mu(i_3) = s_1$. Note that μ is feasible and Pareto efficient (which can be verified by examining all possible matchings). However, the SD does not produce μ with any order since no student is assigned to her best school at μ . Thus, we cannot check the Pareto efficiency of μ by the SD.

1.2 Our contribution

This paper provides SD-type algorithms to check whether a given matching is Pareto efficient under given constraints. First, we identify which feature of $\mathcal{F}_{s_1}$ in Example 1 prevents the SD from finding the Pareto efficient matching μ : $\mathcal{F}_{s_1}$ is not a *matroid*. Specifically, we show that the SD can find all Pareto efficient matchings if the constraint is a matroid (even when it is distributional). To prove this, we develop a generalized top trading cycles (TTC) algorithm. Moreover, we prove the converse direction: the matroid structure is also a necessary condition to find all Pareto efficient matchings via the SD.

Our second result provides an algorithm to check Pareto efficiency under general constraints, including the constraint $\mathcal{F}_{s_1}$ in Example 1 and all constraints listed in the first paragraph of Introduction. We provide an algorithm based on the SD, which is called the constrained serial dictatorship (CSD). Unlike the SD, the CSD is defined on the set of feasible matchings and works under general constraints. In the CSD, a student is assigned to her best school to the extent that the remaining students can be feasibly assigned. The CSD can be used to check the Pareto efficiency of a given matching. To do this, we add an individual rational constraint to the set of feasible matchings in the CSD: every student is weakly better than a given matching.

¹This procedure can also be interpreted as the top trading cycle (TTC) mechanism, that is, a self-loop is selected in each iteration of the mechanism.

If a matching produced by the CSD is the same as a given matching, then it is Pareto efficient. We also discuss the computational aspects of CSD.

As an application of the CSD, we generalize the efficiency-adjusted deferred acceptance (EADA) mechanism. [Kesten \(2010\)](#) introduce the EADA in school choice with responsive choice functions. The EADA Pareto dominates the DA by removing *interrupting pairs* and applying the DA sequentially. We generalize the EADA to the general setting, such as college admissions with budget constraints. A key property of the CSD, along with individual rationality, is that it produces a matching that weakly Pareto dominates an initial matchings. Therefore, the CSD Pareto dominates the DA by setting an initial matching to the one produced by the DA. Then, by utilizing the order over the students obtained by the EADA, the CSD Pareto dominates the DA as the EADA does. In fact, these mechanisms are the same under responsive choice functions.

1.3 Related literature

This paper contributes to the literature of matching with constraints. There are two approaches to represent constraints. The first approach considers constraints imposed separately on individual schools. The second approach considers distributional constraints, which are imposed jointly on subsets of schools. This paper considers both those approaches.

The class of matroid constraints, which we first study, subsumes many constraints in the literature. A typical example of individual matroid constraints is the *type-specific quotas*, which is introduced for diversity concerns in school choice ([Abdulkadiroğlu and Sönmez, 2003](#); [Abdulkadiroğlu, 2005](#)). As a distributional matroid constraint, [Kamada and Kojima \(2015\)](#) study *regional quotas* and introduce a stability concept inspired by medical residency matching in Japan. [Kamada and Kojima \(2018\)](#) consider a more general matroidal case where regions are hierarchical. [Fleiner \(2001, 2003\)](#) shows some results including the existence of a stable matching under the matroid constraints.

General constraints that are not a matroid have also been studied in the literature. Recently, [Kamada and Kojima \(2019\)](#) introduce *general upper-bound*. The following constraints studied in recent works as real-life applications belong to general upper-bound: refugee resettlement ([Delacrétaz et al., 2016](#)), college admissions with budgets constraints ([Abizada, 2016](#)), school choice with bullying ([Kasuya, 2016](#)), and daycare allocation ([Okumura, 2019](#)). Lower bound constraints ([Ehlers et al., 2014](#)) and proportionality constraints ([Nguyen and Vohra, 2019](#)) are examples that do not belong to general-upper bound. A class of generalized-matroid (g-matroid) constraints has a good property as matroid constraints and includes upper and lower bound constraints ([Yokoi, 2017](#)).

Relative to the literature, we focus on Pareto efficiency, not stability (or fairness). While a few papers study Pareto efficient mechanisms under constraints, their motivations are different from ours: our paper studies how to check the Pareto efficiency of a matching. As noted, our algorithms would help in developing a mechanism that satisfies desirable properties; thus, these studies are complementary to each other.

This paper also relates to works that study indivisible goods allocation problems by SD-type algorithms. [Fragiadakis et al. \(2016\)](#) modify the SD to accommodate minimum quotas. Our model is more general than theirs, and their algorithm is special case of our CSD. However, [Fragiadakis et al. \(2016\)](#) also deal with strategic issues. We also provide a characterization of the set of Pareto efficient matchings under matroid constraints by the SD: a matching is Pareto efficient if and only if it is produced by the SD with some order. Thus, our result is a generalization of the corresponding characterization under capacity constraints. This characterization provides a clear understanding of the structure of the set of Pareto efficient matchings and the

foundation for several theoretical studies. The equivalence between the random SD and the core from random endowments serves as an example (Abdulkadiroğlu and Sönmez, 1998; Sönmez and Ünver, 2005; Pathak and Sethuraman, 2011; Carroll, 2014; Bade, 2020). The equivalence builds upon the characterization.² Another example is the study by Manea (2007). Motivated by the characterization, Manea (2007) investigates the relationship between Pareto efficiency and the SD in a general setting where each agent receives multiple goods.

Subsequently, we generalize two matching mechanisms. First, we generalize the TTC introduced by Shapley and Scarf (1974). One branch of the literature generalizes TTC (Abdulkadiroğlu and Sönmez, 1999, 2003; Suzuki et al., 2018). Suzuki et al. (2018) is closely related to our work. Suzuki et al. (2018) generalizes TTC by assuming the distributional constraints are represented as an M-convex set, which is a generalization of a matroid. There is a modeling difference between Suzuki et al. (2018) and ours. They restrict attention to constraints over the numbers of students in different schools, while we allow constraints to depend on the identity of the students. Note that although our model is more general than theirs, they also deal with strategic issues. Second, we generalize the EADA, which is introduced by Kesten (2010). Bando (2014) and Tang and Yu (2014) propose equivalent mechanisms to EADA in the standard model with responsive choice functions. Ehlers and Morrill (2020) generalize EADA to the settings where choice functions satisfy substitutability and the law of aggregate demand. Our work subsumes the settings of all these works, and therefore, our results include them as special cases.

2 Preliminaries

2.1 Model

A market is a tuple $(I, S, (\succ)_{i \in I}, \mathcal{F})$. I is a finite set of students, and S is a finite set of schools. Each student i has a strict preference $\succ_i$ over $S \cup \{\emptyset\}$, where $\emptyset$ means being unmatched (or an outside option). We write $s \succeq_i s'$ if either $s \succ_i s'$ or $s = s'$ holds. $\mathcal{F}$ is a family of subsets of student-school pairs $I \times S$ that reflects distributional constraints of school side.

