

Envy-Free Coalitions of Fixed Size[★]

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Abstract. We consider the problem of partitioning a set of agents into triples based on the agents’ additively separable preferences. This can be viewed as a hedonic game. We apply three successively weaker solution concepts: envy-freeness, weakly justified envy-freeness, and justified envy-freeness. In a model in which coalitions may be any size, trivial solutions exist for these concepts, which provides a strong motivation for considering a partition into coalitions of fixed size. We study the existence of such partitions and the computational complexity of finding such partitions, if they exist.

Keywords: Coalition formation · Hedonic games · Multidimensional roommates · Envy-freeness.

1 Introduction

In this paper we study a general model that involves partitioning a set of agents into disjoint coalitions. In relation to existing literature, this model can be described as a *hedonic game* in which coalitions must have size three or a *multidimensional roommates* problem. A *hedonic game*, involves a set of agents N that will be partitioned into disjoint sets, called coalitions. Each agent has some preference between possible coalitions, which must be respected in some way [2]. The two characteristics of a hedonic game are thus the system of *preference representation* and the *solution concept*, i.e. the criteria, defined in terms of the agents’ preferences, of an acceptable coalition partition. A rich literature of hedonic games exists and a wide variety of preference representations and solution concepts have been proposed. Our model can be phrased as a hedonic game in which the size of any coalition in any acceptable partition is fixed to 3.

In this paper, agents have *additively separable preferences*, which in our context means each agent α_i supplies an numerical valuation $v_{\alpha_i}(\alpha_j)$ of each other agent α_j , and an agent’s preference between two agents depends on its valuation

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of each and its preference between two triples depends on the sum of the valuations of the agents in each triple. For some agent and some triple S , we define this agent's *utility* $u_{\alpha_i}(S)$ in the triple S as the sum of its valuations for the other agents in S . We define some possible restrictions on agents' preferences. If $v_{\alpha_i}(\alpha_j) = v_{\alpha_j}(\alpha_i)$ for any pair of agents α_i, α_j then we say that preferences are *symmetric*. If $v_{\alpha_i}(\alpha_j) \in \{0, 1\}$ for any pair of agents α_i, α_j then we say that the preferences are *binary*. Similarly, if $v_{\alpha_i}(\alpha_j) \in \{0, 1, 2\}$ for any such α_i, α_j we say that preferences are *ternary*. If preferences are symmetric and binary then they reveal an undirected graph, which we refer to as the *underlying graph*, (N, E) . In this case, we use $\Delta_{(N, E)}$ to signify the maximum degree of the underlying graph.

This model, as described above, is identical to an additively separable hedonic game (ASHG) [21]. In this paper we thus refer to a set of agents and preferences as an ASHG, with an acknowledgement that unlike most work on hedonic games, we require each coalition in an acceptable coalition partition to have a fixed size.

The three solution concepts that we explore are all related to “envy”. Suppose π is a partition of the agent set N into triples. We write $\pi(\alpha_i)$ to mean the triple in π that contains α_i , and $u_{\alpha_i}(\pi)$ meaning $u_{\alpha_i}(\pi(\alpha_i))$. We say that an agent α_i has *envy* for another agent α_j if the former agent would gain a utility higher than its utility in the partition if it were to replace the latter agent in its triple. We say that a partition in which no agent has envy for another is *envy-free*. Now, suppose α_i has envy for α_j in some partition of N into triples π . We say that this envy is *justified envy* (j-envy) if $v_{\alpha_k}(\alpha_i) > v_{\alpha_k}(\alpha_j)$ for each $\alpha_k \in \pi(\alpha_i) \setminus \{\alpha_i\}$. Informally, the other agents in the triple of the envied agent all prefer the envying agent to the envied agent. Similarly, we say that envy is *weakly justified envy* (wj-envy) if $v_{\alpha_k}(\alpha_i) \geq v_{\alpha_k}(\alpha_j)$ for each $\alpha_k \in \pi(\alpha_i) \setminus \{\alpha_i\}$. Now each other agent in the triple of the envied agent is at least indifferent between the envying agent and the envied agent. The definitions of *justified envy-free* (j-envy-free) and *weakly justified envy-free* (wj-envy-free) partitions into triples are analogous to the definition of an envy-free partition.

1.1 Related work

We discuss three sets of related literature. The first is work involving both fixed-size coalitions and envy-freeness. The second is work involving envy-freeness and general settings of coalition formation, with no restriction on coalition size (this includes hedonic games). The third is work on coalition formation involving coalitions of fixed size but not necessarily involving envy-freeness. For more general models of coalition formation and hedonic games that involve solution concepts other than those related to envy-freeness, a wealth of literature exists [1, 2, 3, 6].

Envy-freeness, fixed coalition size. The only existing work is by Li et al [17], who consider a slightly more general model. Rather than a fixed coalition size $k \geq 1$, a partition must be *balanced*, meaning the number of coalitions $n \geq k$ is fixed and $\lfloor n/k \rfloor \leq X \leq \lceil n/k \rceil$ for any coalition X in an acceptable partition. They consider additively separable binary and symmetric preferences and focus on “approximate-envy free” partitions and some restricted sets of instances, such

as trees. Notably, they apply results from discrepancy theory to show that an approximate-envy free partition with a particular fixed asymptotic bound must exist, and can be found in polynomial time.

Envy-freeness, arbitrary coalition size. Aziz et al. [1] define envy-freeness as a solution concept in ASHG. They note that the “singleton” coalition partition $\{\{1\}, \{2\}, \dots, \{|N|\}\}$ is trivially envy-free and thus consider the existence of partitions that simultaneously satisfy envy-freeness and other established solution concepts. We believe that the existence of trivially envy-free partitions provides a strong motivation for the additional requirement in our model that each coalition has a fixed size. Building on this, Ueda considers envy-freeness and j-envy-freeness in hedonic games [24]. He too observed that the singleton coalition partition and the “grand” coalition partition $\{\{N\}\}$ are trivially envy-free, although there exist instances in which no non-trivial coalition partition is envy-free, and additional instances in which no non-trivial partition is j-envy-free. He also observed that any core stable partition is also justified-envy-free. Barrot and Yokoo [4] note Ueda’s observation and continue exploring the existence of coalition partitions that satisfy a combination of solution concepts including envy-freeness, wj-envy-freeness, and j-envy-freeness. As well as some non-existence results relating to ASHG, they also consider more general systems of preference representation. Notably, they present results relating to the existence of such coalition partitions in a setting in which preferences satisfy *top responsiveness* and *bottom responsiveness*, two restrictions already well-established in the hedonic games literature.

Fixed coalition size, other solution concepts. A central solution concept is *stability* [2]. The *Stable Roommates problem* (SR) involves matching a set of agents into pairs. It is a classical problem of matching under preferences and has been studied extensively [18]. The generalisation of SR to higher dimensions creates a model which can be viewed as a hedonic game with a fixed coalition size. A number of works such consider models [7, 14, 16, 20]. Cechlárová and Hajduková define the *Stable Partitions problem* [10, 11]. Aziz et al. [1] consider a variety of solution concepts in the setting of ASHG. Cseh et al. [12] present work in relation to partitions that are *Pareto Optimal*. Bilò et al. [5] study ASHG with binary and symmetric preferences in relation to *swap stability* and its variants. We remark that envy-freeness implies strict swap stability.

1.2 Our contribution

We summarise our results in Table 1.

We consider the existence of envy-free, wj-envy-free, and j-envy-free partitions into triples and the complexity of the associated decision and construction problems. For each solution concept, we consider possible restrictions on the agents’ numerical preferences. Our results reveal a number of dichotomies between polynomial-time solvability and NP-hardness (see Table 1).

In Section 2, we show that an envy-free partition into triples may not exist, even if the preferences are binary and the maximum degree of the underlying graph is 2. We describe a polynomial-time algorithm for this restricted case,

solution concept	input settings		results		
	preference restriction	must exist?	decision	shown in	
envy	sym., binary, $\Delta = 2$	X	P	Theorem 1	
"	sym., binary, $\Delta = 3$	X	NP-c.	Theorem 2	
weakly-justified envy	sym., binary, $\Delta = 2$	X	P	Theorem 3	
"	sym., binary, $\Delta = 3$	X	NP-c.	Theorem 4	
justified envy	sym., binary	✓	P	Observation 1	
"	binary	✓	P	Theorem 5	
"	ternary	X	NP-c.	Theorem 6	
"	sym., valuations $\{0, \dots, 6\}$	X	NP-c.	Theorem 7	

Table 1. Our main results.

which can decide if an envy-free partition into triples exists and construct such a partition if so (Theorem 1). We also show that the same decision problem is **NP**-complete even if the maximum degree of the underlying graph is 3 (Theorem 2).

In Section 3, we show a similar dichotomy for *wj*-envy-freeness: that an envy-free partition into triples may not exist, even if the preferences are binary and the maximum degree of the underlying graph is 2. Notably, the set of ASHG that do not contain *wj*-envy-free partitions into triples is strictly smaller than the set of ASHG that do not contain envy-free partitions into triples. We describe a more complex polynomial-time algorithm for this case, which can decide if an *wj*-envy-free partition into triples exists and construct such a partition if so (Theorem 3). As for envy-freeness, we also show that this decision problem is **NP**-complete even if the maximum degree of the underlying graph is 3 (Theorem 4).

In Section 4 we consider *j*-envy-freeness. We show that if preferences are binary but not necessarily symmetric, a *j*-envy-free partition into triples must exist and can be found in polynomial time (Theorem 5). We complement this result with two hardness results. The first is that a *j*-envy-free partition into triples may not exist, even if preferences are ternary but not symmetric, and to decide is **NP**-complete (Theorem 6). The second is that a *j*-envy-free partition into triples may not exist even if preferences are symmetric and the maximum valuation is 6, and the associated decision problem is **NP**-complete (Theorem 7).

2 Envy-freeness

Here we show that an ASHG with binary and symmetric preferences need not contain an envy-free partition into triples. We show that if $\Delta_{(N,E)} = 2$ then we can either construct a envy-free partition into triples in polynomial time or report that no such partition exists, but the associated decision problem is **NP**-complete even if $\Delta_{(N,E)} = 3$.

We remark that this threshold on maximum degree Δ is lower than the corresponding threshold for Partition Into Triangles, which is solvable in polynomial

time in graphs of maximum degree 3 and NP-complete for graphs of maximum degree 4 [8].

We begin by presenting a necessary and sufficient condition for the existence of an envy-free partition into triples in an ASHG in which $\Delta_{(N,E)} = 2$.

Lemma 1. *Consider some ASHG with binary and symmetric preferences in which $\Delta_{(N,E)} = 2$. Let P be the set of isolated agents, $\mathcal{Q}$ be the set of connected components each of size $3k_1 - 2$ for some $k_1 > 1$, and $\mathcal{R}$ be the set of connected components each of size $3k_2 - 1$ for some $k_2 \geq 1$. (N, E) has an envy-free partition into triples if and only if $2|\mathcal{Q}| + |\mathcal{R}| > |P|$.*

Proof sketch. An essential observation is that in a partition into triples, any agent not in P has envy if it has utility 0. Conversely, since the maximum degree is 2, if any agent in P has utility 1 or 2 then it must not be envious. The strategy for connected components with size divisible by 3 is straightforward: assign agents to triples sequentially such that each triple forms a path or triangle. For connected components in $\mathcal{Q}$, a similar strategy can be used to assign all but two agents to triples, but at least two isolated agents are needed for triples containing the leftover two agents. Similarly, for connected components in $\mathcal{R}$, at least one isolated agent is needed. $\square$

In the appendix (Section A.1) we describe formally an algorithm implementing Lemma 1 that can decide, in $O(|N|)$ time, if a given ASHG meets this condition, and if so constructs an envy-free partition into triples.

Theorem 1. *Consider an ASHG with binary and symmetric preferences in which $\Delta_{(N,E)} = 2$. In $O(|N|)$ time, we can either find an envy-free partition into triples or report that no such partition exists.*

We now consider ASHGs with binary and symmetric preferences in which the maximum degree of the underlying graph is 3. In contrast with Theorem 1, Theorem 2 shows that deciding if such an ASHG contains an envy-free partition into triples is NP-complete.

Theorem 2. *It is NP-complete to decide if an envy-free partition into triples exists in a given ASHG, even when preferences are symmetric and binary and $\Delta_{(N,E)} = 3$.*

Proof sketch. The reduction is from the following variant of 3-Satisfiability: “Given a boolean formula C in conjunctive normal form in which each clause $c_r \in C$ contains exactly three positive literals and each variable occurs in exactly three clauses, does there exist a truth assignment to the formula such that each clause contains exactly one true literal?” This problem, which we refer to as 3- CNF_+^3 -XSAT, is NP-complete [22]. Let $X(C)$ be the set of variables and m be the number of variables, which is also the number of clauses.

The reduction is one of local replacement [13]. For each variable $x_i \in X(C)$ construct three agents v_i^1, v_i^2, v_i^3 , which we shall refer to as the i^{th} variable gadget. Add $\{v_i^1, v_i^2\}$, $\{v_i^2, v_i^3\}$, and $\{v_i^3, v_i^1\}$ to E . Next, for each clause $c_r \in C$ construct

eight agents $d_r^1, \dots, d_r^8$, which we shall refer to as the r^{th} clause gadget. Add $\{d_r^1, d_r^4\}$, $\{d_r^2, d_r^5\}$, $\{d_r^3, d_r^8\}$, $\{d_r^4, d_r^6\}$, $\{d_r^5, d_r^7\}$, $\{d_r^6, d_r^8\}$, and $\{d_r^7, d_r^1\}$. We now connect the variable and clause gadgets. Consider each clause $c_r = \{x_i, x_j, x_k\}$. If c_r contains the j^{th} occurrence of x_i then add the edge $\{d_r^1, v_i^j\}$. Similarly, add an edge between d_r^3 and an agent in V_j depending on the index of the occurrence of x_j in the clause c_r and an edge between d_r^4 and an agent in V_k depending on the index of the occurrence of x_k in the clause c_r . Proving that the reduction is correct involves showing a satisfying truth assignment $f : W(C) \rightarrow \{\text{true}, \text{false}\}$ exists for C if and only if an envy-free partition into triples exists in the instance represented by the underlying graph (N, E) (which has maximum degree 3). $\square$

3 Weakly justified envy-freeness

In this section we consider wj-envy-freeness. As before in Section 2, we show that an ASHG may not contain a wj-envy-free partition into triples, even if preferences are symmetric and binary. We show that if $\Delta_{(N,E)} = 2$ then we can either construct a envy-free partition into triples in polynomial time or report that no such partition exists, but the associated decision problem is NP-complete even if $\Delta_{(N,E)} = 3$.

By definition, the requirement of a partition to be wj-envy-free is weaker than the requirement to be envy-free. This relationship is reflected in our results: considering ASHGs in which the maximum degree is 2, the set of ASHGs that do not contain a wj-envy-free partition is strictly larger than the set of ASHGs that do not contain an envy-free partition. In fact, we show that any ASHG with binary and symmetric preferences and maximum degree 2 has a wj-envy-free partition into triples except those belonging to a particularly restricted class.

3.1 Symmetric and binary preferences with maximum degree 2

We begin by defining a class of ASHGs, $\mathcal{I}_{\text{no}}^{\text{wj}}$, such that membership in the class is a necessary and sufficient condition for the non-existence of a wj-envy-free partition into triples, among ASHGs with binary and symmetric preferences.