Let $E = \{(i, s) \in I \times S : s \succ_i \emptyset\}$ be the set of acceptable student-school pairs. A *matching* μ is a subset of E such that each student appears at most one pair of μ ; that is, for any $(i, s), (i', s') \in \mu$, we have $s = s'$ if $i = i'$. We write $\mu(s) = \{i \in I : (i, s) \in \mu\}$ for each $s \in S$. Also, for each $i \in I$, we write $\mu(i)$ to denote the partner of i at μ , that is, $\mu(i) = s$ if $(i, s) \in \mu$ and $\mu(i) = \emptyset$ if $(i, s) \notin \mu$ for all $s \in S$. A matching is called *feasible* if $\mu \in \mathcal{F}$. For notational simplicity, we sometimes add an unmatched pair $(i, \emptyset)$ to a matching, but we ignore such a pair, e.g., for a matching μ , we treat $\mu \cup \{(i, \emptyset)\}$ as μ .

A matching μ *Pareto dominates* μ' if $\mu(i) \succeq_i \mu'(i)$ for all $i \in I$ and $\mu(i) \succ_i \mu'(i)$ for some $i \in I$. Let $\mathcal{M}$ be a set of matchings. A matching $\mu \in \mathcal{M}$ is called *Pareto efficient* with respect to $\mathcal{M}$ if there exists no feasible matching $\mu' \in \mathcal{M}$ that Pareto dominates μ .

For a student $i \in I$ and a subset of schools $S' \subseteq S$, we define $\arg \max_{\succ_i} S'$ to be the school or the outside option that i prefers the most among $S' \cup \{\emptyset\}$. Note that if S' is empty, we have $\arg \max_{\succ_i} S' = \emptyset$.

²We need to consider all Pareto efficient matchings to get an equivalent mechanism to the core from random endowment. The idea of the core from random endowment consists of two steps. First, it randomly chooses a feasible matching. Then, it improves this initial matching by the TTC. Since any feasible matching can be an initial matching, this mechanism assigns positive probabilities to all Pareto efficient matchings. This characterization guarantees that the SD can find all Pareto efficient matchings.

2.2 Constraints

We say that distributional constraints $\mathcal{F}$ is *individual* if there exists $\mathcal{F}_s \subseteq 2^I$ for $s \in S$ such that $\mathcal{F} = \{X \subseteq I \times S : X(s) \in \mathcal{F}_s \ (\forall s \in S)\}$, where $X(s) = \{i' : (i', s) \in X\}$. We sometimes write $(\mathcal{F}_s)_{s \in S}$ to represent such an individual $\mathcal{F}$. Also, we call $\mathcal{F}$ as the aggregated constraints of $(\mathcal{F}_s)_{s \in S}$.

An important class of constraints is the matroids. A family of subsets $\mathcal{F}$ is a *matroid* if it satisfies the following three properties: (I1) $\emptyset \in \mathcal{F}$, (I2) if $X \in \mathcal{F}$ and $X' \subseteq X$, then $X' \in \mathcal{F}$, and (I3) if $X, Y \in \mathcal{F}$ and $|X| < |Y|$, there is $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{F}$. We note that an aggregated constraint $\mathcal{F}$ is also a matroid if $\mathcal{F}_s$ is a matroid for every $s \in S$.

A special case of matroid constraints is a *capacity constraint*: a feasibility of a school $\mathcal{F}_s$ is a capacity constraint if there exists a positive integer q such that $\mathcal{F}_s = \{X \subseteq 2^I : |X| \leq q\}$. Matroid constraints include many other constraints besides a capacity constraint. Let us provide real-life examples of constraints. A school district requires specific diversity of the student body at each school. [Abdulkadiroğlu and Sönmez \(2003\)](#) formalize this requirement is to impose type-specific quotas for each school.³ Each $\mathcal{F}_s$ is a matroid and thus the aggregate constraints $\mathcal{F}$ is also a matroid. [Kamada and Kojima \(2015\)](#) study the regional maximum quotas in the context of medical residency matching in Japan. Under the regional maximum quotas, each school belongs to a region, and there is an upper bound on the number of students who can be matched in each region.

Of course, not all constraints of interest are matroids. [Kamada and Kojima \(2019\)](#) introduce *general upper-bound*. A constraint $\mathcal{F}$ belongs to general upper-bound if $X' \subseteq X \in \mathcal{F}$ implies $X' \in \mathcal{F}$ (i.e., property (I2)). Thus, a matroid belongs to general upper-bound. A class of constraints that belongs to general upper-bound but not to the matroid is the budget constraints.⁴ Another important class of constraints is the g-matroids (which is also called the *$M^\sharp$ -convex families* ([Murota, 2003](#))). A nonempty family of subsets $\mathcal{F}$ is a *g-matroid* if for any $X, Y \in \mathcal{F}$ and $e \in X \setminus Y$, we have (i) $X \setminus \{e\}, Y \cup \{e\} \in \mathcal{F}$ or (ii) there is $e' \in Y \setminus X$ such that $(X \setminus \{e\}) \cup \{e'\}$ and $(Y \cup \{e\}) \setminus \{e'\}$ are in $\mathcal{F}$. Note that a g-matroid $\mathcal{F}$ is a matroid if $\emptyset \in \mathcal{F}$.

Before completing this section, we present some properties related to matroids that will be used later. For a matroid $\mathcal{F} \subseteq 2^E$ and $X \subseteq E$, the *contraction* of $\mathcal{F}$ by X is defined as $\mathcal{F}/X := \{Y \subseteq E \setminus X : Y \cup X \in \mathcal{F}\}$. It is well known that the set family $\mathcal{F}/X$ is also a matroid.

Lemma 1 (see e.g. [Lee \(2004\)](#)). Let $(E, \mathcal{F})$ be a matroid with $X \in \mathcal{F}$. Suppose that $Y \subseteq E$, $|Y| = |X|$, and the bipartite graph $G = (X, E \setminus X; \{(x, y) \in X \times (E \setminus X) : (X \setminus \{x\}) \cup \{y\} \in \mathcal{F}\})$ contains a unique perfect matching between $X \setminus Y$ and $Y \setminus X$. Then, $Y \in \mathcal{F}$.

2.3 Serial dictatorship mechanism

Now, we introduce the *serial dictatorship* mechanism (SD) for our setting. The SD mechanism, which is formally described in Algorithm 1, considers the students in a certain order and assigns to each student her best school among available ones. Let Σ be the set of all permutations of the students. For $\sigma \in \Sigma$, we denote SD^σ to be the outcome matching of the SD mechanism.

Since the SD mechanism allocates students greedily, it does not always output a feasible matching in general.⁵ To overcome this issue, we define a modified version of the SD, which

³Formally, there exist a partition of students $(I_t)_{t \in T}$ with types T , i.e., $I = \bigcup_{t \in T} I_t$ and $I_t \cap I_{t'} = \emptyset$ for any distinct $t, t' \in T$. A feasible constraint $\mathcal{F}_s$ for a school s satisfies the following: there exist a capacity $q \in \mathbb{N}$ and a type- t quota $q_t \in \mathbb{N}$ for every $t \in T$ such that $\mathcal{F}_s = \{X \subseteq I : |X| \leq q \text{ and } |X \cap I_t| \leq q_t\}$.