Definition 1. *An ASHG belongs to $\mathcal{I}_{\text{no}}^{\text{wj}}$ if and only if the underlying graph comprises a set of disjoint 4-cycles and a single isolated agent.*

Lemma 2. *If an ASHG belongs to $\mathcal{I}_{\text{no}}^{\text{wj}}$ then it does not contain a wj-envy-free partition into triples.*

Proof sketch. In the full proof (in the appendix, Section A.2.2), we consider an arbitrary ASHG in $\mathcal{I}_{\text{no}}^{\text{wj}}$, represented by its underlying graph (N, E) , and a wj-envy-free partition into triples π . We deduce that any triple must contain agents in exactly two 4-cycles in (N, E) and that the agents in any 4-cycle in (N, E) must belong to exactly two or four triples. With reference to the definition of $\mathcal{I}_{\text{no}}^{\text{wj}}$, we show that these facts together lead to a contradiction. $\square$

Building on Lemma 2, we present Theorem 3. We describe a polynomial-time algorithm that, given an ASHG with binary and symmetric preferences in which

$\Delta_{(N,E)} = 2$, either returns a wj -envy-free partition into triples π or reports that (N, E) belongs to $\mathcal{I}_{no}^{wj}$. Together with Lemma 2, this establishes the fact that $\mathcal{I}_{no}^{wj}$ is a necessary and sufficient condition for the non-existence of a wj -envy-free partition into triples in ASHG with with binary and symmetric preferences.

Theorem 3. *In $O(|N|)$ time, we can either find a wj -envy-free partition into triples or report that (N, E) belongs to $\mathcal{I}_{no}^{wj}$ (and thus contains no wj -envy-free partition into triples).*

Proof sketch. In some aspects the approach taken is straightforward. For example, paths or cycles that contain a number of agents divisible by 3 are broken up into triples of three successively adjacent agents. This ensures no agent in such a component will be wj -envied. Other paths and cycles, except 4-cycles, are broken up in a similar fashion leaving one or two “surplus” agents per connected component. It is straightforward to show that if a surplus agent has wj -envy then it must be for another surplus agent. In fact, finding a maximal matching helps to ensure the utility of surplus agents is maximised, which in turn ensures that no surplus agent has wj -envy. More care is required in the assignment of the agents in 4-cycles to triples. The 12 agents in three 4-cycles can be assigned to four triples in a relatively straightforward way that ensures no agent is wj -envied in some resulting partition into triples. The main complexity of the algorithm stems from the case when the number of 4-cycles is not divisible by 3. We describe this algorithm in full in the appendix (Section A.2.1). $\square$

3.2 Symmetric and binary preferences with maximum degree 3

As before in Theorem 2, here we consider ASHG with binary and symmetric preferences in which $\Delta_{(N,E)} = 3$. Also as before, we reduce from $3\text{-CNF}_+^3\text{-XSAT}$.

Theorem 4. *It is NP-complete to decide if a wj -envy-free partition into triples exists in a given ASHG, even when preferences are symmetric and binary and $\Delta_{(N,E)} = 2$*

Proof sketch. There are three main differences between this reduction and the reduction for the ASHG envy-free decision problem shown in Section 2. The first

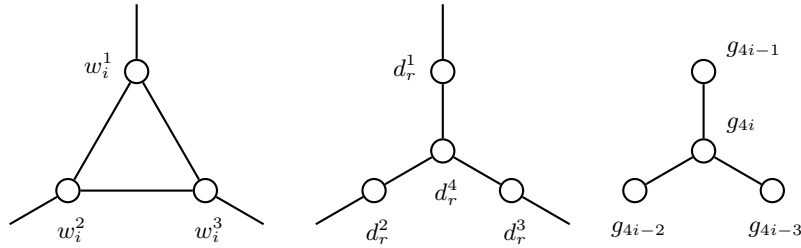


Fig. 1. A variable gadget V_i , clause gadget D_r , and enforcer gadget G_i , represented as undirected graphs, in the reduction relating to wj -envy-free partitions into triples.

difference is that we assume the number of variables is divisible by four. The second difference is that a number of “garbage collector” gadgets are constructed. The third is in the “only if” direction, in which we show that C is exactly satisfiable if a wj-envy-free partition π exists in the constructed instance (N, E) . In particular, we associate true literals with variable gadgets that belong to one triple (rather than three) and false literals with variable gadgets that belong to three triples (rather than one). In the full proof, we show that there is only one possible configuration of the triples of agents in the garbage collector gadgets in a wj-envy-free partition.

Given an instance C of $3\text{-CNF}_+^3\text{-XSAT}$, the reduction constructs an ASHG with binary and symmetric preferences (N, E) . The reduction, illustrated in Figure 1, is as follows. For each variable $x_i \in X(C)$ construct three agents v_i^1, v_i^2, v_i^3 , which we shall refer to as the i^{th} *variable gadget*. Add the edges $\{v_i^1, v_i^2\}$, $\{v_i^2, v_i^3\}$, and $\{v_i^3, v_i^1\}$ to E . Next, for each clause c_r in C construct four agents $d_r^1, \dots, d_r^4$, which we shall refer to as the r^{th} *clause gadget*. Add the edges $\{d_r^1, d_r^4\}$, $\{d_r^2, d_r^3\}$, and $\{d_r^3, d_r^4\}$. Recall that 3 divides m and $m = 4l$ for some integer $l > 1$. Construct $12l$ agents labelled $g_1, \dots, g_{12l}$. E . For any $1 \leq i \leq 3l$, we shall refer to $\{g_{4i-3}, g_{4i-2}, g_{4i-1}, g_{4i}\}$ as the i^{th} *garbage collector gadget*. For each $1 \leq i \leq 3l$, add the edges $\{g_{4i}, g_{4i-1}\}$, $\{g_{4i}, g_{4i-2}\}$, and $\{g_{4i}, g_{4i-3}\}$ to E . Note that each garbage collector gadget is an isolated claw $K_{1,3}$.

The variable and clause gadgets are connected as before: consider each clause $c_r = \{x_i, x_j, x_k\}$. If c_r contains the j^{th} occurrence of x_i then add the edge $\{d_r^1, v_i^j\}$. Similarly, add an edge between d_r^3 and an agent in V_j depending on the index of the occurrence of x_j in the clause c_r and an edge between d_r^4 and an agent in V_k depending on the index of the occurrence of x_k in the clause c_r .

Like before, we show that a satisfying truth assignment f exists for C if and only if a wj-envy-free partition into triples exists in the instance represented by the underlying graph (N, E) . The proof of the “only if” direction is similar to the envy-free reduction, except the agents in a true variable gadgets are assigned to one triple and the agents in a false variable gadget are assigned to three triples. Each agent in garbage collector gadget is assigned to a triple with one agent in a clause gadget and an adjacent agent in a variable gadget. The proof of the “if” direction is also similar to the analogous proof in the previous reduction, except we first show that any garbage collector agent must belong to a triple containing two adjacent non-garbage collector agents. $\square$

4 Justified envy-freeness

In this section we consider justified envy-freeness (j-envy-freeness). By definition, any partition that is envy-free or wj-envy-free is also j-envy-free, so we say that the latter solution concept is weaker than the two former concepts. In fact, Ueda observes that in a hedonic game, j-envy-freeness is also weaker than core stability [24]. In our setting, of fixed-size coalitions, an analogous observation is also true. These relationships are reflected in our complexity results.

In this section, we first show that any ASHG with binary preferences has a j-envy-free partition into triples, even if the preferences are not symmetric,

and that such a partition can be found in polynomial time. This contrasts with Theorems 2 and 4, which show that the same ASHG may not contain an envy-free or wj-envy-free partition into triples and that the associated decision problems are both NP-complete.

We then consider the existence of j-envy-free partitions into triples in ASHG with ternary preferences that are not necessarily symmetric. We show that such an ASHG may not contain a j-envy-free partition into triples, and the associated decision problem is NP-complete. This result implies the general case, showing that an arbitrary ASHG may not contain a j-envy-free partition into triples and that deciding if it does is NP-complete.

From this result, a natural question arises: is it the symmetry of agents' preferences that guarantees the existence of a j-envy-free partition into triples? We finally show that this is not the case, a j-envy-free partition into triples may not exist even if agents' preferences are symmetric, and the corresponding decision problem is NP-complete. We remark, however, that the instances shown that do not contain j-envy-free partitions into triples are relatively contrived and involve non-negative integer valuations up to 6. We leave open the case in which the preferences are symmetric and have maximum valuation between 1 and 6.

4.1 Binary preferences

A partition into triples π is *core stable* if it contains no blocking triple: a set of three agents S where $u_{\alpha_i}(S \setminus \{\alpha_i\}) > u_{\alpha_i}(\pi)$ for each $\alpha_i \in S$ [20]. It is straightforward to show that if a partition into triples is core stable then it is j-envy-free. From the existing result that a core stable partition must exist and can be found in polynomial time [19], Observation 1 follows.

Observation 1. *Given an ASHG with binary and symmetric preferences, a j-envy-free partition into triples must exist and can be found in $O(|N|^3)$ time.*

From the literature, we know that when preferences are binary but not necessarily symmetric, a stable partition into triples need not exist and the associated decision problem is NP-complete [19]. In contrast, we show that a j-envy-free partition into triples is bound to exist and can be found in polynomial time.

Theorem 5. *Given an ASHG with binary, but not necessarily symmetric preferences, a j-envy-free partition into triples must exist and can be found in $O(|N|^3)$ time.*

Proof sketch. Suppose (N, V) is an ASHG with binary preferences. From (N, V) , first construct another ASHG (N, V') with binary and symmetric preferences by deleting asymmetric arcs. Next, find a partition into triples π that is core stable in (N, V') and meets the following condition: if there exists a pair of agents $\alpha_i, \alpha_j \in N$ where $v'_{\alpha_i}(\alpha_j) = 1$ and $u'_{\alpha_i}(\pi) = u'_{\alpha_j}(\pi) = 0$ then no triple contains three agents each with utility 0. To do this, we can use Algorithm **findStableUW** as defined in [19] (this algorithm finds a core stable partition into triples that has at least half of the maximum welfare of any core stable partition into triples, which is equivalent). We claim that π is j-envy-free in (N, V) . The proof is

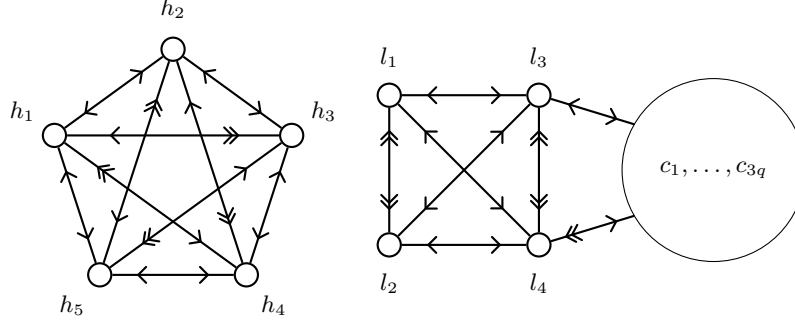


Fig. 2. The reduction from Directed Triangle Cover to the problem of deciding if a j -envy-free partition into triples exists given an ASHG with ternary (and not necessarily symmetric) preferences.

relatively direct. In it, we show that if there exists an agent with j -envy in (N, V) then that agent belongs to a triple that blocks the same partition in (N, V') , which is a contradiction. $\square$

4.2 Ternary preferences

A natural next step would be to ask if the polynomial-time algorithm described in Theorem 5 works if preferences are ternary, i.e. $v_{\alpha_i}(\alpha_j) \in \{0, 1, 2\}$. We show that this is not the case and the associated decision problem is NP-complete.

Theorem 6. *It is NP-complete to decide if a j -envy-free partition into triples exists in a given ASHG, even if preferences are ternary (and not necessarily symmetric).*

Proof sketch. We reduce from *Directed Triangle Cover* (DTC), which is as follows: “Given a simple directed graph $G = (W, A)$ where $W = \{w_1, w_2, \dots, w_{3q}\}$ for some integer q , does a *directed triangle cover* exist, i.e. a partitioning of W into q disjoint sets $X = \{X_1, X_2, \dots, X_q\}$, each set containing exactly three vertices, such that each $X_p = \{w_i, w_j, w_k\}$ in X is a directed 3-cycle, i.e. the arcs (w_i, w_j) , (w_j, w_k) , and (w_k, w_i) belong to A ?” We reduce from the special case of DTC in which G is antisymmetric (i.e. it contains no “bidirectional” arcs) [9,12].

The reduction constructs $3q+9$ agents, with valuations as shown in Figure 2. Note that $H = \{h_1, \dots, h_5\}$, $L = \{l_1, \dots, l_4\}$, and $C = \{c_1, \dots, c_{3q}\}$. Note also that $v_{c_i}(l_3) = v_{l_3}(c_i) = v_{l_4}(c_i) = 1$ and $v_{c_i}(l_4) = 2$ for each i where $1 \leq i \leq 3q$. For each i and j where $1 \leq i, j \leq 3q$, $v_{c_i}(c_j) = 2$ if $(w_i, w_j) \in A$ and 1 otherwise.

We remark that H can be used to derive an ASHG that contains no j -envy-free partition into triples. This ASHG comprises H as well as a single “isolated” agent α_z , where $v_{\alpha_z}(h_i) = v_{h_i}(\alpha_z) = 0$ for each i where $1 \leq i \leq 5$. In fact, the proof that this instance contains no j -envy-free partition into triples can be directly derived from the proof of Lemma 39 in the appendix.

The main aim of the full proof is to show that a directed triangle cover exists in G if and only if a j -envy-free partition into triples exists in the instance (N, V) . The proof of the “only if” direction is straightforward. In it, a partition π is constructed such that $\{h_1, h_2, h_3\}$, $\{h_4, l_1, l_2\}$, and $\{h_5, l_3, l_4\}$ belong to π and for each directed 3-cycle $X_p = \{w_i, w_j, w_k\}$ in X , the corresponding triple $\{c_i, c_j, c_k\}$ belongs to π . The proof of the “if” direction is more complex, but can be summarised as three deductions. First, by the design of H , at least two triples in π each contain exactly one agent in H . Second, these two triples each contain exactly one agent in H and also either both l_1 and l_2 or both l_3 and l_4 . The third deduction, which follows directly from the second, is that each agent in C must have utility 3 in π . With reference to the construction of C , from this third deduction it is straightforward to identify a directed triangle cover in G . $\square$

4.3 Symmetric and non-binary preferences

In this section we show that a j -envy-free partition into triples may not exist in an ASHG in which the preferences are ternary but not necessarily symmetric.

The proof of this is very similar to the proof presented in Section 4.2. In the reduction, shown in Figure 3, H is different and we reduce from Partition Into Triangles (PIT) [13] rather than Directed Triangle Cover.

Theorem 7. *It is NP-complete to decide if a j -envy-free partition into triples exists in a given ASHG, even when preferences are symmetric and the maximum possible valuation is 6.*

Proof sketch. This reduction constructs $3q + 15$ agents, with valuations as depicted in Figure 3. Like before, $H = \{h_1, \dots, h_{11}\}$, $L = \{l_1, \dots, l_4\}$, and $C =$

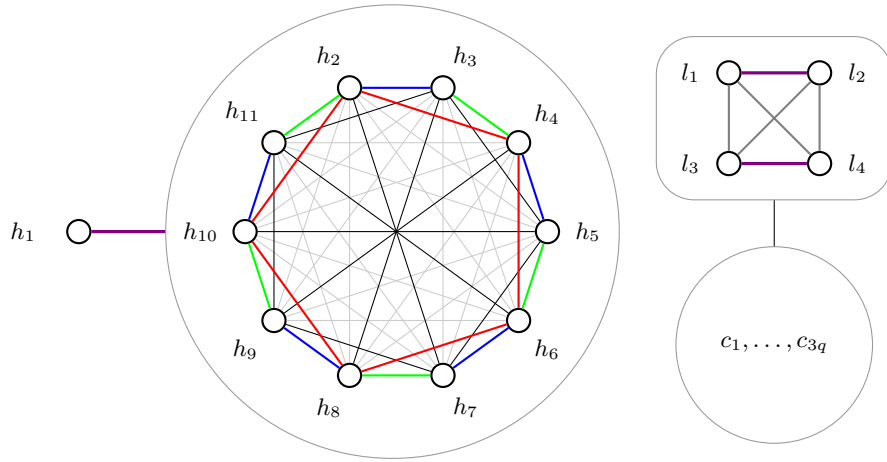


Fig. 3. The reduction from Partition Into Triangles to the problem of deciding if a j -envy-free partition into triples exists in an ASHG with symmetric preferences. Valuation colour key: red: 6, green: 5, blue: 4, black: 3, purple: 2, grey: 1.

$\{c_1, \dots, c_{3q}\}$. Note that $v_{h_1}(h_i) = 2$ for each $2 \leq i \leq 11$, $v_{l_i}(c_j) = 1$ for each $1 \leq i \leq 4$ and $1 \leq j \leq 3q$, and $v_{c_i}(l_j) = 1$ for each i where $1 \leq i \leq 3q$ and j where $1 \leq j \leq 4$. For each i and j where $1 \leq i, j \leq 3q$, $v_{c_i}(c_j) = 3$ if $\{w_i, w_j\} \in E$ and 2 otherwise.

Like the reduction for Theorem 6, here the proof of the “only if” direction is straightforward. Like before, a j -envy-free partition into triples is constructed that contains three triples that each contain three agents in H ; two triples that contain one agent in H and two agents in L ; and q triples in C , each one corresponding to a triangle in the partition into triangles. The “if” direction is also similar to the previous reduction. Like before, we deduce that two triples in π each contain one agent in H and either both l_1 and l_2 or both l_3 and l_4 . From this deduction it is then straightforward to show that each agent in C has utility 6 in π . Like before, with reference to the structure of C , it is then straightforward to show the existence of a partition into triangles. $\square$

Remark. We designed the reduction in Theorem 7 around a particular instance that contains no j -envy-free partition into triples (which can be constructed by removing l_2, l_3, l_4 and C from any instance produced by the reduction). This particular instance was discovered using a guided computer search using an Integer Linear Programming [25] model to test candidate solutions. One assumption made was the valuations of h_1 , in the spirit of an “undesired guest” [6]. We believe that this was an effective technique to identify ASHG’s that have no partition into triples satisfying a particular solution concept, although it remains open if the instance described contains a minimal number of agents.

5 Conclusion

We considered three successively weaker solution concepts: envy-freeness, weakly justified envy-freeness, and justified envy-freeness, and studied the existence of partitions into triples that satisfy each concept and the associated decision and construction problems.

From a practical perspective, the polynomial-time algorithms for envy-freeness and weakly-justified envy-freeness impose a strict requirement on the agents’ preferences, which may not be achievable in practice. On the other hand, the polynomial-time algorithm for justified envy-freeness can be applied more generally in situations involving “friends” and “neutrals”.

The scope for future work is wide. The immediate open problem relates to justified envy-freeness and preferences that are symmetric and ternary (a strengthening of Theorem 7). Weaker solution concepts, for example α -envy-freeness (see [17]) may lead to practical and interesting efficient algorithms. More generally, it would also be interesting to investigate cardinal preferences, for example considering complete, $\mathcal{B}$ -, $\mathcal{W}$ -, or lexicographic preferences.

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A Proofs in full

A.1 Envy-freeness

Lemma 1. *Consider some ASHG with binary and symmetric preferences in which $\Delta_{(N,E)} = 2$. Let P be the set of isolated agents, $\mathcal{Q}$ be the set of connected components each of size $3k_1 - 2$ for some $k_1 > 1$, and $\mathcal{R}$ be the set of connected components each of size $3k_2 - 1$ for some $k_2 \geq 1$. (N, E) has an envy-free partition into triples if and only if $2|\mathcal{Q}| + |\mathcal{R}| > |P|$.*

Proof. Consider an arbitrary ASHG represented by its underlying graph (N, E) . Label $P = \{p_1, p_2, \dots, p_{|P|}\}$, $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_{|\mathcal{Q}|}\}$, and $\mathcal{R} = \{R_1, R_2, \dots, R_{|\mathcal{R}|}\}$. Let $\mathcal{S} = \{S_1, S_2, \dots, S_{|\mathcal{S}|}\}$ be the set of connected components each of size $3k_3$ for some $k_3 \geq 1$.

To show the first direction, suppose $2|\mathcal{Q}| + |\mathcal{R}| \leq |P|$. We shall construct a partition into triples π and demonstrate that it is envy-free. We make two observations. First, no agent in P is envious in any partition into triples. Second, any agent with degree 1 or 2 must gain utility 1 or 2 in π , for otherwise that agent is immediately envious.

Construct π as follows. First, for each $1 \leq i \leq |\mathcal{S}|$, consider $S_i = (s_1, s_2, \dots, s_{3k_3})$. Add to π the triple $\{s_{3j-2}, s_{3j-1}, s_{3j}\}$ for each $1 \leq j \leq k_3$. It follows that each agent in each component S_i has utility either 1 or 2. By construction, for any agent in each S_i with utility 1 and two neighbours those two neighbours belong to different triples in π . It follows that no such agent is envious. Each agent in each S_i with utility 2 is trivially not envious.

Second, for each $1 \leq i \leq |\mathcal{R}|$, consider $R_i = (r_1, r_2, \dots, r_{3k_2-1})$. First, add to π the triple $\{r_{3j-2}, r_{3j-1}, r_{3j}\}$ for each $1 \leq j \leq k_2 - 1$. Second, add to π the triple $\{r_{3k_2-2}, r_{3k_2-1}, p_{2|\mathcal{Q}|+i}\}$ (recalling that $|P| \geq 2|\mathcal{Q}| + |\mathcal{R}|$). As before, it follows that each agent in each component R_i has utility either 1 or 2. Since the maximum degree of the underlying graph is 2, any such agent has at most one

neighbour not in its triple, and is therefore not envious in π (each agent in each Q_i with utility 2 is trivially not envious).

Third, for each $1 \leq i \leq |\mathcal{Q}|$, consider $Q_i = (q_1, q_2, \dots, q_{3k_1-2})$. First, add to π the triple $\{q_{3j-2}, q_{3j-1}, q_{3j}\}$ for each $1 \leq j \leq k_1 - 2$. Second, add to π the triple $\{q_{3k_1-5}, q_{3k_1-4}, p_i\}$ and $\{q_{3k_1-3}, q_{3k_1-2}, p_{2i}\}$. As before, it follows that each agent in each component Q_i has utility either 1 or 2. Since the maximum degree of the underlying graph is 2, any such agent has at most one neighbour not in its triple, and is therefore not envious in π (each agent in each Q_i with utility 2 is trivially not envious).

In total, we have added $|\mathcal{S}|k_3 + |\mathcal{R}|k_2 + |\mathcal{Q}|k_1$ triples to π . There remain $|P| - 2|\mathcal{Q}| - |\mathcal{R}|$ agents, in P , that have yet to be placed into triples in π . It must be that $|P| - 2|\mathcal{Q}| - |\mathcal{R}|$ is divisible by three. Arbitrarily add these agents to triples in π . Note that these remaining agents have degree 0 and utility 0 in π so are not envious. This completes the construction of π , which we have shown to be envy-free.

To show the second direction, suppose for a contradiction that (N, E) has an envy-free partition into triples π but does belong to $\mathcal{I}_{\text{eno}}$, and thus that $2|\mathcal{Q}| + |\mathcal{R}| > |P|$.

Since (N, E) has maximum degree 2, any triple in π either contains three agents with utility 2, two agents with utility 1 and one agent with utility 2, two agents with utility 1 and one agent with utility 0, or three agents with utility 0.

Consider some $Q_i \in \mathcal{Q}$. Note that $|Q_i| = 3k_1 - 2$ for some $k_1 > 1$ and no agent in Q_i has utility 0, for otherwise that agent is envious. It follows that there are two triples in π that contain at least one agent in $Q_i \in \mathcal{Q}$ and at least one agent with utility 0. In total, there must exist $2|\mathcal{Q}|$ triples in π each containing two agents in some $Q_i \in \mathcal{Q}$ and at some agent with utility 0. Similarly, since each $R_i \in \mathcal{R}$ has size $3k_2 - 1$ for some $k_2 \geq 1$, there exist at least $|\mathcal{R}|$ triples in π that contain two agents in some $R_i \in \mathcal{R}$ and at least one agent with utility 0. Since $2|\mathcal{Q}| + |\mathcal{R}| > |P|$, there exists at least one triple in π that contains no agent in P and at least one agent α_j with utility 0. It follows that α_j has degree 1 or more and therefore is envious, which is a contradiction. $\square$

Theorem 1. *Consider an ASHG with binary and symmetric preferences in which $\Delta_{(N,E)} = 2$. In $O(|N|)$ time, we can either find an envy-free partition into triples or report that no such partition exists.*

Proof. Lemma 1 gives a necessary and sufficient condition for the existence of an envy-free partition into triples in (N, E) , based on the number of connected components of different sizes. Suppose P is the set of isolated agents, $\mathcal{Q}$ is the set of connected components each of size $3k_1 - 2$ for some $k_1 > 1$, and $\mathcal{R}$ is the set of connected components each of size $3k_2 - 1$ for some $k_2 \geq 1$. Lemma 1 shows that an envy-free partition exists in the instance represented by (N, E) if and only if $2|\mathcal{Q}| + |\mathcal{R}| \leq |P|$. We outline one possible linear-time algorithm based on the constructive proof in Lemma 1. The output of the algorithm will be either $\perp$, if no envy-free partition into triples exists, or a labelling τ of each agent $\alpha_i \in N$ with some index $1 \leq \tau(\alpha_i) \leq |N|/3$ that represents the index

of $\pi(\alpha_i)$ in some arbitrary ordering of the triples in an envy-free partition into triples π .

The algorithm has three phases, as follows.

In the first phase, the algorithm constructs a stack P that contains all isolated agents in (N, E) . It also constructs a stack T , which contains exactly one agent per connected component of size 2 or more, such that if a connected component is a path T contains one of its endpoints. The construction of T can thus be performed in linear time.

The algorithm now enters the second phase, which is as follows. The algorithm maintains a counter r to track the label of the agent last labelled. Initially, $r = 1$. The algorithm pops an unmarked agent m_i from the stack T and marks $\tau(m_i) = 1$. It sets a counter c to 1, which will track the size of the connected component that contains m_i . It then identifies successive adjacent agents and labels each one with r , incrementing r by one every third agent, following the path or cycle in the underlying graph. The successive agents are therefore marked $1, 1, 1, 2, 2, 2, 3, 3, 3, \dots$. The counter c is updated to ensure that c is the size of this connected component. Eventually, either some agent with degree 1 or some previously labelled agent is discovered. In this case, there are three possibilities.

The first possibility is that $c = 3k_3$ for some $k_3 \geq 1$. In this case the algorithm pops some yet unlabelled m_i from the stack T and repeats the above process.

The second possibility is that $c = 3k_2 - 1$ for some $k_2 \geq 1$. In this case the algorithm pops some isolated agent p_i from the stack P and labels $\tau(p_i) = r$. If the stack P is empty then it must be that $2|\mathcal{Q}| + |\mathcal{R}| > |P|$ and thus the algorithm returns $\perp$. The algorithm then pops some yet unlabelled m_i from the stack T and repeats the above process.

The third possibility is that $c = 3k_1 - 2$ for some $k_1 > 1$. In this case it must be that exactly one agent α_j has been marked with r . The algorithm identifies the last agent α_l that was labelled with $r - 1$, which must be adjacent to α_j . It relabels $\tau(\alpha_l) = r$. It follows that exactly two adjacent agents are labelled with $r - 1$ and exactly two adjacent agents are labelled with r . The algorithm then pops two isolated agents p_g, p_h from the stack P and labels $\tau(p_g) = r$ and $\tau(p_h) = r - 1$. If $|P| < 2$ then it must be that $2|\mathcal{Q}| + |\mathcal{R}| > |P|$ and thus the algorithm returns $\perp$. The algorithm then pops some yet unlabelled m_i from the stack T and repeats the above process.

In the third phase, since the algorithm has not yet returned $\perp$, it must be that each agent with degree 1 or more has been labelled and therefore assigned to some triple in π . The algorithm arbitrarily assigns the remaining agents in P to triples in π by popping successive agents p_i from the stack P and labelling $\tau(p_i) = r$, incrementing r every third agent. $\square$

A.1.1 Proof of Theorem 2

We first provide some background on $3\text{-CNF}_+^3\text{-XSAT}$ and introduce the problem formally. $3\text{-CNF}_+^3\text{-XSAT}$ is a restricted version of the Exact Satisfiability problem [13] (XSAT). An instance of XSAT is a boolean formula in conjunctive normal form (CNF). We write $C = \{c_1, c_2, \dots, c_m\}$ to represent such a formula

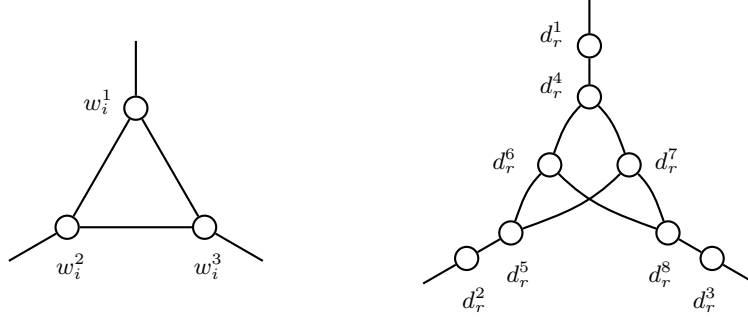


Fig. 4. A variable gadget V_i and clause gadget D_r , represented as undirected graphs, in the reduction relating to envy-free partitions into triples.

as the set of its clauses. We represent each clause $c_r \in C$ as a set of literals. Each literal is either an occurrence of a single variable or its negation. Given a formula, we write $X(C)$ to mean the set of variables contained in the formula. A *truth assignment* $f : W(C) \rightarrow \{\text{true}, \text{false}\}$ is an assignment of values to the set of variables. We say that an *exact model* is a truth assignment to the variables such that each clause contains exactly one true literal (and therefore exactly two false literals). Given an instance C , if an exact model exists then we say that C is *exactly satisfiable*. Otherwise we say that C is *exactly unsatisfiable*. Deciding if an instance C of XSAT is exactly satisfiable is **NP**-complete [23] even when each clause contains exactly three literals. Porschen et al. show that this problem remains **NP**-complete even when every literal is positive and each variable occurs in exactly three clauses [22]. They refer to this problem as $3\text{-CNF}_+^3\text{-XSAT}$. Note that in this problem $|V(C)| = m$.

We construct a polynomial-time reduction from $3\text{-CNF}_+^3\text{-XSAT}$ [22]. Given a formula instance C of $3\text{-CNF}_+^3\text{-XSAT}$, this reduction constructs an ASHG with binary and symmetric preferences represented by a graph (N, E) , such that C is exactly satisfiable if and only if an envy-free partition into triples exists in the instance represented by (N, E) . The reduction is illustrated in Figure 4.