⁴Formally, there exist weights $(a_i)_{i \in S}$ and a budget b , a feasible constraint $\mathcal{F}_s$ for a school s is represented as $\{X \subseteq I : \sum_{i \in X} a_i \leq b\}$.

⁵For example, let us consider a market $(I, S, (\succ)_{i \in I}, \mathcal{F})$ where $I = \{i_1, i_2\}$, $S = \{s\}$, s is acceptable for both students, and $\mathcal{F} = \{\{(i_1, s), (i_2, s)\}\}$. Then, we have $\text{SD}^\sigma = \emptyset$ for any $\sigma \in \Sigma$.

Algorithm 1: SD

input : matching market $(I, S, (\succ)_{i \in I}, \mathcal{F})$, $\sigma \in \Sigma$
output: a matching SD^σ
1 Let $\mu^{(0)} \leftarrow \emptyset$;
2 **for** $k \leftarrow 1, 2, \dots, |I|$ **do**
3 $\mu^{(k)} \leftarrow \mu^{(k-1)} \cup \{(\sigma(k), \arg \max_{\succ_{\sigma(k)}} \{s \in S : \{(\sigma(k), s)\} \in \mathcal{F} / \mu^{(k-1)}\})\}$;
4 **return** $\mu^{(n)}$;

we call *constrained serial dictatorship* (CSD). Different from the SD, the CSD allocates each student to her most preferred school under the constraint that a feasible matching exists. Fixing a set of feasible matching $\mathcal{M}$, the CSD mechanism is formally described in Algorithm 2. We will set $\mathcal{M}$ not only as the set of school feasible matchings, but also as the set of school feasible matchings that are also individually rational for a given matching.

Algorithm 2: constrained serial dictatorship (CSD)

input : $\sigma \in \Sigma$, preference profile $\succ$, feasible matchings $\mathcal{M}$
output: a matching $\text{CSD}^{\sigma, \mathcal{M}}$
1 Let $\mu^{(0)} \leftarrow \emptyset$;
2 **for** $k \leftarrow 1, 2, \dots, |I|$ **do**
3 Let $\mathcal{M}^{(k)} = \{\mu \cap (\{\sigma(1), \dots, \sigma(k)\} \times S) : \mu \in \mathcal{M}\}$;
4 $\mu^{(k)} \leftarrow \mu^{(k-1)} \cup \{(\sigma(k), \arg \max_{\succ_{\sigma(k)}} \{s \in S : \mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{M}^{(k)}\})\}$;
5 **return** $\mu^{(n)}$;

It is not difficult to see that the SD and the CSD are equivalent (i.e., $\text{SD}^\sigma = \text{CSD}^{\sigma, \mathcal{M}}$) if $\mathcal{F}$ belongs to general upper-bound and $\mathcal{M}$ is the set of school feasible matchings.

Theorem 1. If $\mathcal{F}$ belong to general upper-bound, the outcome matching SD^σ is Pareto efficient for any $\sigma \in \Sigma$.

3 Condition to check Pareto efficiency by SD: matroid

In this section, we characterize the condition that Pareto efficiency of a given matching can be determined by the serial dictatorship mechanism (Algorithm 1). To be precise, we prove that any Pareto efficient matching can be obtained by the serial dictatorship mechanism if $\mathcal{F}$ is a matroid. Also, we show that a Pareto efficient matching may not be obtained by the serial dictatorship mechanism if $\mathcal{F}$ is not a matroid.

Let us first observe that checking Pareto efficiency of a given matching by TTC is not straightforward even when $\mathcal{F}$ is a matroid.

Example 2. Let $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s_1, s_2\}$. Suppose that i_1 and i_2 prefer s_2 to s_1 , while i_3 and i_4 prefer s_1 to s_2 . The constraint $\mathcal{F}$ is a (partition) matroid that is defined by the aggregation of

$$\begin{aligned}\mathcal{F}_{s_1} &= \{I' \subseteq I : |I' \cap \{i_1, i_3\}| \leq 1 \text{ and } |I' \cap \{i_2, i_4\}| \leq 1\}, \\ \mathcal{F}_{s_2} &= \{I' \subseteq I : |I' \cap \{i_1, i_4\}| \leq 1 \text{ and } |I' \cap \{i_2, i_3\}| \leq 1\}.\end{aligned}$$

Consider a matching $\mu = \{(i_1, s_1), (i_2, s_1), (i_3, s_2), (i_4, s_2)\}$. Note that μ is not Pareto efficient because it is Pareto dominated by $\mu^* = \{(i_1, s_2), (i_2, s_2), (i_3, s_1), (i_4, s_1)\}$.

Now, we try to apply a TTC-like mechanism to the market of Example 2. A natural implementation of TTC in our setting is to repeatedly execute the following procedure: (i) each student points to students who are matched to her most preferred school, (ii) identify a cycle, (iii) implement the trade indicated by this cycle, and (iv) remove all the involved students. The cycle obtained by this process at μ is (i_1, i_3) , but the matching after the trade by this cycle, i.e., $\{(i_1, s_2), (i_2, s_1), (i_3, s_1), (i_4, s_2)\}$, is not feasible.

The reason why the above mechanism fails is that it does not take into account the feasibility of the schools. To resolve this issue, we construct a trading graph on $I \times (S \cup \{\emptyset\})$, instead of I . Our TTC mechanism is formally defined in Algorithm 3. Throughout the algorithm, $\mu \cup \tilde{\mu}$ represents the current matching, and $\tilde{\mu}$ represents the fixed part. Intuitively, each student i (with her current partner $\mu(i)$) points to her most preferred school q_i among those that can accept her by rejecting some student. Then, each student-school pair (i, q_i) determines the matched pairs $(i', \mu(i'))$ such that $(\mu \setminus \{(i', \mu(i'))\}) \cup \{(i, q_i)\} \in \mathcal{F}$. Note that the graph must contain at least one cycle because each node has out-degree at least one. Implementing the trade indicated by this cycle must yield a matching. For the instance in Example 2, $C = ((i_1, s_1), (i_1, s_2), (i_4, s_2), (i_4, s_1), (i_2, s_1), (i_2, s_2), (i_3, s_2))$ is the unique cycle in the graph constructed in the first round. Implementing the trade indicated by C yields μ^* .

Here, selecting a shortest cycle is crucial to keep feasibility of the school side when $\mathcal{F}$ is a matroid.⁶ For example, let us consider a market that is almost the same as the one in Example 2, but the school constraints is a matroid $\mathcal{F}'$ which is defined as the aggregation of

$$\mathcal{F}'_{s_1} = \{I' \subseteq I : |I' \cap \{i_3, i_4\}| \leq 1 \text{ and } |I'| \leq 2\} \text{ and } \mathcal{F}'_{s_2} = \{I' \subseteq I : |I'| \leq 2\}.$$

Note that $\mathcal{F}'$ is also a matroid. Then C is also a cycle in the graph constructed in the first round. However, implementing the trade indicated by C yields μ^* , which is infeasible. From a computational perspective, such a shortest cycle can be found in linear time ($O(|U| + |V| + |E|) = O(|I| \cdot |S|)$ time) by the breadth-first search algorithm.