For each variable $x_i \in X(C)$ construct three agents w_i^1, w_i^2, w_i^3 . Add the edges $\{w_i^1, w_i^2\}$, $\{w_i^2, w_i^3\}$, and $\{w_i^3, w_i^1\}$ to E . We shall refer to agents w_i^1, w_i^2 , and w_i^3 as the i^{th} *variable gadget*, W_i . Next, for each clause $c_r \in C$ construct eight agents $d_r^1, \dots, d_r^8$. Add the edges $\{d_r^1, d_r^4\}$, $\{d_r^2, d_r^5\}$, $\{d_r^3, d_r^8\}$, $\{d_r^4, d_r^6\}$, $\{d_r^5, d_r^7\}$, $\{d_r^6, d_r^8\}$, $\{d_r^7, d_r^6\}$, and $\{d_r^8, d_r^7\}$. We shall refer to agents $d_r^1, \dots, d_r^8$ as the r^{th} *clause gadget*, D_r . Finally, we shall connect the variable and clause gadgets. Consider each clause $c_r = \{x_i, x_j, x_k\}$. If c_r contains the first occurrence of x_i then add the edge $\{d_r^1, w_i^1\}$. Alternatively, if c_r contains the second occurrence of x_i then add the edge $\{d_r^1, w_i^2\}$. Finally, if c_r contains the third occurrence of x_i then add the edge $\{d_r^1, w_i^3\}$. Similarly, add an edge between d_r^2 and an agent in W_j depending on the index of the occurrence of x_j in the clause c_r . Finally,

add an edge between d_r^4 and an agent in W_k depending on the index of the occurrence of x_k in the clause c_r . We say that the clause gadget D_r is adjacent to the variable gadgets W_i , W_j , and W_k , and vice-versa.

Now each agent in a variable gadget has degree 3, the agents $d_r^4, d_r^5, d_r^7, d_r^8$ and d_r^1 for each $1 \leq r \leq m$ have degree 3, and the agents d_r^2, d_r^3 for each $1 \leq r \leq m$ have degree 2. It follows that the maximum degree of (N, E) is 3.

Lemma 1. *Suppose π is a partition into triples in (N, E) . If $u_{\alpha_i}(\pi) = 1$ and $\sigma(N(\alpha_i), \pi) = \delta(\alpha_i)$ then α_i is not envious in π .*

Proof. Suppose, to the contrary, that α_i envies some $\alpha_j \in N$. Then $u_{\alpha_i}(\pi(\alpha_j) \setminus \{\alpha_j\}) = 2$. It follows that two agents in $N(\alpha_i)$ belong to the same triple, $\pi(\alpha_j)$, so $\sigma(N(\alpha_i), \pi) < \delta(\alpha_i)$, which is a contradiction. $\square$

Lemma 2. *In the reduction, suppose π is a partition into triples in (N, E) . If $u_{\alpha_i}(\pi) = 0$ then α_i is envious in π .*

Proof. Note that by construction of (N, E) , each agent has degree at least one. Suppose then, for a contradiction, that there exists some $\alpha_i \in N$ where $u_{\alpha_i}(\pi) = 0$. There must exist some α_j where $\{\alpha_i, \alpha_j\} \in E$, so it follows that α_i envies both agents in $\pi(\alpha_j)$, which is a contradiction. $\square$

Lemma 3. *In the reduction, if C is exactly satisfiable then the ASHG represented by (N, E) contains an envy-free partition into triples.*

Proof. Suppose f is an exact model of C . We shall construct a partition into triples π that is envy-free. For each x_i in $X(C)$ where $f(x_i)$ is false, add $\{w_i^1, w_i^2, w_i^3\}$ to π . Next, consider each clause $c_r = \{x_i, x_j, x_k\}$ and the corresponding clause gadget D_r , labelling i, j, k such that W_i contains an agent adjacent to d_r^1 , W_j contains an agent adjacent to d_r^2 , and W_k contains an agent adjacent to d_r^3 . There are three cases: $f(x_i)$ is true while both $f(x_j)$ and $f(x_k)$ are false, $f(x_j)$ is true while both $f(x_i)$ and $f(x_k)$ are false, and $f(x_k)$ is true while both $f(x_i)$ and $f(x_j)$ are false. In the first case, suppose c_r contains the u^{th} occurrence of x_i . Add the triples $\{w_i^u, d_r^1, d_r^4\}$, $\{d_r^2, d_r^5, d_r^7\}$, and $\{d_r^3, d_r^6, d_r^8\}$. The constructions in the second and third cases are symmetric: in the second case, suppose c_r contains the u^{th} occurrence of x_j . Add the triples $\{w_j^u, d_r^2, d_r^5\}$, $\{d_r^1, d_r^4, d_r^6\}$, and $\{d_r^3, d_r^7, d_r^8\}$. In the third case, suppose c_r contains the u^{th} occurrence of x_k . Add the triples $\{w_k^u, d_r^3, d_r^8\}$, $\{d_r^1, d_r^4, d_r^6\}$, and $\{d_r^2, d_r^5, d_r^7\}$.

Now for any variable gadget W_i either $\sigma(W_i, \pi) = 1$ or $\sigma(W_i, \pi) = 3$. For each clause gadget D_r , there exist two triples in π that each contain three agents in D_r and one triple in π that contains two agents in D_r and one agent in some variable gadget. There are therefore three kinds of triple in π : those triples that contain three agents belonging to the same variable gadget; those triples that contain one agent in some variable gadget and two agents in a clause gadget; and those triples that contain three agents in the same clause gadget. We shall show that no triple of each kind contains an envious agent.

First, consider some triple $t \in \pi$ where $t = W_i$ for some variable gadget W_i . Since each agent in t has utility 2, no agent in t is envious.

Second, consider some triple $t \in \pi$ that contains one agent w_i^a in some variable gadget W_i and two agents in some clause gadget D_r . By the construction of π , either the triple contains $\{w_i^u, d_r^1, d_r^4\}$, $\{w_i^a, d_r^2, d_r^5\}$, or $\{w_i^a, d_r^3, d_r^8\}$. Suppose the triple contains $\{w_i^a, d_r^1, d_r^4\}$. Since $u_{d_r^1}(\pi) = 2$ clearly d_r^1 is not envious. By construction, $\sigma(W_i, \pi) = 3$ so since $u_{w_i^a}(\pi) \geq 1$ it follows by Lemma 1 that $w_i^u = a$ is also not envious. Similarly, since $\pi(d_r^6) \neq \pi(d_r^7)$ it follows that $\sigma(N(d_r^4), \pi) = 3$ so by Lemma 1 d_r^4 is also not envious. The proof in the two remaining cases, in which the triple comprises $\{w_i^a, d_r^2, d_r^5\}$ or $\{w_i^a, d_r^3, d_r^8\}$, is symmetric.

Third, consider some triple $t \in \pi$ where $t \subset D_r$ for some clause gadget D_r . There are four cases: either $t = \{d_r^1, d_r^4, d_r^6\}$, $t = \{d_r^2, d_r^5, d_r^7\}$, $t = \{d_r^3, d_r^6, d_r^8\}$, or $t = \{d_r^3, d_r^7, d_r^8\}$. Suppose $t = \{d_r^1, d_r^4, d_r^6\}$. Since $\sigma(N(d_r^1), \pi) = 2$ and $u_{d_r^1}(\pi) = 1$ by Lemma 1 d_r^1 is not envious. Since $\pi(d_r^2) \neq \pi(d_r^8)$, it must be that $\sigma(N(d_r^6), \pi) = 3$ so since $u_{d_r^6}(\pi) = 1$, by Lemma 1 d_r^6 is not envious. Since $u_{d_r^4}(\pi) = 2$, d_r^4 is also not envious. The proof in the three remaining cases, in which $t = \{d_r^2, d_r^5, d_r^7\}$, $t = \{d_r^3, d_r^6, d_r^8\}$, or $t = \{d_r^3, d_r^7, d_r^8\}$, is symmetric. It follows that no agent in t is envious. $\square$

Lemma 4. *In the reduction, if π is a envy-free partition into triples in (N, E) then for any variable gadget W_i , either $\sigma(W_i, \pi) = 1$ or $\sigma(W_i, \pi) = 3$.*

Proof. Since $|W_i| = 3$, clearly $1 \leq \sigma(W_i, \pi) \leq 3$. Suppose for a contradiction that $\sigma(W_i, \pi) = 2$. It must be that some triple in π contains exactly two agents in W_i and some third agent, which we label α_j . Label the remaining agent in the variable gadget w_i^a . Since the maximum degree of the instance is three, it must be that $u_{w_i^a}(\pi) \leq 1$. It follows that w_i^a envies α_j since $u_{w_i^a}(\pi(\alpha_j) \setminus \{\alpha_j\}) = 2$. $\square$

Lemma 5. *In the reduction, if the ASHG represented by (N, E) contains an envy-free partition into triples then C is exactly satisfiable.*

Proof. Suppose π is an envy-free partition into triples in (N, E) . By Lemma 4, for any variable gadget W_i either $\sigma(W_i, \pi) = 1$ or $\sigma(W_i, \pi) = 3$. Construct a truth assignment f in C by setting $f(x_i)$ to be true if $\sigma(W_i, \pi) = 3$ and false otherwise. Each variable x_i corresponds to exactly one variable gadget W_i so it follows that f is a valid truth assignment. By the construction of (N, E) , each clause c_r corresponds to exactly one clause gadget D_r . Each clause gadget is adjacent to three variable gadgets that correspond to the three variables in that clause. To show that f is an exact model of C , it is sufficient to show that for each clause gadget D_r there exists exactly one variable gadget W_i such that D_r is adjacent to W_i and $\sigma(W_i, \pi) = 3$.

Consider an arbitrary clause gadget D_r and the corresponding clause $c_r = \{x_i, x_j, x_k\}$, labelling i, j, k such that d_r^1 is adjacent to some agent in W_i , d_r^2 is adjacent to some agent in W_j and d_r^3 is adjacent to some agent in W_k . We shall show that a contradiction exists if either D_r is adjacent to two variable gadgets W_i, W_j where $\sigma(W_i, \pi) = \sigma(W_j, \pi) = 3$ or D_r is adjacent to three variable gadgets W_i, W_j, W_k where $\sigma(W_i, \pi) = \sigma(W_j, \pi) = \sigma(W_k, \pi) = 1$. The remaining possibility is that D_r is adjacent to exactly one variable gadget W_i

where $\sigma(W_i, \pi) = 3$ and exactly two variable gadgets W_j, W_k where $\sigma(W_j, \pi) = \sigma(W_k, \pi) = 1$.

First, suppose that D_r is adjacent to two variable gadgets W_i and W_j where $\sigma(W_i, \pi) = 3$ and $\sigma(W_j, \pi) = 3$. Suppose c_r contains the a^{th} occurrence of x_i and the b^{th} occurrence of x_j . Consider $\pi(w_i^a)$. By Lemma 2, no agent has utility 0 in π , so either $\{d_r^1, d_r^4\} \in \pi(w_i^a)$, $\{d_r^2, d_r^5\} \in \pi(w_i^a)$, or $\{d_r^3, d_r^8\} \in \pi(w_i^a)$. Similarly, either $\{d_r^1, d_r^4\} \in \pi(w_j^b)$, $\{d_r^2, d_r^5\} \in \pi(w_j^b)$, or $\{d_r^3, d_r^8\} \in \pi(w_j^b)$. By the symmetry of the clause gadget, suppose without loss of generality that $\{d_r^1, d_r^4\} \in \pi(w_i^a)$ and $\{d_r^2, d_r^5\} \in \pi(w_j^b)$. Consider d_r^6, d_r^7 and d_r^8 . Since no agent has utility 0, the only possibility is that $\{d_r^6, d_r^7, d_r^8\} \in \pi$. It then follows that d_r^4 envies d_r^8 , since $u_{d_r^4}(\pi) = 1$ and $u_{d_r^4}(\{d_r^6, d_r^7\}) = 2$, which is a contradiction.

Second, suppose that D_r is adjacent to three variable gadgets W_i, W_j, W_k where $\sigma(W_i, \pi) = \sigma(W_j, \pi) = \sigma(W_k, \pi) = 1$. Since $|D_r| = 8$, there exists triple in π that contains some agent $d_r^a \in D_r$ as well as some agent $\alpha_p \notin D_r$. It follows that either $u_{d_r^a}(\pi) = 0$ or $u_{\alpha_p}(\pi) = 0$, which contradicts Lemma 2. $\square$

Theorem 2. *It is NP-complete to decide if an envy-free partition into triples exists in a given ASHG, even when preferences are symmetric and binary and $\Delta_{(N,E)} = 3$.*

Proof. Suppose we are given an instance (N, E) and a partition into triples π . For any two agents $\alpha_i, \alpha_j \in N$ we can test if α_i envies α_j in constant time. We can therefore decide if π is envy-free in polynomial time, which shows that the above decision problem belongs to NP.

The above reduction from $3\text{-CNF}_+^3\text{-XSAT}$ can be performed in polynomial time. By Lemmas 3 and 5, the formula C is exactly satisfiable if and only if an envy-free partition into triples exists in the constructed instance (N, E) . It follows that the reduction is valid and the above decision problem is NP-hard. $\square$

A.2 Weakly justified envy-freeness

A.2.1 Symmetric and binary preferences with maximum degree 2

We recap the definition of $\mathcal{I}_{\text{no}}^{\text{wj}}$ given in Section 3.1.

Definition 1. *An ASHG belongs to $\mathcal{I}_{\text{no}}^{\text{wj}}$ if and only if the underlying graph comprises a set of disjoint 4-cycles and a single isolated agent.*

We now prove that any ASHG that belongs to $\mathcal{I}_{\text{no}}^{\text{wj}}$ must not contain a wj-envy-free partition into triples. Consider an ASHG with binary and symmetric preferences that belongs to $\mathcal{I}_{\text{no}}^{\text{wj}}$ and label the underlying graph $(\hat{N}, \hat{E})$. Suppose $\hat{\pi}$ is an arbitrary partition into triples in $(\hat{N}, \hat{E})$. Let $\sigma(S, \pi)$ be the number of triples in some set of triples π that contain at least one agent in some set of agents S .

Lemma 6. *For any 4-cycle R in $(\hat{N}, \hat{E})$, if some triple in $\hat{\pi}$ contains three agents in R then $\hat{\pi}$ is not wj-envy-free.*

Proof. Suppose $R = (r_1, r_2, r_3, r_4)$. By definition, $\{r_1, r_2\}, \{r_2, r_3\}, \{r_3, r_4\}, \{r_4, r_1\} \in E$. Without loss of generality we need only consider the case when $\{r_1, r_2, r_3\} \in \hat{\pi}$. In this case it must be that r_4 has wj-envy for r_2 since $u_{r_4}(\hat{\pi}) = 0 < 2 = u_{r_4}(\{r_1, r_3\})$, $v_{r_1}(r_2) = 1 \leq 1 = v_{r_1}(r_4)$, and $v_{r_3}(r_2) = 1 \leq 1 = v_{r_3}(r_4)$. $\square$

Lemma 7. *For any three distinct connected components C_1, C_2, C_3 in $(\hat{N}, \hat{E})$, if some triple in $\hat{\pi}$ contains one agent in each of C_1, C_2 , and C_3 then $\hat{\pi}$ is not wj-envy-free.*

Proof. Suppose some such triple in $\hat{\pi}$ exists. By the construction of $(\hat{N}, \hat{E})$, it must be that at least two of C_1, C_2 , and C_3 are 4-cycle. Assume without loss of generality that C_1 and C_2 are 4-cycles, so label C_1 as $R_1 = (r_1^1, r_1^2, r_1^3, r_1^4)$ and C_2 as $R_2 = (r_2^1, r_2^2, r_2^3, r_2^4)$. We may further assume that this triple is $\{r_1^1, r_2^1, c_3^1\}$ where $r_1^1 \in C_1$, $r_2^1 \in C_2$, and $c_3^1 \in C_3$. Note that $\{r_1^1, c_3^1\} \notin E$ and $\{r_2^1, c_3^1\} \notin E$. Now consider R_1 . If $u_{r_1^1}(\hat{\pi}) = 0$ then r_1^2 has wj-envy for c_3^1 since $u_{r_1^2}(\hat{\pi}) = 0 < 1 = u_{r_1^2}(\{r_1^1, r_2^1\})$, $v_{r_1^1}(c_3^1) = 0 \leq 1 = v_{r_1^1}(r_2^1)$, and $v_{r_2^1}(c_3^1) = 0 \leq 0 = v_{r_2^1}(r_1^2)$. It follows that $u_{r_1^2}(\hat{\pi}) \geq 1$ so it must be that $\hat{\pi}(r_1^2)$ contains r_1^3 . A symmetric argument shows that $u_{r_1^4}(\hat{\pi}) \geq 1$ and thus that $\hat{\pi}(r_1^4)$ must also contain r_1^3 . Now $\hat{\pi}(r_1^2) = \{r_1^2, r_1^3, r_1^4\}$ so by Lemma 6 it follows that $\hat{\pi}$ is not wj-envy-free. $\square$

Lemma 8. *For any triple $t \in \hat{\pi}$, if $\hat{\pi}$ is wj-envy-free then the agents in t belong to exactly two connected components in $(\hat{N}, \hat{E})$.*

Proof. Since t is a triple, the agents in t belong to either 1, 2, or 3 connected components. If $\hat{\pi}$ is wj-envy-free, then by Lemmas 6 and 7 the agents in t do not belong to either 1 or 3 connected components in $(\hat{N}, \hat{E})$. $\square$

Lemma 9. *For any 4-cycle R in $(\hat{N}, \hat{E})$, if $\hat{\pi}$ is wj-envy-free then $\sigma(R, \hat{\pi}) \in \{2, 4\}$.*

Proof. By definition, $2 \leq \sigma(R, \hat{\pi}) \leq 4$ for any 4-cycle R in $(\hat{N}, \hat{E})$. It suffices to show that if $\hat{\pi}$ is wj-envy-free then $\sigma(R, \hat{\pi}) \neq 3$ for any such R . Suppose then, for a contradiction, that $\hat{\pi}$ is wj-envy-free and there exists some 4-cycle R in $(\hat{N}, \hat{E})$ where $\sigma(R, \hat{\pi}) = 3$. Label $R = (r_1, r_2, r_3, r_4)$. Since $\sigma(R, \hat{\pi}) = 3$ there must exist one triple in $\hat{\pi}$ that contains exactly two agents in R and two triples in $\hat{\pi}$ that each contain exactly one agent in R . Label the former triple $\{r_{i_1}, r_{i_2}, \alpha_{j_1}\}$ and the latter two triples $\{r_{i_3}, \alpha_{j_2}, \alpha_{j_3}\}$ and $\{r_{i_4}, \alpha_{j_4}, \alpha_{j_5}\}$, where $\alpha_{j_1}, \dots, \alpha_{j_5} \in N \setminus R$. Since R is a 4-cycle it must be that $\{r_{i_1}, \alpha_{j_1}\} \notin E$ and $\{r_{i_2}, \alpha_{j_1}\} \notin E$. Similarly, $\{r_{i_3}, \alpha_{j_2}\} \notin E$ and $\{r_{i_3}, \alpha_{j_3}\} \notin E$. It must also be that either $\{r_{i_3}, r_{i_1}\} \in E$ or $\{r_{i_3}, r_{i_2}\} \in E$. We can now see that r_{i_3} has wj-envy for α_{j_1} since $u_{r_{i_3}}(\hat{\pi}) = 0 < 1 \leq u_{r_{i_3}}(\{r_{i_1}, r_{i_2}\})$, $v_{r_{i_1}}(\alpha_{j_1}) = 0 \leq v_{r_{i_1}}(r_{i_3})$, and $v_{r_{i_2}}(\alpha_{j_1}) = 0 \leq v_{r_{i_2}}(r_{i_3})$. $\square$

Lemma 2. *If an ASHG belongs to $\mathcal{I}_{n_0}^{wj}$ then it does not contain a wj-envy-free partition into triples.*

Proof. We considered an arbitrary ASHG that belongs to $\mathcal{I}_{\text{no}}^{\text{wj}}$ in which the preferences are binary and symmetric with the underlying graph $(\hat{N}, \hat{E})$, and that $\hat{\pi}$ was an arbitrary partition into triples of $(\hat{N}, \hat{E})$.

Suppose to the contrary that $\hat{\pi}$ is *wj-envy-free*. Construct a graph $(\mathcal{C}, \Gamma)$ where $\mathcal{C}$ is the set of connected components in $(\hat{N}, \hat{E})$. Γ is constructed as follows. For any triple $t \in \hat{\pi}$, it must be that the agents in t belong to exactly two connected components in $(\hat{N}, \hat{E})$ (Lemma 8). For each triple $t \in \hat{\pi}$ identify the two such connected components C_i, C_j in $(\hat{N}, \hat{E})$ and add the edge $\{C_i, C_j\}$ to Γ . By the design of $(\hat{N}, \hat{E})$, there exists exactly one connected component in $(\hat{N}, \hat{E})$ that is not a 4-cycle, which contains exactly one agent. Label this component C_1 . Label the remaining connected components in $(\hat{N}, \hat{E})$, which are all 4-cycles, $C_2, \dots, C_{|\mathcal{C}|}$. Since $|C_1| = 1$ it must be that exactly one triple in $\hat{\pi}$ contains the agent in C_1 so the degree of vertex C_1 in the graph $(\mathcal{C}, \Gamma)$ is 1. Consider the 4-cycles $C_2, \dots, C_{|\mathcal{C}|}$ in $(\hat{N}, \hat{E})$. By Lemma 9, it must be that $\sigma(C_i, \hat{\pi}) \in \{2, 4\}$ for each such 4-cycle C_i (where $2 \leq i \leq |\mathcal{C}|$). It follows that the degree of each vertex C_i in $\mathcal{C}$ where $2 \leq i \leq |\mathcal{C}|$ is either 2 or 4. It follows that the sum of the degrees of all vertices in $\mathcal{C}$ is odd, which is impossible. $\square$

Recall Theorem 3. We present Algorithm **wjPathsCycles** in full. Given an ASHG with binary and symmetric preferences in which the underlying graph (N, E) has maximum degree 2, Algorithm **wjPathsCycles** either returns a *wj-envy-free* partition into triples π or reports that (N, E) belongs to $\mathcal{I}_{\text{no}}^{\text{wj}}$.

In some aspects the approach taken by Algorithm **wjPathsCycles**. For example, paths or cycles that contain a number of agents divisible by 3 are broken up into triples of three successively adjacent agents. Other paths and cycles, except 4-cycles, are broken up in a similar fashion leaving one or two “surplus” agents per connected component. More care is required in the assignment of the agents in 4-cycles to triples. The 12 agents in three 4-cycles can be assigned to four triples in a relatively straightforward way that ensures no agent is *wj-envied* in some resulting partition into triples. The main complexity of Algorithm **wjPathsCycles** stems from the case when the number of 4-cycles is not divisible by 3. The pseudocode contains calls to five subroutines, which we describe in prose. The first four subroutines can be thought of as procedures to construct *wj-envy-free* partitions into triples in particular types of instances.

The first subroutine is Subroutine **nonC4Components**, shown in Algorithm 1. This subroutine takes as input a set of connected components $\mathcal{C}$ in (N, E) , none of which are 4-cycles. It returns a pair (T, S) where T is a set of triples of agents and S is a set of agents. For each component in $\mathcal{C}$, the corresponding set of triples in T is constructed in a straightforward way by breaking up the component into triples of three successively adjacent agents. This procedure leaves remaining at most two agents from each component, which are then added to S . It follows that the maximum degree of the subgraph induced by S in (N, E) is 1.

Algorithm 1 Subroutine **nonC4Components**

Input: a set of connected components $\mathcal{C}$ that are not 4-cycles

Output: a pair (T, S) where T is a set of triples of agents in $\mathcal{C}$ and S is a set of agents

$T \leftarrow \emptyset$

$S \leftarrow \emptyset$

for each connected component C in $\mathcal{C}$, labelling $C = (c_1, c_2, \dots, c_{|C|})$ **do**

for $i = 1$ to $\lfloor |C|/3 \rfloor$ **do**

$T \leftarrow T \cup \{c_{3i-2}, c_{3i-1}, c_{3i}\}$

end for

if $|C| \bmod 3 > 0$ **then**

$S \leftarrow S \cup \{c_{3\lfloor |C|/3 \rfloor + 1}, \dots, c_{|C|}\}$

end if

end for

return (T, S)

Lemma 10. Suppose $\mathcal{C}$ is a set of connected components in (N, E) that are not 4-cycles. Suppose (T, S) is returned by a call **nonC4Components**($\mathcal{C}$) and π is a partition into triples in (N, E) . For any agent $c_i \in \bigcup T$, if $\pi(c_i) \in T$ then c_i is not *wj-envious* in π .

Proof. Suppose $c_i \in \bigcup T$ belongs to some connected component $C \in \mathcal{C}$, which must not be a 4-cycle. By the construction of T in Subroutine **nonC4Components**, it must be that the triple in T that contains c_i either contains c_{i-1} or c_{i+1} . Since $\pi(c_i) \in T$ by assumption, it follows that $u_{c_i}(\pi) \geq 1$. If c_i has *wj-envy* in π then it must be that two agents not in $\pi(c_i)$ are adjacent to c_i in (N, E) . Since $u_{c_i}(\pi) \geq 1$ it follows that c_i has degree 3 in (N, E) , which is a contradiction. $\square$

Lemma 11. Suppose $\mathcal{C}$ is a set of connected components in (N, E) that are not 4-cycles. Suppose (T, S) is returned by a call **nonC4Components**($\mathcal{C}$) and π is a partition into triples in (N, E) . For any agent $c_j \in \bigcup T$, if $\pi(c_j) \in T$ then c_j is not *wj-envied* in π .

Proof. Suppose for a contradiction that some $c_j \in \bigcup T$ where $\pi(c_j) \in T$ is *wj-envied* in π . Consider the pseudocode of Subroutine **nonC4Components**. Let $i = \lceil j/3 \rceil$. It must be that the triple $t = \{c_{3i-2}, c_{3i-1}, c_{3i}\}$, which contains c_j , was added to T in the i^{th} iteration of the inner **for** loop, in the particular iteration of the outer **for** loop in which component C was identified. Note that $\{c_{3i-2}, c_{3i-1}\} \in E$ and $\{c_{3i-1}, c_{3i}\} \in E$, by definition. There are now three possibilities: $j = 3i - 2$, $j = 3i - 1$, and $j = 3i$.

Suppose either $j = 3i - 2$ or $j = 3i$. Since α_k has *wj-envy* for c_j it must be that $v_{c_{3i-1}}(\alpha_k) \geq v_{c_{3i-1}}(c_j) = 1$. It follows that $v_{c_{3i-1}}(\alpha_k) = v_{c_{3i-1}}(c_{3i-2}) = v_{c_{3i-1}}(c_{3i}) = 1$ and thus that c_{3i-1} has degree 3 in (N, E) , which is a contradiction.

Suppose then that $j = 3i - 1$. Since α_k has *wj-envy* for c_j it must be that $v_{c_{3i-2}}(\alpha_k) \geq v_{c_{3i-2}}(c_{3i-1}) = 1$ and $v_{c_{3i}}(\alpha_k) \geq v_{c_{3i}}(c_{3i-1}) = 1$. It follows that

$v_{c_{3i-2}}(\alpha_k) = v_{c_{3i}}(\alpha_k) = 1$. The only possibility is that C is a 4-cycle comprising $(c_{3i-2}, c_{3i-1}, c_{3i}, \alpha_k)$, which contradicts the statement of the lemma. $\square$

Lemma 12. *Subroutine `nonC4Components` terminates in $O(|\bigcup \mathcal{C}|)$ time and returns a pair (T, S) where $\bigcup \mathcal{C} = S \cup \bigcup T$.*

Proof. There are $O(|\mathcal{C}|)$ iterations of the outer **for** loop. In each iteration of the outer loop, some connected component C is identified, and the number of iterations of the inner **for** loop is $O(|C|)$. It follows that the total number of iterations of the inner **for** loop is $\sum_{C \in \mathcal{C}} |C| = O(|\bigcup \mathcal{C}|)$. In each iteration of the inner loop, a set of three agents is added to a set T , which can be performed in constant time, using an appropriate data structure for T . In each iteration of the outer loop, at most two agents are added to S , which can also be performed in constant time. It follows that the running time of Subroutine `nonC4Components` is thus $O(|\bigcup \mathcal{C}|)$.

For each connected component $C \in \mathcal{C}$, each agent in that component is either added to T in the inner **for** loop or is later added to S . It follows that $\bigcup \mathcal{C} = S \cup \bigcup T$. $\square$

The second subroutine is Subroutine `oneC4TwoSingles`, shown in Algorithm 2. This subroutine takes as input three connected components in (N, E) . The first, R , is a 4-cycle in (N, E) . The second and third, w_1 and w_2 , are other agents in (N, E) . It returns two triples in C , each of which contains two agents in R and either w_1 or w_2 .

Algorithm 2 Subroutine `oneC4TwoSingles`

Input: a 4-cycle $R = (r_1, r_2, r_3, r_4)$ and two other agents w_1, w_2

Output: a set of two triples

return $\{\{w_1, r_1, r_2\}, \{w_2, r_3, r_4\}\}$

Lemma 13. *Consider an arbitrary 4-cycle R and two other arbitrary agents w_1, w_2 in (N, E) . Suppose T is returned by a call `oneC4TwoSingles`(R, w_1, w_2). If π is a partition into triples in (N, E) where $T \subseteq \pi$ then no agent in $R \cup \{w_1, w_2\}$ is *wj-envied* in π .*

Proof. By the design of Subroutine `oneC4TwoSingles`, it must be that $T = \{\{w_1, r_1, r_2\}, \{w_2, r_3, r_4\}\}$, for some labelling of R where $R = (r_1, r_2, r_3, r_4)$. Suppose for a contradiction that some agent $\alpha_k \in N$ has *wj-envy* for some agent in $R \cup \{w_1, w_2\}$. By symmetry, we need only consider two cases: either α_k has *wj-envy* for r_1 or α_k has *wj-envy* for w_1 .

If α_k has *wj-envy* for r_1 then consider r_2 . Since $r_2 \in \pi(r_1)$ it must be that $v_{r_2}(\alpha_k) \geq v_{r_2}(r_1) = 1$ and thus that $v_{r_2}(\alpha_k) = 1$. The only possibility is that $\alpha_k = r_3$. This is a contradiction since $u_{r_3}(\pi) = 1 = u_{r_3}(\{r_1, w_2\})$, so r_3 does not have *wj-envy* for r_1 .