Algorithm 3: Top Trading Cycle (TTC)

input : a market $(I, S, (\succ)_{i \in I}, \mathcal{F})$ and a feasible matching μ

output: a matching TTC^μ

1 Let $\tilde{\mu} \leftarrow \emptyset$ and $R \leftarrow I$;

2 **while** $R \neq \emptyset$ **do**

3 Construct a bipartite graph $G = (U, V; E)$ as follows:

- $U \leftarrow \{u_i : i \in R\}$ where $u_i = (i, \mu(i))$,
- $V \leftarrow \{v_i : i \in R\}$ where $v_i = (i, \arg \max_{\succ_i} \{s \in S : \{(i, s)\} \in \mathcal{F}/\tilde{\mu}\})$,
- $E \leftarrow \{(u_i, v_i) : i \in R\} \cup \{(v, u) \in V \times U : (\mu \setminus \{u\}) \cup \{v\} \in \mathcal{F}/\tilde{\mu}\}$;

Identify a shortest cycle in G and let $(u_{\tau(1)}, v_{\tau(1)}, \dots, u_{\tau(k)}, v_{\tau(k)})$ be the cycle;

Let $\mu \leftarrow \mu \setminus \{u_{\tau(1)}, \dots, u_{\tau(k)}\}$ and $\tilde{\mu} \leftarrow \tilde{\mu} \cup \{(v_{\tau(1)}, \dots, v_{\tau(k)})\}$;

Remove all the involved students, i.e., $R \leftarrow R \setminus \{\tau(1), \dots, \tau(k)\}$;

4 **return** $\tilde{\mu}$;

In what follows, we prove that our TTC mechanism always outputs a Pareto efficient matching TTC^μ that Pareto dominates the given matching μ if $\mathcal{F}$ is a matroid. Moreover, we show that SD^σ is equal to TTC^μ when we set σ as the order that the students are removed in the

⁶For constraints more general than matroids, selecting a shortest cycle does not guarantee the feasibility of the school side.

TTC mechanism. As a consequence, we conclude that any Pareto efficient matching can be obtained by the SD if $\mathcal{F}$ is a matroid.

We first prove that TTC^μ Pareto dominates μ .

Lemma 2. If $\mathcal{F}$ is a matroid, then TTC^μ is a feasible matching that Pareto dominates the given matching μ .

Proof. We show that, at the beginning of any iteration of the while-loop in Algorithm 3, $\mu \cup \tilde{\mu}$ is feasible matching that Pareto dominates μ . At the first round, the statement clearly holds from the definition of μ and $\tilde{\mu}$. In each iteration, $\mu \cup \tilde{\mu}$ remains contained in $\mathcal{F}$ by Lemma 1. In addition, the partner of a student i at $\mu \cup \tilde{\mu}$ does not get worse because $\{(i, \mu(i))\}$ must be in $\mathcal{F}/\tilde{\mu}$ if $\mu(i) \in S$ by $\mu \cup \tilde{\mu} \in \mathcal{F}$. $\square$

Next, we show that TTC^μ can be represented as SD^σ where σ is the order that the students are removed in the TTC mechanism.

Lemma 3. If $\mathcal{F}$ is a matroid, $\text{SD}^\sigma = \text{TTC}^\mu$ when we set σ as the order in which the students are removed within the TTC mechanism.

Proof. We prove the lemma by induction on the while-loop iterations. At the beginning, $\tilde{\mu} = \emptyset$, which is consistent the outcome of the SD when no student is assigned. In each iteration, student $\tau(j)$ ($j \in \{1, \dots, k\}$) is assigned to $\text{TTC}^\mu(\tau(j)) = \arg \max_{\succ_{\tau(j)}} \{s \in S : \{(\tau(j), s)\} \in \mathcal{F}/\hat{\mu}\}$. Meanwhile, the SD mechanism assigns $\tau(j)$ to $\text{SD}^\sigma(\tau(j)) = \arg \max_{\succ_{\tau(j)}} \{s \in S : \{(\tau(j), s)\} \in \mathcal{F}/(\hat{\mu} \cup (\{v_{\tau(1)}, \dots, v_{\tau(j-1)}\}))\}$. Here, as TTC^μ is feasible by Lemma 2, we have $\{(\tau(j), \text{TTC}^\mu(\tau(j)))\} \in \mathcal{F}/(\hat{\mu} \cup (\{v_{\tau(1)}, \dots, v_{\tau(j-1)}\}))$. Hence, we obtain $\text{TTC}^\mu(\tau(j)) = \text{SD}^\sigma(\tau(j))$. Therefore, $\text{SD}^\sigma = \text{TTC}^\mu$. $\square$

Theorem 2. If $\mathcal{F}$ is a matroid, we have

$$\{\text{SD}^\sigma : \sigma \in \Sigma\} = \{\mu : \mu \text{ is a Pareto efficient matching}\}.$$

Proof. By Theorem 1, it is sufficiently to prove that any Pareto efficient matching μ can be represented as SD^σ for some $\sigma \in \Sigma$. Fix such an efficient matching μ . By Lemma 2, TTC^μ must be equal to μ . Thus, by taking σ as shown in Lemma 3, we can conclude that $\text{SD}^\sigma = \text{TTC}^\mu = \mu$. $\square$

Conversely, we show that the matroid structure is a necessary condition to represent every Pareto efficient matching as an outcome of the SD mechanism. The class of matroid constraints subsumes many practical cases, but there are some constraints that are not matroid, such as school admissions with budget constraints. A natural question is whether the conclusion of Theorem 2 holds without the assumption of matroid. The following result shows that the answer to this question is negative.

Theorem 3. Fix a set of students I , a set of schools S with $|S| \geq 2$, and a school $s \in S$ and its constraint $\mathcal{F}_s$. Suppose $\mathcal{F}_s$ is not a matroid. Then there exist a preference profile $(\succ)_{i \in I}$ and a capacity constraint profile $\mathcal{F}_{-s}$ such that

$$\{\text{SD}^\sigma : \sigma \in \Sigma\} \neq \{\mu : \mu \text{ is a Pareto efficient matching}\}. \quad (1)$$

This result shows that Theorem 2 cannot be generalized further in the sense that the class of matroids is a “maximal domain.” Specifically, a matroid is the most permissive restriction on constraints imposed on individual schools which guarantees that the SD finds all Pareto efficient matchings.

4 Checking Pareto efficiency

In this section, we show that Pareto efficiency of a given matching can be checked by the CSD mechanism (Algorithm 2) for general constraints. Specifically, we apply the CSD with $\mathcal{M}$ as the set of school feasible matchings that are also individually rational for the given matching. We prove that the outcome of the CSD is a Pareto efficient matching that weakly Pareto dominates the given matching. As a consequence, we can conclude that Pareto efficiency of a given matching can be checked by the CSD.