If α_k has wj-envy for w_1 then it must be that $u_{\alpha_k}(\{r_1, r_2\}) \geq 1$. The only possibility is that either $\alpha_k = r_3$ or $\alpha_k = r_4$. Since $u_{r_3}(\pi) = 1 = u_{r_3}(\{r_1, r_2\})$ and $u_{r_4}(\pi) = 1 = u_{r_4}(\{r_1, r_2\})$ it follows that neither r_3 nor r_4 have wj-envy for w_1 , which is a contradiction. $\square$

Lemma 14. *Subroutine `oneC4TwoSingles` terminates in constant time.*

Proof. A suitable choice of data structure for π allows the asymptotic running time of this subroutine to be $O(1)$. $\square$

The third subroutine is Subroutine `multipleOfThreeC4s`, shown in Algorithm 3. It takes as input a set $\mathcal{R}$ of 4-cycles in (N, E) , where the number of 4-cycles is $3q$ for some integer $q \geq 1$. It returns $4q$ triples, each of which contains two agents in one 4-cycle and one agent in a different 4-cycle. The agents in each 4-cycle are assigned to either two or four triples.

Algorithm 3 Subroutine `multipleOfThreeC4s`

Input: a set $\mathcal{R}$ of 4-cycles in (N, E) , where 3 divides $|\mathcal{R}|$, labelled $R_1, R_2, \dots, R_{3q}$ where $R_i = (r_i^1, r_i^2, r_i^3, r_i^4)$ ($1 \leq i \leq 3q$) **Output:** a set T of $4q$ triples

$T \leftarrow \emptyset$

for $d = 1$ to q **do**

$T \leftarrow T \cup \{ \{r_{3d-2}^1, r_{3d-2}^2, r_{3d-1}^1\}, \{r_{3d-2}^3, r_{3d-2}^4, r_{3d-1}^4\}, \{r_{3d-1}^2, r_{3d}^1, r_{3d}^2\}, \{r_{3d-1}^3, r_{3d}^3, r_{3d}^4\} \}$

end for

return T

Lemma 15. *Consider an arbitrary set $\mathcal{R}$ of 4-cycles in (N, E) where $|\mathcal{R}| = 3q$ for some $q \geq 1$. Suppose T is returned by a call `multipleOfThreeC4s`($\mathcal{R}$). If π is a partition into triples in (N, E) where $T \subseteq \pi$ then no agent in $\bigcup \mathcal{R}$ is wj-envied in π .*

Proof. Suppose to the contrary that some agent in $\bigcup \mathcal{R}$ is wj-envied in π . By the design of Subroutine `multipleOfThreeC4s` it must be that some such agent was labelled r_j^i for some i where $1 \leq i \leq 4$ and j where $1 \leq j \leq 3q$. By the pseudocode of this subroutine, it must be that some triple containing r_j^i was added to T in the d'^{th} iteration of the **for** loop, where $d' = \lceil j/3 \rceil$. We show that no agent in any of the four triples added to T in this iteration is wj-envied.

In fact, by the symmetric construction of the four triples in T in this d'^{th} iteration of the loop, it suffices to consider only the triple $\{r_{3d'-2}^1, r_{3d'-2}^2, r_{3d'-1}^1\}$.

Suppose first that some agent $\alpha_k \in N$ has wj-envy for $r_{3d'-2}^1$. Since $v_{r_{3d'-2}^2}^{r_{3d'-2}^1}(r_{3d'-2}^1) = 1$ it must be that $v_{r_{3d'-2}^2}(\alpha_k) = 1$. The only possibility is that $\alpha_k = r_{3d'-2}^3$. This is a contradiction since, by construction, $\pi(r_{3d'-2}^3) = \{r_{3d'-2}^3, r_{3d'-2}^4, r_{3d'-1}^4\}$ so $u_{r_{3d'-2}^3}(\pi) = 1 = u_{r_{3d'-2}^3}(\{r_{3d'-2}^2, r_{3d'-1}^1\})$ and thus $r_{3d'-2}^3$ does not have wj-envy for $r_{3d'-2}^1$. A symmetric argument shows that if some α_k has wj-envy for

$r_{3d'-2}^2$ then it must be that $\alpha_k = r_{3d'-2}^4$, which leads to a contradiction since $u_{r_{3d'-2}^4}(\pi) = 1$.

Suppose finally that that some agent $\alpha_k \in N$ has wj-envy for $r_{3d'-1}^1$. It follows that $u_{\alpha_k}(\{r_{3d'-2}^1, r_{3d'-2}^2\}) \geq 1$ so either $\alpha_k = r_{3d'-2}^3$ or $\alpha_k = r_{3d'-2}^4$. If $\alpha_k = r_{3d'-2}^3$ then by construction $\pi(r_{3d'-2}^3) = \{r_{3d'-2}^3, r_{3d'-2}^4, r_{3d'-1}^4\}$ and thus $u_{r_{3d'-2}^3}(\pi) = 1$. Since $u_{r_{3d'-2}^3}(\pi) = 1 = u_{r_{3d'-2}^3}(\{r_{3d'-2}^1, r_{3d'-2}^2\})$ it follows that $r_{3d'-2}^3$ does not in fact wj-envy $r_{3d'-1}^1$, which is a contradiction. A similar argument applies if $\alpha_k = r_{3d'-2}^4$. $\square$

Lemma 16. *Subroutine `multipleOfThreeC4s` terminates in $O(|\mathcal{R}|)$ time.*

Proof. Consider the `for` loop in the subroutine. In each iteration, four triples are added to T , which can be performed in constant time. Since there are exactly q iterations it follows that the running time of this subroutine is $O(|\mathcal{R}|)$. $\square$

The fourth subroutine is Subroutine `configureSurplusAgents`, shown in Algorithm 4. It takes as input a set of agents $\hat{S} \subseteq N$ of agents where $3 \mid |\hat{S}|$ and the subgraph induced by $\hat{S}$ in (N, E) has maximum degree 1. It returns a set T of $|\hat{S}|/3$ triples. In the context of Algorithm `wjPathsCycles`, this subroutine will be called with a subset of the “surplus” agents in the second element of the tuple returned by Subroutine `nonC4Components`.

Algorithm 4 Subroutine `configureSurplusAgents`

Input: a set $\hat{S} \subseteq N$ of agents where $3 \mid |\hat{S}|$ and the maximum degree of the subgraph induced by $\hat{S}$ in (N, E) is 1

Output: a set of $|\hat{S}|/3$ triples

$P \leftarrow$ the set of agents with degree 0 in the subgraph induced by $\hat{S}$ in (N, E)

labelling $P = \{p_1, p_2, \dots, p_{|P|}\}$

$\mathcal{Q} \leftarrow$ a set containing each pair of agents $\{q_i, q_j\} \subset \hat{S}$ where $\{q_i, q_j\} \in E$, labelling

$\mathcal{Q} = \{Q_1, Q_2, \dots, Q_{|\mathcal{Q}|}\}$

$X \leftarrow \emptyset$

if $|\mathcal{Q}| \geq |\hat{S}|/3$ **then**

$X \leftarrow \{q_1, \dots, q_{|\hat{S}|/3}\}$

else

$\triangleright$ by Lemma 17, $|P| > 2(|\hat{S}|/3 - |\mathcal{Q}|)$

$\mathcal{W} \leftarrow \{\{p_i, p_{2i}\} : 1 \leq i \leq |\hat{S}|/3 - |\mathcal{Q}|\}$

$X = \mathcal{Q} \cup \mathcal{W}$

end if

$Y \leftarrow \hat{S} \setminus \bigcup X$ $\triangleright$ note that X is a set of pairs of agents while Y is a set of individual agents
 $\triangleright$ suppose $X = \{x_1, x_2, \dots, x_{|\hat{S}|/3}\}$ and $Y = \{y_1, y_2, \dots, y_{|\hat{S}|/3}\}$

return $\{x_i \cup y_i : 1 \leq i \leq |\hat{S}|/3\}$

Lemma 17. *In Subroutine `configureSurplusAgents`, after initialising P and $\mathcal{Q}$, if $|\mathcal{Q}| < |\hat{S}|/3$ then $|P| > 2(|\hat{S}|/3 - |\mathcal{Q}|)$.*

Proof.

$$\begin{aligned}
 |P| &= |\hat{S}| - 2|\mathcal{Q}| && \text{by definition of } P \text{ and } \mathcal{Q} \\
 &> \frac{2|\hat{S}|}{3} - 2|\mathcal{Q}| && \text{since } |\hat{S}| \geq 1 \\
 &= 2\left(\frac{|\hat{S}|}{3} - |\mathcal{Q}|\right) . && \square
 \end{aligned}$$

Lemma 18. *Consider an arbitrary set $\hat{S} \subseteq N$ where $3 \mid \hat{S}$ and the maximum degree of the subgraph induced by $\hat{S}$ in (N, E) is 1. Suppose T is returned by a call `configureSurplusAgents`($\hat{S}$). If π is a partition into triples in (N, E) where $T \subseteq \pi$ then no agent in $\hat{S}$ has *wj-envy* in π for any other agent in $\hat{S}$.*

Proof. From the pseudocode, it is straightforward to show that Subroutine `configureSurplusAgents` is bound to terminate and must return a set T of disjoint triples where $\bigcup T = \hat{S}$.

Suppose for a contradiction that some agent $\alpha_{j_1} \in \hat{S}$ has *wj-envy* for an agent $\alpha_{k_1} \in \hat{S}$ where $\pi(\alpha_{j_1}) = \{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}\}$ and $\pi(\alpha_{k_1}) = \{\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3}\}$. It follows that $u_{\alpha_{j_1}}(\{\alpha_{k_2}, \alpha_{k_3}\}) \geq 1$. Without loss of generality assume that $v_{\alpha_{j_1}}(\alpha_{k_2}) = 1$, or equivalently that $\{\alpha_{j_1}, \alpha_{k_2}\} \in E$. By the definition of $\mathcal{Q}$, it must be that $\{\alpha_{j_1}, \alpha_{k_2}\} \in \mathcal{Q}$.

Note that if $|\mathcal{Q}| < |\hat{S}|/3$ then, by the pseudocode, for every $\{q_a, q_b\} \in \mathcal{Q}$ it must be that $q_a \in \pi(q_b)$. Since $\{\alpha_{j_1}, \alpha_{k_2}\} \in \mathcal{Q}$ and $\alpha_{j_1} \notin \pi(\alpha_{k_2})$ it follows that $|\mathcal{Q}| \geq |\hat{S}|/3$.

By the pseudocode, for each triple $r \in T$ there exists some pair $\{q_a, q_b\} \in \mathcal{Q}$ where $\{q_a, q_b\} \subset r$. Since $\mathcal{Q}$ is agent-disjoint (and we established that $\{\alpha_{j_1}, \alpha_{k_2}\} \in \mathcal{Q}$) the only possibility is that $\{\alpha_{k_1}, \alpha_{k_3}\} \in \mathcal{Q}$. By the definition of $\mathcal{Q}$ it must be that $v_{\alpha_{k_1}}(\alpha_{k_3}) = 1$. Since α_{j_1} has *wj-envy* for α_{k_1} it must be that $v_{\alpha_{k_3}}(\alpha_{j_1}) \geq v_{\alpha_{k_3}}(\alpha_{k_1})$ so it follows that $v_{\alpha_{k_3}}(\alpha_{j_1}) = 1$. Now $\{\alpha_{k_3}, \alpha_{k_1}\} \in E$ and $\{\alpha_{k_3}, \alpha_{j_1}\} \in E$ so α_{k_3} has degree 2 in the subgraph induced by $\hat{S}$ in (N, E) , which is a contradiction. $\square$

Lemma 19. *Subroutine `configureSurplusAgents` terminates in $O(|\hat{S}|)$ time.*

Proof. The sets of pairs X and W , the set of agents Y , and the returned set of triples T can all be constructed in $O(|\hat{S}|)$ time. $\square$

The fifth subroutine is Subroutine `pickLowDegree`. This subroutine takes as input a set S and integer $k \geq 1$, such that the maximum degree of the subgraph induced by S in (N, E) is 1. It returns a set of k agents in S such that the sum of the degrees of the agents returned in the subgraph induced by S in (N, E) is minimised. Since the maximum degree of the subgraph induced by S in (N, E) is 1, this subroutine can be implemented to run in $O(|N|)$ time.

We now present Algorithm `wjPathsCycles`, shown in Algorithm 5. The overall strategy of this algorithm is as follows. First Subroutine `nonC4Components` is used to break up connected components that are not 4-cycles into a set of triples

T , in which each triple contains three successively adjacent agents, and a set S of surplus agents. The algorithm then constructs a set $\mathcal{R}$ of all 4-cycles in (N, E) . If $|\mathcal{R}|$ is divisible by three, Subroutine **multipleOfThreeC4s** is called and all agents in $\bigcup \mathcal{R}$, i.e. all agents belonging to 4-cycles, are assigned to triples in π . If $|\mathcal{R}|$ is not divisible by three then there are two cases. In the first, the ASHG is identified as belonging to $\mathcal{I}_{no}^{wj}$. In the second, a set of surplus agents in S are used to assign the agents belonging to either one or two 4-cycles to triples in π . This set is chosen using Subroutine **pickLowDegree**, which (as we shall see later) ensures that no agent in this set will be wj -envious in π . Next, the remaining 4-cycles (the number of which is divisible by three) are then added to triples in π using Subroutine **multipleOfThreeC4s**. The final step (in the case in which the ASHG does not belong to $\mathcal{I}_{no}^{wj}$) is a call to Subroutine **configureSurplusAgents** and the assignment of all remaining agents in S to triples in π .

Algorithm 5 Algorithm `wjPathsCycles`

Input: an ASHG with binary and symmetric preferences represented by its underlying graph (N, E) with maximum degree 2

Output: either a wj -envy-free partition into triples π or “belongs to $\mathcal{I}_{no}^{wj}$ ”

$\pi \leftarrow \emptyset; \hat{T} \leftarrow \emptyset; \hat{S} \leftarrow \emptyset$

$\mathcal{C} \leftarrow$ the set of all connected components in (N, E) that are not 4-cycles

$(T, S) \leftarrow \text{nonC4Components}(\mathcal{C})$

$\mathcal{R} \leftarrow$ the set of 4-cycles in (N, E) , labelling $\mathcal{R} = \{R_1, R_2, \dots, R_{|\mathcal{R}|}\}$

$l \leftarrow 0$

if $|\mathcal{R}| \bmod 3 = 2$ **then**

if $|S| \geq 4$ **then**

$\{w_1, w_2, w_3, w_4\} \leftarrow \text{pickLowDegree}(S, 4)$

$\hat{S} \leftarrow S \setminus \{w_1, \dots, w_4\}$

$\hat{T} \leftarrow T$

else if $|T| \geq 1$ **then**

$\triangleright$ it must be that $|S| = 1$ (see Lemma 21)

$w_1 \leftarrow$ the agent in S

$\hat{t} \leftarrow$ some triple in T

$\{w_2, w_3, w_4\} \leftarrow \hat{t}$

$\hat{S} \leftarrow \emptyset$

$\hat{T} \leftarrow T \setminus \hat{t}$

else

return “ (N, E) belongs to $\mathcal{I}_{no}^{wj}$ ”

end if

$\pi \leftarrow \pi \cup \text{oneC4TwoSingles}(R_1, w_1, w_2) \cup \text{oneC4TwoSingles}(R_2, w_3, w_4)$

$l \leftarrow 2$

else if $|\mathcal{R}| \bmod 3 = 1$ **then**

$\triangleright$ it must be that $|S| \geq 2$ (see Lemma 22)

$\{w_1, w_2\} \leftarrow \text{pickLowDegree}(S, 2)$

$\hat{S} \leftarrow S \setminus \{w_1, w_2\}$

$\hat{T} \leftarrow T$

$\pi \leftarrow \pi \cup \text{oneC4TwoSingles}(R_1, w_1, w_2)$

$l \leftarrow 1$

else

$\triangleright$ it must be that $|\mathcal{R}| \bmod 3 = 0$

$\hat{S} \leftarrow S$

$\hat{T} \leftarrow T$

end if

$\mathcal{R}' \leftarrow \{R_{l+1}, R_{l+2}, \dots, R_{|\mathcal{R}|}\}$

$\triangleright$ note it must be that $3 \mid (|\mathcal{R}| - l)$

$\pi \leftarrow \pi \cup \text{multipleOfThreeC4s}(\mathcal{R}') \cup \hat{T} \cup \text{configureSurplusAgents}(\hat{S})$

return π

Lemma 20. *In Algorithm `wjPathsCycles`, $S \cup \bigcup T$ is the set of agents that do not belong to 4-cycles in (N, E) .*

Proof. This follows immediately by Lemma 12. □

We first prove two lemmas that show that S is large enough to extract the number of agents required.