Let us first observe that if a matching is Pareto efficient under an individual rational constraint, then the matching is also Pareto efficient without the constraint.

Lemma 4. For a set of matchings $\mathcal{M}$ and a matching $\mu \in \mathcal{M}$, let $\mathcal{M}' = \{\mu' \in \mathcal{M} : \mu'(i) \succeq_i \mu(i) \ (\forall i \in I)\}$. If μ is Pareto efficient within $\mathcal{M}'$, then it is also Pareto efficient within $\mathcal{M}$.

Proof. We prove the contraposition. Let $\mu^* \in \mathcal{M}'$ be a matching that is Pareto inefficient within $\mathcal{M}$. Then, there exists a matching $\mu' \in \mathcal{M}$ that is a Pareto improvement of μ^* . Here, μ' must be in $\mathcal{M}'$ because $\mu'(i) \succeq_i \mu^*(i) \succeq_i \mu(i)$ for every $i \in I$. Hence, μ^* is also Pareto inefficient within $\mathcal{M}'$. $\square$

Theorem 4. Let $\mathcal{M}$ be the set of school feasible matchings that are also individually rational for a given matching μ . Suppose that $\mathcal{M} \neq \emptyset$. Then, for any $\sigma \in \Sigma$, the matching CSD $^{\sigma, \mathcal{M}}$ is Pareto efficient and weakly Pareto dominate μ .

Finally, let us discuss computational issues. We observe that Algorithm 2 can be implemented to run in polynomial time if $\mathcal{F}$ is a g-matroid (provided a membership oracle and an initial independent set). To do this, it is sufficient to show that the condition “ $\mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{M}^{(k)}$ ” at line 4 can be checked in polynomial time. We reduce the checking problem to the g-matroid intersection problem defined as follows: given two g-matroids $\mathcal{F}_1$ and $\mathcal{F}_2$, determining whether $\mathcal{F}_1 \cap \mathcal{F}_2$ is nonempty. The g-matroid intersection problem can be solved in polynomial time (Murota, 1996b,a). Define

$$\mathcal{F}' = \left\{ X \subseteq I \times S : \begin{array}{l} |X \cap (\{i\} \times S)| \leq 1 \ (\forall i \in I), \\ (i, s) \notin X \text{ if } \mu(i) \succ_i s \ (\forall i \in I, \forall s \in S), \\ \mu^{(k-1)} \cup \{(\sigma(k), s)\} \subseteq X \end{array} \right\},$$

which is a g-matroid representing the set of individually rational matchings that contains $\mu^{(k-1)} \cup \{(\sigma(k), s)\}$. Hence, the condition “ $\mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{M}^{(k)}$ ” holds if and only if $\mathcal{F} \cap \mathcal{F}'$ is nonempty.

Unfortunately, it is coNP-hard to distinguish whether a given matching is Pareto efficient or not even when $\mathcal{F}_s$ is a budget constraint for each $s \in S$. To observe this, we reduce the *subset sum problem* known to be NP-complete (Garey and Johnson, 1979), defined below: given positive integers $a_1, a_2, \dots, a_\ell$ and a_0 ($\leq \sum_{j=1}^\ell a_j$), deciding whether there exists a subset $J \subseteq \{1, 2, \dots, \ell\}$ such that $\sum_{j \in J} a_j = a_0$. Given an instance of the subset sum problem, we construct a matching market where $I = \{i_0, i_1, \dots, i_\ell\}$, $S = \{s_1, s_2\}$, $s_1 \succ_{i_0} s_2$, $s_2 \succ_{i_k} s_1$ ($k = 1, 2, \dots, \ell$), and

$$\mathcal{F}_{s_1} = \left\{ I' \subseteq I : \sum_{i_k \in I'} a_k \leq \sum_{k=1}^\ell a_k \right\} \text{ and } \mathcal{F}_{s_2} = \left\{ I' \subseteq I : \sum_{i_k \in I'} a_k \leq t \right\}.$$

Consider a matching $\mu = \{(i_0, s_2), (i_1, s_1), \dots, (i_\ell, s_1)\}$. If the subset sum instance is a yes-instance, i.e., there exists $J \subseteq \{1, 2, \dots, \ell\}$ such that $\sum_{j \in J} a_j = t$, then $\{(i, s_1) : i \in I \setminus J\} \cup \{(i, s_2) : i \in J\}$ Pareto dominates μ . On the other hand, if a matching μ' Pareto dominates μ , then the subset sum instance must be a yes-instance since $\sum_{(i_j, s_2) \in \mu'} a_j = t$ and $(i_0, s_2) \notin \mu'$. Hence, μ is not Pareto efficient if and only if the subset sum instance is a yes-instance.

Theorem 5. Checking Pareto efficiency of a given matching is coNP-hard even if $\mathcal{F}$ is individual budget constraints.

Although Pareto efficiency is computationally hard to check, the CSD can be implemented to run practically fast in most cases by using an integer programming solver such as Gurobi or CPLEX if the distributional constraints can be represented as a set of linear inequalities. For example, for individual budgets constraints, $\arg \max_{\succ_{\sigma(k)}} \{s \in S : \mu^{(k-1)} \cup \{(\sigma(k), s)\} \in \mathcal{M}^{(k)}\} \in S \cup \{\emptyset\}$ at line 4 can be computed by the following integer programming:

$$\begin{array}{ll} \max & \sum_{s \in S} \sum_{s' \in S: s' \succeq_s} x_{\sigma(k), s'} \\ \text{s.t.} & \sum_{s \in S} x_{i, s} \leq 1 \quad \forall i \in I, \\ & \sum_{i \in I} a_{i, s} x_{i, s} \leq b_s \quad \forall s \in S, \\ & x_{i, s} = 1 \quad \forall (i, s) \in \mu^{(k-1)}, \\ & x_{i, s} = 0 \quad \forall (i, s) \in I \times S \text{ with } \mu(i) \succ_i s, \\ & x_{i, s} \in \{0, 1\} \quad \forall (i, s) \in I \times S. \end{array}$$

Here, $x_{i, s}$ means that i is matched with s or not.

5 Application of the CSD: generalized EADA

In this section, we study a model of indivisible goods allocation with priorities. As an application of the CSD, we provide a mechanism that Pareto dominates the DA. The mechanism not just simply utilize the CSD, but also calculate an appropriate order of the students based on the priorities. We show that this mechanism is an extension of the EADA mechanism.