Lemma 21. *In Algorithm `wjPathsCycles`, after initialising $\mathcal{R}$, if $|\mathcal{R}| \bmod 3 = 2$ and $|S| < 4$ then $|S| = 1$.*

Proof. Suppose $|\mathcal{R}| \bmod 3 = 2$ and $|S| < 4$ after initialising $\mathcal{R}$. Then there exists some constant $k_1 \geq 0$ such that $|\mathcal{R}| = 3k_1 + 2$, so the number of agents in N that belong to 4-cycles is $4|\mathcal{R}| = 12k_1 + 8$. It follows that the number of agents in N that do not belong to 4-cycles is $|N| - 12k_1 - 8$. Since $|N| - 12k_1 - 8 \bmod 3 = 1$ there exists some constant $k_2 \geq 0$ such that the number of agents in N that do not belong to 4-cycles is $3k_2 + 1$. By Lemma 20, $S \cup \bigcup T$ is the set of agents that do not belong to 4-cycles. Since $|S \cup \bigcup T| = 3k_2 + 1$, T is a set of disjoint triples, and $|S| < 4$, it must be that $k_2 = 0$ and $|S| = 1$. $\square$

Lemma 22. *In Algorithm `wjPathsCycles`, after initialising $\mathcal{R}$, if $|\mathcal{R}| \bmod 3 = 1$ then $|S| \geq 2$.*

Proof. Suppose $|\mathcal{R}| \bmod 3 = 1$ after initialising $\mathcal{R}$. Then there exists some constant $k_1 \geq 0$ such that $|\mathcal{R}| = 3k_1 + 1$, so the number of agents in N that belong to 4-cycles is $4|\mathcal{R}| = 12k_1 + 4$. It follows that the number of agents in N that do not belong to 4-cycles is $|N| - 12k_1 - 4$. Since $(|N| - 12k_1 - 4) \bmod 3 = 2$ there exists some constant $k_2 \geq 0$ where the number of agents in N that do not belong to 4-cycles is $3k_2 + 2$. By Lemma 20, $S \cup \bigcup T$ is the set of agents that do not belong to 4-cycles. Since $|S \cup \bigcup T| = 3k_2 + 2$ and T is a set of disjoint triples it must be that $|S| \geq 2$. $\square$

We now show that Algorithm `wjPathsCycles` is bound to terminate and has a linear running time with respect to the number of agents.

Lemma 23. *Algorithm `wjPathsCycles` terminates in $O(|N|)$ time.*

Proof. The pseudocode describes the algorithm at a high level. To analyse the worst-case asymptotic time complexity we describe one possible system of data structures and analyse the algorithm with respect to the number of basic operations on these data structures. We begin the analysis at the start of the pseudocode.

The initialisation of π , $\hat{T}$ and $\hat{S}$ can be performed in constant time. The set of connected components $\mathcal{C}$ that are not 4-cycles can be identified in $O(|N|)$ time using breadth-first search, since the maximum degree of (N, E) is two.

By Lemma 12, the call to Subroutine `nonC4Components` takes $O(|\bigcup \mathcal{C}|) = O(|N|)$ time.

Like $\mathcal{C}$, the set of connected components $\mathcal{R}$ that are 4-cycles can be constructed in $O(|N|)$ time. Each nested branch of the `if/else` statement involves removing a constant number of elements from S , at most two calls to Subroutine `oneC4TwoSingles` (which has constant running time by Lemma 16), and an assignment to $\hat{T}$ and $\hat{S}$ (which can be performed in $O(|N|)$ time). It follows that the total running time of the `if/else` statement is $O(|N|)$.

By Lemma 16, Subroutine `multipleOfThreeC4s` has $O(|\mathcal{R}|) = O(|N|)$ running time. By Lemma 19, Subroutine `configureSurplusAgents` has $O(|S|) = O(|N|)$ running time. It follows that the asymptotic worst-case running time of Algorithm `wjPathsCycles` is $O(|N|)$. $\square$

Having established that Algorithm `wjPathsCycles` is bound to terminate, we prove its correctness using a sequence of lemmas. First we show that if (N, E) belongs to $\mathcal{I}_{no}^{wj}$ then the algorithm correctly identifies it as such.

Lemma 24. *If (N, E) belongs to $\mathcal{I}_{no}^{wj}$ then Algorithm `wjPathsCycles` returns “ (N, E) belongs to $\mathcal{I}_{no}^{wj}$ ”.*

Proof. Suppose (N, E) belongs to $\mathcal{I}_{no}^{wj}$. In the algorithm, the set of connected components $\mathcal{C}$ that are not 4-cycles contains exactly one element C_1 where C_1 contains a single agent c_1 in (N, E) . By Lemma 12, Subroutine `nonC4Components` must return $(\emptyset, \{c_1\})$ so $T = \emptyset$ and $S = \{c_1\}$. Consider the outermost `if/else` statement in the algorithm. By Definition 1, it must be that $|N| = 4|\mathcal{R}| + 1$. Since $|N|$ is divisible by three, it must be that $4|\mathcal{R}| + 1 \bmod 3 = 0$. This implies that $4|\mathcal{R}| + 4 \bmod 3 = 0$ so $4(|\mathcal{R}| + 1) \bmod 3 = 0$. Since $3 \nmid 4$ it must be that $|\mathcal{R}| + 1 \bmod 3 = 0$ and thus that $|\mathcal{R}| \bmod 3 = 2$. It follows that the algorithm enters the first branch of the outermost `if/else` statement. Since $|S| = 1 < 4$ and $T = \emptyset$ the algorithm must then return “ (N, E) belongs to $\mathcal{I}_{no}^{wj}$ ”. $\square$

We now consider the case in which (N, E) does not belong to $\mathcal{I}_{no}^{wj}$. We first show that in this case Algorithm `wjPathsCycles` returns a partition into triples.

Lemma 25. *If (N, E) does not belong to $\mathcal{I}_{no}^{wj}$ then Algorithm `wjPathsCycles` returns a partition into triples π .*

Proof. Consider an arbitrary connected component C in (N, E) . We show that each agent in C is added to exactly one triple in π .

Suppose C is not a 4-cycle. By the design of Algorithm `wjPathsCycles`, exactly one call is made to Subroutine `nonC4` with argument C . Consider an arbitrary agent $c_i \in C$. There are two cases: either $i \leq \lfloor |C|/3 \rfloor$ or $i > \lfloor |C|/3 \rfloor$. In the former case, exactly one triple containing c_i is added to T in Subroutine `nonC4`, which is then added to π in the main algorithm. In the latter case, c_i is eventually added to S . We can see from the pseudocode of Subroutine `configureSurplusAgents` that c_i is therefore eventually added to exactly one triple in π .

Suppose C is a 4-cycle, so by definition $C \in \mathcal{R}$. If $C \in \mathcal{R}'$ then each agent in C is added to exactly one triple in π in some call to Subroutine `multipleOfThreeC4s`. If $C \notin \mathcal{R}'$ then some call to Subroutine `oneC4TwoSingles` was made with the first argument equal to C and the returned set of two triples was then added to π . It follows that each agent in each 4-cycle is added to exactly one triple in π . $\square$

We now show that if (N, E) does not belong to $\mathcal{I}_{no}^{wj}$ then the algorithm returns a partition into triples π that is `wj-envy-free`. In the next four lemmas we consider

subsets of N and show that in each subset no agent is wj -envied in π . The results of these lemmas are then combined in Lemma 30, in which we show that if the algorithm returns a partition into triples π then π is wj -envy-free.

Lemma 26. *If Algorithm `wjPathsCycles` returns a partition into triples π then no agent in $\bigcup T$ is wj -envied in π .*

Proof. Suppose Algorithm `wjPathsCycles` has returned some partition into triples π . Consider an arbitrary triple $t \in T$. By the pseudocode of Algorithm `wjPathsCycles` there are two possibilities: either $t \in \hat{T}$ or t was labelled $\hat{t}$. If $t \in \hat{T}$ then by Lemma 11 no agent in t is wj -envied in π . Suppose then that t was labelled $\hat{t}$. By the pseudocode of Algorithm `wjPathsCycles`, for any agent c_i in $\hat{t}$ it must be that some call was made to Subroutine `oneC4TwoSingles` in which the second or third argument was equal to c_i and then the two triples returned by the subroutine were added to π . By Lemma 13, it follows that no agent in $\hat{t}$ is wj -envied in π . $\square$

Lemma 27. *If Algorithm `wjPathsCycles` returns a partition into triples π then no agent in $\hat{S}$ is wj -envied in π .*

Proof. Suppose Algorithm `wjPathsCycles` has returned some partition into triples π in which some agent $\alpha_i \in N$ has wj -envy for some $\hat{s}_{j_1} \in \hat{S}$. By the pseudocode, it must be that $\pi(\hat{s}_{j_1})$ contains three agents in $\hat{S}$ so we label $\pi(\hat{s}_{j_1}) = \{\hat{s}_{j_1}, \hat{s}_{j_2}, \hat{s}_{j_3}\}$. Note that since $|\hat{S}| > 0$ it must be that $\hat{T} = T$.

Since α_i has wj -envy for $\hat{s}_{j_1}$ it must be that $u_{\alpha_i}(\{\hat{s}_{j_2}, \hat{s}_{j_3}\}) \geq 1$ so without loss of generality assume that $\{\alpha_i, \hat{s}_{j_2}\} \in E$. We now consider two possibilities: $\alpha_i \in S$ and $\alpha_i \notin S$.

First, suppose $\alpha_i \in S$. If $\alpha_i \in \hat{S}$ then Lemma 18 is contradicted, so it must be that $\alpha_i \in S \setminus \hat{S}$. By the pseudocode, α_i was labelled either w_1, w_2, w_3 or w_4 during algorithm execution and must belong to some set of agents returned by a call to Subroutine `pickLowDegree`. Since $\{\alpha_i, \hat{s}_{j_2}\} \subset S$ the degree of α_i in the subgraph induced by S in (N, E) is 1. By the definition of Subroutine `pickLowDegree` it must be that the degree of each agent in $\hat{S}$ is also 1. With this in mind, consider the call `configureSurplusAgents`($\hat{S}$). It must be that $|\mathcal{Q}| = |\hat{S}|/2$. It follows that $X \subset \mathcal{Q}$. By the pseudocode of Subroutine `configureSurplusAgents`, We shall consider the values of the variables in Subroutine `configureRemainingAgents` inside this call. It must be that for each triple in the set of triples returned by this subroutine contains two agents that are adjacent in (N, E) . If $\{\hat{s}_{j_2}, \hat{s}_{j_1}\} \in E$ or $\{\hat{s}_{j_2}, \hat{s}_{j_3}\} \in E$ then the degree of $\hat{s}_{j_2}$ in the subgraph induced by S in (N, E) is 2, which is a contradiction. It remains that $\{\hat{s}_{j_1}, \hat{s}_{j_3}\} \in E$. Since w_k has wj -envy for $\hat{s}_{j_1}$ it must be that $v_{\hat{s}_{j_3}}(\hat{s}_{k_1}) \geq v_{\hat{s}_{j_3}}(\hat{s}_{j_1}) = 1$. It follows that $\hat{s}_{j_3}$ has degree 2 in the subgraph induced by S in (N, E) , which is a contradiction.

Second, suppose $\alpha_i \notin S$. Since $\{\alpha_i, \hat{s}_{j_2}\} \in E$, by the pseudocode of Algorithm `wjPathsCycles` it must be that α_i belongs to the same connected component in (N, E) as $\hat{s}_{j_2}$. Since $\hat{s}_{j_2} \in S$ it must be that the connected component that contains α_i and $\hat{s}_{j_2}$ is not a 4-cycle and belongs to $\mathcal{C}$. Since $\alpha_i \notin S$ it must

also be that $\pi(\alpha_i) \in T$. Since $\hat{T} = T$ it follows that $\pi(\alpha_i) \in \hat{T}$. Lemma 10 now implies that α_i is not wj -envious in π , which is a contradiction. $\square$

Lemma 28. *If Algorithm `wjPathsCycles` returns a partition into triples π then no agent in S is wj -envious in π .*

Proof. Suppose Algorithm `wjPathsCycles` has returned some partition into triples π . Consider an arbitrary agent $s_i \in S$. If $s_i \in \hat{S}$ then by Lemma 27 it must be that s_i is not wj -envious in π .

Suppose then that $s_i \notin \hat{S}$. There are three cases: either $|\mathcal{R}| \bmod 3 = 2$, $|S| \geq 4$, and s_i was labelled w_1, w_2, w_3 , or w_4 ; $|\mathcal{R}| \bmod 3 = 2$, $|T| \geq 1$, and s_i was labelled w_1 ; or $|\mathcal{R}| \bmod 3 = 1$ and s_i was labelled either w_1 or w_2 . In each of the three cases, some call was then made to Subroutine `oneC4TwoSingles` in which the second or third argument was equal to s_i and then two triples returned by the subroutine were added to π . By Lemma 13 it follows that s_i is not wj -envious in π . $\square$

Lemma 29. *If Algorithm `wjPathsCycles` returns a partition into triples π then no agent in $\bigcup \mathcal{R}$, i.e. in some 4-cycle in (N, E) , is wj -envious in π .*

Proof. Consider an arbitrary $R_j \in \mathcal{R}$ where $1 \leq j \leq |\mathcal{R}|$. We show that no agent in R_j is wj -envious in π . In this case, let l' be the final value assigned to the variable l before the algorithm terminated. There are two possibilities: either $j > l'$ or $j \leq l'$.

Suppose $j > l'$. It must be that $R_j \in \mathcal{R}'$, by the construction of $\mathcal{R}'$ in Algorithm `wjPathsCycles`. By Lemma 15 it follows that no agent in R_j is wj -envious in π .