Suppose that each school s is endowed with a priority, which is modeled as choice function over sets of students. Let $\text{Ch}_s: 2^I \rightarrow 2^I$ be the choice function of $s \in S$, where $\text{Ch}_s(I') \subseteq I'$ for all $I' \subseteq I$. The aggregate choice function $\text{Ch}: 2^{I \times S} \rightarrow 2^{I \times S}$ for $(\text{Ch}_s)_{s \in S}$ is defined as

$$\text{Ch}(X) = \{(i, s) : s \in S, i \in \text{Ch}_s(X(s))\} \quad \text{for } X \subseteq I \times S.$$

A matching μ is called *individually rational* if $\text{Ch}(\mu) = \mu$. Note that this condition can be seen as the feasibility of the school side. The *stability* leads a fairness notion in indivisible goods allocation problems: no student prefers some school to her assigned one and has higher priority than some student who is assigned to that school. Formally, a matching μ is stable if it is individually rational and there exists no $(i, s) \in I \times S$ such that $s \succ_i \mu(i)$ and $i \in \text{Ch}_s(\mu(s) \cup \{i\})$.

We introduce conditions that restrict the priorities. A choice function Ch satisfies: *substitutability* if for every $x \in X \subseteq Y$, $x \in \text{Ch}(Y)$ implies $x \in \text{Ch}(X)$, *path-independence* if for every X and Y , $\text{Ch}(X \cup Y) = \text{Ch}(\text{Ch}(X) \cup \text{Ch}(Y))$, and *the law of aggregate demand (LAD)* if for every $X \subseteq Y$, $|\text{Ch}(X)| \leq |\text{Ch}(Y)|$. Path-independence leads to substitutability, and substitutability and LAD together lead to path-independence. For each property, the aggregate choice function satisfies the property if and only if all the choice functions of the schools satisfy the property.

The *responsive* choice functions are the most standard ones that satisfies substitutability and LAD. For school s , let q_s denote the capacity and $\succ_s$ denote a priority order of the students. The responsive choice function Ch_s with respect to q_s and $\succ_s$ is defined as

$$\text{Ch}_s(I') = \begin{cases} I' & \text{if } |I'| \leq q_s, \\ \{i \in I' : i \succeq_s i^*\} & \text{otherwise,} \end{cases}$$

where $i^* \in I'$ is the student such that $|\{i \in I' : i \succeq_s i^*\}| = q_s$.

When every choice function satisfies path-independence, a stable matching exists, and the DA, which is formally defined in Algorithm 4, finds a stable matching (Roth, 1984; Aygün and

Algorithm 4: Deferred Acceptance (DA)

input: a market $(I, S, (\succ)_{i \in I}, \mathcal{F})$ and an aggregate choice function Ch

```
1 Let  $R \leftarrow I \times S$ ;  
2 while true do  
3    $Y \leftarrow \{(i, \arg \max_{\succ_i} \{s : (i, s) \in R\}) : i \in I\}$ ;  
4    $Z \leftarrow \text{Ch}(Y)$  and  $R \leftarrow R \setminus (Y \setminus Z)$ ;  
5   if  $Y = Z$  then return  $Y$ ;
```

Sönmez, 2013). Moreover, it finds the *student-optimal* stable matching, i.e., the stable matching that Pareto dominates any other stable matchings. Assuming both substitutability and LAD, we obtain some more useful properties.⁷

We provide crucial properties for the generalized EADA by Ehlers and Morrill (2020). For a matching μ , a student i is *irrelevant* if either $\mu(i) = \emptyset$ or for $\mu(i) = s \in S$

$$\text{Ch}_s(\{j \in I : s \succeq_j \mu(j)\} \setminus \{i\}) \subseteq \mu(s).$$

Given a matching μ and a student i , we say that student i is *Pareto improvable* if there exists a individual rational matching ν such that $\nu(i) \succ_i \mu(i)$ and $\nu(j) \succeq_j \mu(j)$ for all $j \in I$.

Lemma 5 (Ehlers and Morrill (2020)). Suppose that the choice function Ch_s satisfies substitutability and LAD for every school $s \in S$. Let μ be the matching produced by the DA.

1. There always exists a student who is irrelevant for μ .
2. If a student i is irrelevant for the matching μ , then i is not Pareto improvable.

The first property holds for path-independent choice functions. In fact, Ehlers and Morrill (2020) prove that the last students assigned in DA are irrelevant for μ without the assumption of the LAD. However, without the LAD, the second property does not hold: a student i is irrelevant for μ , but is Pareto improvable. The following example illustrates this fact.

Example 3. Let $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s_1, s_2\}$. Suppose that i_1, i_2 , and i_3 prefer s_1 to s_2 , while i_4 prefer s_2 to s_1 . Ch_{s_2} is a responsive choice function with $q_{s_2} = 2$ and $\succ_{s_2}$ such that $i_2 \succ_{s_2} i_3 \succ_{s_2} i_4 \succ_{s_2} i_1$. Ch_{s_1} is similar to a responsive choice function with $q_{s_1} = 1$. Different from a responsive choice function, s_2 has a weak priority order $i_4 \succ_{s_1} i_1 \succ_{s_1} i_2 \sim_{s_1} i_3$ and accepts multiple students at the borderline given q_{s_1} .⁸ Formally, $\text{Ch}_{s_1}(I')$ is I' if $|I'| \leq q_{s_1}$ and $\{i \in I' : i \succeq_{s_1} i^*\}$ otherwise, where i^* is the highest priority student such that $|\{i \in I' : i \succeq_{s_1} i^*\}| \geq q_{s_1}$. Note that Ch_{s_1} satisfies path-independence, but not LAD. The outcome of the DA is $\mu = \{(i_1, \emptyset), (i_2, s_2), (i_3, s_2), (i_4, s_1)\}$, and only i_1 is irrelevant for μ . However, i_1 is Pareto improvable. In fact, $\nu = \{(i_1, s_2), (i_2, s_1), (i_3, s_1), (i_4, s_2)\}$ Pareto dominates μ and $\nu(i_1) \succ_{i_1} \mu(i_1)$. Note that every student is strictly better off at ν than at the matching μ produced by the DA.

Even without the LAD, the generalized EADA is well-defined since the first condition of Lemma 5 holds. However, it may produce a Pareto inefficient matching. In Example 3, the generalized EADA produces $\{(i_1, \emptyset), (i_2, s_1), (i_3, s_1), (i_4, s_2)\}$, which is Pareto inefficient. Nevertheless, the order over the students obtained in the generalized EADA is still useful.

⁷The DA is strategy-proof; the rural hospital theorem holds; the set of stable matchings is a lattice (Hatfield and Milgrom, 2005).

⁸This choice function is introduced by Fleiner and Jankó (2014) and studied in college admissions in Hungary (Biró and Kiselgof, 2015).

Algorithm 5: CSD with priorities

input: a market $(I, S, (\succ)_{i \in I}, \text{Ch})$

- 1 $R \leftarrow I$ and $\sigma \leftarrow ()$;
- 2 **for** $k \leftarrow 1, 2, \dots, |I|$ **do**
- 3 let $\mu^{(k)}$ be the outcome of the DA for the current restricted market;
- 4 let $i^{(k)} \in R$ be a student who is irrelevant for $\mu^{(k)}$;
- 5 $\sigma(k) \leftarrow i^{(k)}$ and $R \leftarrow R \setminus \{i^{(k)}\}$;
- 6 letting all schools except $\mu^{(k)}(i^{(k)})$ are unacceptable for $i^{(k)}$;
- 7 Let $\mathcal{M}$ be the set of individually rational matchings that Pareto dominates $\mu^{(1)}$;
- 8 **return** $\text{CSD}^{\sigma, \mathcal{M}}$;

Our algorithm is formally described in Algorithm 5. Intuitively, it computes an ordering of the students by iteratively processing the following: identify an irrelevant student for the outcome of the DA, insert her to the head of the list, and remove her from the market. Then, it applies the CSD with the obtained order, where we set the set of feasible matchings $\mathcal{M}$ as the set of individually rational matchings that Pareto dominates the outcome of DA for the original market. In Example 3, the Algorithm 5 produces ν which is Pareto efficient. Moreover, these mechanisms coincide if the choice functions satisfy substitutability and LAD.

Theorem 6. The output of Algorithm 5 is equal to that of the generalized EADA by Ehlers and Morrill (2020) if the choice functions satisfy substitutability and LAD.

From Theorem 6, the outcome of Algorithm 5 is uniquely determined regardless of the selection of students at line 4 if the choice functions satisfy substitutability and LAD. However, this property does not hold without the LAD (see Example 4 in Appendix A). In addition, we cannot obtain the same matching by SD^σ even when the choice functions satisfy substitutability and LAD (see Example 5 in Appendix A).

6 Conclusion

We studied methods to check Pareto efficiency in indivisible goods allocation problems under general constraints. We started with the observation that the standard approach by the SD does not work in general. Our first main result characterizes the constraints that the SD can use to check whether a given matching is Pareto efficient. These are matroid constraints. Our result also generalizes the characterization of Pareto efficient matchings by the SD in the standard model with capacity constraints. Our second main result provides an algorithm to check the Pareto efficiency of a given matching under general constraints. Finally, as an application of the CSD, we propose a mechanism for a matching model with priorities. We observed that this mechanism is a generalization of the EADA.

A study on the incentive properties of a mechanism under constraints can be considered in the future. Let us fix an order σ of the students. Then, the $\text{CSD}^{\sigma, \mathcal{M}}$ is strategy-proof if reporting preferences cannot affect the set of feasible matchings $\mathcal{M}$. However, the CSD is not strategy-proof under general constraints. For example, the CSD would not be strategy-proof by adding the individual rationality constraint to $\mathcal{M}$. Fragiadakis et al. (2016) assume no outside option and show that the CSD is strategy-proof in the model with minimal quotas. Thus, a natural question is which class of constraints guarantees the CSD is strategy-proof. More generally, it is important to design a Pareto efficient and strategy-proof mechanism under general constraints. We leave such investigations for future research.

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A Examples for Algorithm 5

We first provide an example such that the output of Algorithm 5 is not unique.

Example 4. Let $I = \{i_1, i_2, i_3, i_4, i_5\}$ and $S = \{s_1, s_2\}$. Suppose that i_1, i_2, i_3 , and i_4 prefer s_1 to s_2 , while i_5 prefer s_2 to s_1 . School s_1 has a quota $q_{s_1} = 1$ and a weak priority order $i_5 \succ_{s_1} i_3 \sim_{s_1} i_4 \succ_{s_1} i_1 \sim_{s_1} i_2$. Also, school s_2 has a quota $q_{s_2} = 2$ and a weak priority order $i_3 \sim_{s_2} i_4 \succ_{s_2} i_1 \sim_{s_2} i_2 \succ_{s_2} i_5$. The choice functions Ch_{s_1} and Ch_{s_2} are defined as $\text{Ch}_s(I')$ is I' if $|I'| \leq q_s$ and $\{i \in I' : i \succeq_s i^*\}$ otherwise, where i^* is the highest priority student such that $|\{i \in I' : i \succeq_s i^*\}| \geq q_s$. Note that Ch_{s_1} and Ch_{s_2} satisfy path-independence, but not LAD.

Let us apply Algorithm 5 to this market. The outcome of the DA in the first round is

$$\mu^{(1)} = \{(i_1, \emptyset), (i_2, \emptyset), (i_3, s_2), (i_4, s_2), (i_5, s_1)\},$$

and the irrelevant students for $\mu^{(1)}$ are i_1 and i_2 . If i_1 is chosen as $i^{(1)}$, the outcome of the algorithm is

$$\{(i_1, s_2), (i_2, \emptyset), (i_3, s_1), (i_4, s_1), (i_5, s_2)\}.$$

On the other hand, if i_2 is chosen as $i^{(1)}$, the outcome of the algorithm is

$$\{(i_1, \emptyset), (i_2, s_2), (i_3, s_1), (i_4, s_1), (i_5, s_2)\}.$$

Next, we present an example in which Algorithm 5 must use the CSD instead of SD to obtain a matching that Pareto dominate the outcome of the DA.

Example 5. Let $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s_1, s_2, s_3\}$. Suppose that the student preferences are

$$s_2 \succ_{i_1} s_3 \succ_{i_1} s_1, \quad s_3 \succ_{i_2} s_2 \succ_{i_2} s_1, \quad s_1 \succ_{i_3} s_2 \succ_{i_3} s_3, \quad \text{and} \quad s_1 \succ_{i_4} s_3 \succ_{i_4} s_2.$$

Every school has a quota of 1, and the weak priority orders are

$$i_1 \sim_{s_1} i_2 \succ_{s_1} i_4 \succ_{s_1} i_3, \quad i_4 \succ_{s_2} i_2 \succ_{s_2} i_3 \succ_{s_2} i_1, \quad \text{and} \quad i_3 \succ_{s_3} i_1 \succ_{s_3} i_2 \succ_{s_3} i_4.$$

The choice function Ch_s for each school s is defined with the quota and the weak priority order in the same way as in Example 4. Note that the choice functions satisfy substitutability and LAD.

For this market, the outcome of the DA is

$$\mu = \{(i_1, s_1), (i_2, s_1), (i_3, s_2), (i_4, s_3)\}.$$

In addition,

$$\nu = \{(i_1, s_1), (i_2, s_1), (i_3, s_3), (i_4, s_2)\}.$$

is a unique Pareto efficient matching that Pareto dominates μ . As no student is assigned to the best school at ν , we cannot obtain this matching by the SD.

B Omitted proofs

B.1 Proof of Theorem 1

Proof. To obtain a contradiction, suppose that $\mu := \text{SD}^\sigma$ is Pareto inefficient for some $\sigma \in \Sigma$. Let μ' be a feasible matching that Pareto dominates μ and let k^* be the minimum index such that $\mu'(\sigma(k^*)) \neq \mu(\sigma(k^*))$. By definition, we have $\mu'(\sigma(k)) = \mu(\sigma(k))$ for all $k = 1, 2, \dots, k^* - 1$ and $\mu'(\sigma(k^*)) \succ_{\sigma(k^*)} \mu(\sigma(k^*))$. As $\mu' \in \mathcal{F}$ and $\mathcal{F}$ belongs to general upper-bound, the student $\sigma(k^*)$ must be matched with a school that is no worse than $\mu'(\sigma(k^*))$ at line 3 in Algorithm 1, a contradiction. $\square$

B.2 Proof of Theorem 3

First, suppose that $\mathcal{F}_s$ violates property (I1). Then we have

$$\{\text{SD}^\sigma : \sigma \in \Sigma\} = \{\emptyset\} \neq \emptyset = \{\mu : \mu \text{ is a Pareto efficient matching}\}$$

when every school is unacceptable for all students.

Next, suppose that $\mathcal{F}_s$ violates property (I2), i.e., there exist $X, Y \subseteq I$ such that $X \subseteq Y$, $Y \in \mathcal{F}_s$, and $X \notin \mathcal{F}_s$. Consider a preference profile such that each student $i \in Y$ only accepts s and each student $i \notin Y$ accepts no school. Then, there is a unique Pareto efficient matching $\{(i, s) : i \in Y\}$. On the other hand, for σ such that $\sigma^{-1}(x) < \sigma^{-1}(y)$ for any $x \in X$ and $y \in Y \setminus X$, some student in X must be rejected by s in SD^σ . Hence, $\text{SD}^\sigma \neq \{(i, s) : i \in Y\}$ for such σ , and (I1) holds.

Finally, suppose that $\mathcal{F}_s$ violates only property (I3), i.e., there exist $X, Y \in \mathcal{F}_s$ such that $|X| < |Y|$ and $X \cup \{i\} \notin \mathcal{F}_s$ for all $i \in Y \setminus X$. Let s' be a school that is different from s . We will choose a capacity constraint $\mathcal{F}_{s'}$, a preference profile $(\succ)_{i \in I}$, and a matching μ such that

- μ is Pareto efficient;
- $\text{SD}^\sigma \neq \mu$ for any $\sigma \in \Sigma$.

By $Y \in \mathcal{F}_s$ and (I2), there exists $Z \subseteq X \cup Y$ such that $Z \in \mathcal{F}_s$ and $|Z| = |X| + 1$. Choose $Z \in \arg \max_{Z \subseteq X \cup Y} \{|Z \cap X| : Z \in \mathcal{F}_s, |Z| = |X| + 1\}$. Note that $X \not\subseteq Z$ by the assumption that $X \cup \{i\} \notin \mathcal{F}_s$ for all $i \in Y \setminus X$. Let $\mathcal{F}_{s'}$ be a capacity constraint with capacity $|X \setminus Z|$. Construct a matching μ and a preference profile $(\succ)_{i \in I}$ as follows:

- for $i \in X \setminus Z$, $s \succ_i s' \succ_i \emptyset \succ_i t$ for any school $t \in S \setminus \{s, s'\}$ and $\mu(i) = s'$,
- for $i \in Z \cap X$, $s \succ_i \emptyset \succ_i t$ for any school $t \in S \setminus \{s\}$ and $\mu(i) = s$,
- for $i \in Z \setminus X$, $s' \succ_i s \succ_i \emptyset \succ_i t$ for any school $t \in S \setminus \{s, s'\}$ and $\mu(i) = s$,
- for $i \in I \setminus (X \cup Z)$, $\emptyset \succ_i t$ for any school $t \in S$ and $\mu(i) = \emptyset$.

We show that μ is Pareto efficient by contradiction. Suppose that μ' Pareto dominates μ . As μ' is a feasible matching, we have $\mu'(s) \subseteq X \cup Z$ and $\mu'(s) \in \mathcal{F}_s$. Also, as μ' Pareto dominates μ , we have $\mu'(i) = s$ for all $i \in Z \cap X$ and $\mu'(i) = s$ for some $i \in X \setminus Z$. Hence, we have $|\mu'(s) \cap X| > |Z \cap X|$. This implies $|\mu'(s)| < |Z|$ by the maximality of Z . Thus, we obtain $|\mu'(s')| = |X \cup Z| - |\mu'(s)| > |X \cup Z| - |Z| = |X \setminus Z|$, which contradicts the feasibility of μ' . What is left is to show that $\text{SD}^\sigma \neq \mu$ for any $\sigma \in \Sigma$. If $\sigma^{-1}(i) < \sigma^{-1}(i')$ for some $i \in Z \setminus X$ and $i' \in X \setminus Z$, we have $\text{SD}^\sigma(i) = s'$ but $\mu(i) = s$. Otherwise (i.e., $\sigma^{-1}(i) > \sigma^{-1}(i')$ for any $i \in Z \setminus X$ and $i' \in X \setminus Z$), we have $\text{SD}^\sigma(i') = s$ for all $i' \in X \setminus Z$ by $X \in \mathcal{F}_s$ and (I2), but $\mu(i') = s'$. Hence, $\text{SD}^\sigma \neq \mu$ for any $\sigma \in \Sigma$.

B.3 Proof of Theorem 4

Proof. By Lemma 4, it is sufficient to prove that $\text{CSD}^{\sigma, \mathcal{M}}$ is Pareto efficient within $\mathcal{M}$. To obtain a contradiction, suppose that $\mu := \text{SD}^\sigma$ is Pareto inefficient for some $\sigma \in \Sigma$. Let $\mu' \in \mathcal{M}$ Pareto dominates μ and let k^* be the minimum index such that $\mu'(\sigma(k^*)) \neq \mu(\sigma(k^*))$. By definition, we have $\mu'(\sigma(k)) = \mu(\sigma(k))$ for all $k = 1, 2, \dots, k^* - 1$ and $\mu'(\sigma(k^*)) \succ_{\sigma(k^*)} \mu(\sigma(k^*))$. As $\mu' \in \mathcal{M}$, the student $\sigma(k^*)$ must be matched with a school that is no worse than $\mu'(\sigma(k^*))$ at line 4 in Algorithm 2, a contradiction. $\square$

B.4 Proof of Theorem 6

Proof. Suppose that the choice functions satisfy substitutability and LAD. Then, the matching $\mu^{(|I|)} = \{(i^{(k)}, \mu^{(k)}(i^{(k)})) : k \in \{1, 2, \dots, |I|\}\}$ in Algorithm 5 is the one produced by the generalized EADA. By Lemma 5, $i^{(1)}$ is not Pareto improvable at $\mu^{(1)}$. Hence, $i^{(1)}$ must be assigned to $\mu^{(1)}(i^{(1)})$ in $\text{CSD}^{\sigma, \mathcal{M}}$. Suppose that $i^{(k)}$ is assigned to $\mu^{(k)}(i^{(k)})$ in $\text{CSD}^{\sigma, \mathcal{M}}$ for $k = 1, 2, \dots, t - 1$. Then, $i^{(t)}$ must be assigned to $\mu^{(t)}(i^{(t)})$ in $\text{CSD}^{\sigma, \mathcal{M}}$, because $i^{(t)}$ is not Pareto improvable at $\mu^{(k)}$ when all schools except $\mu^{(k)}(i^{(k)})$ are unacceptable for $i^{(k)}$ for $k = 1, 2, \dots, t - 1$. Thus, we obtain $\text{CSD}^{\sigma, \mathcal{M}} = \mu^{(I)}$. $\square$