Suppose $j \leq l'$. By the design of Algorithm `wjPathsCycles` there are two possibilities: either $|\mathcal{R}| \bmod 3 = 2$ and $l' = 2$, or $|\mathcal{R}| \bmod 3 = 1$ and $l' = 1$. In either case, by the pseudocode of Algorithm `wjPathsCycles` it must be that some call to Subroutine `oneC4TwoSingles` was made with the first argument equal to R_j and the returned set of two triples was then added to π . It follows by Lemma 13 that no agent in R_j is wj -envious in π . $\square$

Lemma 30. *If (N, E) does not belong to $\mathcal{I}_{no}^{wj}$ then Algorithm `wjPathsCycles` returns a partition into triples π that is wj -envy-free.*

Proof. By definition, $\mathcal{C} \cup \mathcal{R}$ is the set of all connected components in (N, E) . By Lemma 25, Algorithm `wjPathsCycles` returns a partition π of N into triples. By Lemma 20, the set of agents $\bigcup \mathcal{C} = S \cup \bigcup T$. By Lemma 26, no agent in $\bigcup T$ is wj -envious in π . By Lemma 28, no agent in S is wj -envious in π . By Lemma 29, no agent in $\bigcup \mathcal{R}$ is wj -envious in π . $\square$

Theorem 3. *In $O(|N|)$ time, we can either find a wj -envy-free partition into triples or report that (N, E) belongs to $\mathcal{I}_{no}^{wj}$ (and thus contains no wj -envy-free partition into triples).*

Proof. Lemma 23 shows that Algorithm `wjPathsCycles` terminates in $O(|N|)$ time. Lemmas 24 and 30 establish the correctness of this algorithm and show that the algorithm either returns a wj-envy-free partition into triples or reports that “ (N, E) belongs to $\mathcal{I}_{\text{no}}^{\text{wj}}$ ”. In the latter case, Lemma 2 shows that the supplied instance contains no wj-envy-free partition into triples. $\square$

A.2.2 Symmetric and binary preferences with maximum degree 3

The polynomial time reduction from $3\text{-CNF}_+^3\text{-XSAT}$ is similar to the reduction used to show that the envy-free decision problem is NP-complete (see Theorem 2 and the corresponding full proof in Section A.1.1). In this reduction, we suppose that the number of clauses m satisfies $m = 4l$ for some $l \geq 1$. The decision problem $3\text{-CNF}_+^3\text{-XSAT}$ remains NP-complete under this restriction. This can be shown using a simple polynomial-time reduction from the general problem, as follows. Construct four distinct copies of the set of variables $X(C)$ and formula C . Let the constructed formula C' be the union of the four copies of C . It is straightforward to show that C' is exactly satisfiable if and only if each of the four copies is exactly satisfiable, which is true if and only if the original formula C is exactly satisfiable.

The main difference between this reduction and the reduction for the ASHG envy-free decision problem is in the second direction, in which we show that C is exactly satisfiable if a wj-envy-free partition π exists in the constructed instance (N, E) . Here we associate true literals with variable gadgets that belong to one triple (rather than three) and false literals with variable gadgets that belong to three triples (rather than one). Another difference is that we construct a number of “garbage collector” gadgets. We will show that there is only one possible configuration of the triples of agents in the garbage collector gadgets in a wj-envy-free partition.

Given an instance C of $3\text{-CNF}_+^3\text{-XSAT}$, the reduction constructs an ASHG with binary and symmetric preferences (N, E) such that C is exactly satisfiable if and only if a wj-envy-free partition into triples exists in (N, E) .

The reduction, illustrated in Figure 1, is as follows. For each variable $x_i \in X(C)$ construct three agents w_i^1, w_i^2, w_i^3 . Add the edges $\{w_i^1, w_i^2\}$, $\{w_i^2, w_i^3\}$, and $\{w_i^3, w_i^1\}$ to E . We shall refer to agents w_i^1, w_i^2 , and w_i^3 as the i^{th} *variable gadget*, V_i . Next, for each clause c_r in C construct four agents $d_r^1, \dots, d_r^4$. Add the edges $\{d_r^1, d_r^4\}$, $\{d_r^2, d_r^4\}$, and $\{d_r^3, d_r^4\}$. We shall refer to agents $d_r^1, \dots, d_r^4$ as the r^{th} *clause gadget*, D_r . Recall that $3 \mid m$ and $m = 4l$. Construct $12l$ agents labelled $g_1, \dots, g_{12l}$. For each $1 \leq i \leq 3l$, add the edges $\{g_{4i}, g_{4i-1}\}$, $\{g_{4i}, g_{4i-2}\}$, and $\{g_{4i}, g_{4i-3}\}$ to E . For any $1 \leq i \leq 3l$, we shall refer to $\{g_{4i-3}, g_{4i-2}, g_{4i-1}, g_{4i}\}$ as the i^{th} *garbage collector gadget*, G_i . Note that each G_i is an isolated claw $K_{1,3}$. Let $G = \bigcup_{1 \leq i \leq 3l} G_i$.

Finally, we shall connect the variable and clause gadgets. Consider each clause $c_r = \{x_i, x_j, x_k\}$ and the corresponding variable gadgets W_i, W_j, W_k . If c_r contains the first occurrence of x_i then add the edge $\{d_r^1, w_i^1\}$. Alternatively, if c_r contains the second occurrence of x_i then add the edge $\{d_r^1, w_i^2\}$. Finally, if c_r contains the third occurrence of x_i then add the edge $\{d_r^1, w_i^3\}$. Similarly, add an

edge between d_r^2 and an agent in W_j depending on the index of the occurrence of x_j in the clause c_r . Finally, add an edge between d_r^3 and an agent in W_k depending on the index of the occurrence of x_k in the clause c_r . We say that the clause gadget D_r is adjacent to the variable gadgets W_i , W_j , and W_k , and vice-versa.

Now, each agent in a variable gadget has degree 3, d_r^1, d_r^2, d_r^3 for each $1 \leq r \leq m$ have degree 2, d_r^4 for each $1 \leq r \leq m$ has degree 3, g_i where $1 \leq i \leq 12l$ and $4 \nmid i$ has degree 1, and g_j where $1 \leq j \leq 12l$ and $4 \mid j$ has degree 3. It follows that the maximum degree of the instance is 3.

As before, for some set $S \subset N$, we let $\sigma(S, \pi)$ be the number of triples in π that each contain at least one agent in S . By definition, $\lceil |S|/3 \rceil \leq \sigma(S, \pi) \leq |S|$ for any $S \subseteq N$.

Lemma 31. *In the reduction, if C is exactly satisfiable then the ASHG (N, E) contains an wj -envy-free partition into triples.*

Proof. Suppose f is an exact model in C . We shall construct a partition into triples π in (N, E) that is wj -envy-free. For each variable x_i in $X(C)$ where $f(x_i)$ is true, add $\{w_i^1, w_i^2, w_i^3\}$ to π . Next, consider each clause $c_r = \{x_i, x_j, x_k\}$ and the corresponding clause gadget D_r , labelling i, j, k such that W_i contains an agent adjacent to d_r^1 , W_j contains an agent adjacent to d_r^2 , and W_k contains an agent adjacent to d_r^3 . There are three cases: $f(x_i)$ is true while both $f(x_j)$ and $f(x_k)$ are false, $f(x_j)$ is true while both $f(x_i)$ and $f(x_k)$ are false, and $f(x_k)$ is true while both $f(x_i)$ and $f(x_j)$ are false. In the first case, suppose c_r contains the a^{th} occurrence of x_j and the b^{th} occurrence of x_k . Add the triples $\{d_r^1, d_r^4, g_{3r}\}$, $\{d_r^2, w_j^a, g_{3r-1}\}$, and $\{d_r^3, w_k^b, g_{3r-2}\}$. The constructions in the second and third cases are symmetric: in the second case, suppose c_r contains the a^{th} occurrence of x_i and the b^{th} occurrence of x_k . Add the triples $\{d_r^2, d_r^4, g_{3r}\}$, $\{d_r^1, w_i^a, g_{3r-1}\}$, and $\{d_r^3, w_k^b, g_{3r-2}\}$. In the third case, suppose c_r contains the a^{th} occurrence of x_i and the b^{th} occurrence of x_j . Add the triples $\{d_r^3, d_r^4, g_{3r}\}$, $\{d_r^1, w_i^a, g_{3r-1}\}$, and $\{d_r^2, w_j^b, g_{3r-2}\}$.

Note that for any triple $t \in \pi$, either $t = W_i$ for some variable gadget W_i ; $t = \{d_r^4, d_r^a, g_{3r}\}$ for some $1 \leq r \leq m$ and $1 \leq a \leq 3$; or $t = \{d_r^a, w_i^b, g_j\}$ for some $1 \leq i, r \leq m$, $1 \leq a, b \leq 3$ and $1 \leq j \leq 12l$ where $3 \nmid j$. We shall show that in each case t does not contain an agent with wj -envy.

First, consider some triple $t \in \pi$ where $t = W_i$ for some variable gadget W_i . Since each agent in t has utility 2, no agent in t is envious.

Second, consider some triple $t \in \pi$ where $t = \{d_r^4, d_r^a, g_{3r}\}$ for some $1 \leq r \leq m$ and $1 \leq a \leq 3$. Since $\pi(d_r^1) \neq \pi(d_r^2)$, $\pi(d_r^1) \neq \pi(d_r^3)$, and $\pi(d_r^2) \neq \pi(d_r^3)$, it must be that $\sigma(N(d_r^4), \pi) = 3$. Since $u_{d_r^4}(\pi) = 1$, it follows by Lemma 1 that d_r^4 is not envious. Similarly, since $\pi(d_r^a) = \{d_r^4, d_r^a, g_{3r}\}$ it follows that $\sigma(N(d_r^a), \pi) = 2$ so since $u_{d_r^a}(\pi) = 1$ by Lemma 1 it must be that d_r^a is also not envious. Suppose for a contradiction that g_{3r} wj -envies some agent α_j . It must be that $u_{g_{3r}}(\pi(\alpha_j) \setminus \{\alpha_j\}) \geq 1$ so $\pi(\alpha_j)$ contains some g_q where $\{g_{3r}, g_q\} \in E$. Let $\pi(\alpha_j) = \{\alpha_j, g_q, \alpha_k\}$. By construction of π , it must be that $\alpha_j, \alpha_k \subset D_s$ for some $1 \leq s \leq m$ and $u_{\alpha_j}(\pi) = u_{\alpha_k}(\pi) = 1$. It follows that $\{\alpha_k, g_q\} \notin E$ so

$u_{\alpha_k}(\pi) < u_{\alpha_k}(\{g_q, g_{3r}\})$, which contradicts the supposition that g_{3r} wj-envies α_j .

Third, consider some triple $t \in \pi$ where $t = \{d_r^a, w_i^b, g_j\}$ for some $1 \leq i, r \leq m$, $a \in \{1, 2, 3\}$, $b \in \{1, 2, 3\}$ and $1 \leq j \leq 12l$ where $3 \nmid j$. By construction, $\sigma(N(w_i^b), \pi) = 3$ so by Lemma 1 w_i^b is not envious. Similarly, since $\pi(d_r^a) = \{d_r^a, w_i^b, g_j\}$ it follows that $\sigma(N(d_r^a), \pi) = 2$ so since $u_{d_r^a}(\pi) = 1$ by Lemma 1 it must be that d_r^a is also not envious. As before, suppose for a contradiction that g_j wj-envies some agent α_k . It must be that $u_{g_j}(\pi(\alpha_k) \setminus \{\alpha_k\}) \geq 1$ so $\pi(\alpha_k)$ contains some g_q where $\{g_j, g_q\} \in E$. Let $\pi(\alpha_k) = \{\alpha_k, g_q, \alpha_h\}$. By construction of π , it must be that $\alpha_k, \alpha_h \in D_s$ for some $1 \leq s \leq m$ and $u_{\alpha_k}(\pi) = u_{\alpha_h}(\pi) = 1$. It follows that $\{\alpha_h, g_q\} \notin E$ so $u_{\alpha_h}(\pi) < u_{\alpha_h}(\{g_q, g_j\})$, which contradicts the supposition that g_j wj-envies α_k . $\square$

Lemma 32. *In the reduction, if π is a wj-envy-free partition into triples in (N, E) then for any variable gadget W_i , either $\sigma(W_i, \pi) = 1$ or $\sigma(W_i, \pi) = 3$.*

Proof. The proof is similar to Lemma 4. If some triple in π contains exactly two agents in W_i then the third agent in W_i is wj-envious. $\square$

Lemma 33. *In the reduction, if π is a wj-envy-free partition into triples in (N, E) then $u_{g_i}(\pi) = 0$ for any $1 \leq i \leq 12l$.*

Proof. Suppose π is a wj-envy-free partition into triples. By construction, the structure of G_i for each $1 \leq i \leq 3l$ is identical, so to simplify the proof we assume, without loss of generality, that $i = 1$ and $G_i = \{g_1, g_2, g_3, g_4\}$. We shall prove that $\sigma(G_1, \pi) = 4$, from which it follows directly that $u_{g_1}(\pi) = u_{g_2}(\pi) = u_{g_3}(\pi) = u_{g_4}(\pi) = 0$. Since $|G_1| = 4$ clearly $\sigma(G_1, \pi) \leq 4$. Suppose for a contradiction that $\sigma(G_1, \pi) \leq 3$. Then there exists two agents $g_a, g_b \in G_1$ where $g_a \in \pi(g_b)$. Label the third agent in $\pi(g_b)$ as α_j . By symmetry, we need only consider two cases. In the first case, $a = 1$ and $b = 4$. In the second case, $a = 1$ and $b = 2$. We will show that in both cases it is relatively straightforward to identify an envious agent, which is a contradiction.

First, suppose $a = 1$ and $b = 4$. Since $\{g_1, g_4\} \subset \pi(g_4)$, by construction it must be that either $u_{g_2}(\pi) = 0$ or $u_{g_3}(\pi) = 0$. Suppose without loss of generality that $u_{g_2}(\pi) = 0$. It follows that g_2 wj-envies g_1 , since $u_{g_4}(\{g_1, \alpha_j\}) = u_{g_4}(\{g_2, \alpha_j\})$ and, by construction of G_1 , $u_{\alpha_j}(\{g_4, g_1\}) = u_{\alpha_j}(\{g_4, g_2\})$.

Second, suppose $a = 1$ and $b = 2$. There are two cases: $g_4 = \alpha_j$ or $g_4 \neq \alpha_j$. If $g_4 = \alpha_j$ then g_3 wj-envies g_2 , since $u_{g_1}(\{g_2, g_4\}) = u_{g_1}(\{g_3, g_4\})$ and $u_{g_4}(\{g_1, g_2\}) = u_{g_4}(\{g_2, g_3\})$. If $g_4 \neq \alpha_j$ then $u_{g_4}(\pi) \leq 1$ so g_4 wj-envies α_j , since $u_{g_4}(\{g_1, g_2\}) = 2$, $u_{g_1}(\pi) = u_{g_2}(\pi) = 0$, and $u_{g_1}(\{g_2, g_4\}) = u_{g_2}(\{g_1, g_4\}) = 1$. $\square$

Lemma 34. *In the reduction, if π is a wj-envy-free partition into triples in (N, E) then $\pi(g_i) = \{g_i, \alpha_a, \alpha_b\}$ where $\{\alpha_a, \alpha_b\} \in E$ for any $1 \leq i \leq 12l$.*

Proof. Suppose π is a wj-envy-free partition into triples. Consider an arbitrary g_i where $1 \leq i \leq 12l$. Let g_j be some agent for which $\{g_i, g_j\} \in E$. By Lemma 33,

$u_{g_i}(\pi) = u_{g_j}(\pi) = 0$. It remains to show $\{\alpha_a, \alpha_b\} \in E$. Suppose for a contradiction that $\{\alpha_a, \alpha_b\} \notin E$. It follows that $u_{\alpha_a}(\pi) = u_{\alpha_b}(\pi) = u_{g_i}(\pi) = 0$. Now g_j envies α_a , since $u_{g_j}(\{g_i, \alpha_b\}) = 1$, $u_{\alpha_b}(\{g_j, g_i\}) = u_{\alpha_b}(\pi)$, and $u_{g_i}(\{g_j, \alpha_b\}) = 1$, which is a contradiction. $\square$

Lemma 35. *In the reduction, suppose π is an wj -envy-free partition into triples in (N, E) . For each $1 \leq r \leq m$, there exist exactly three triples t_1, t_2, t_3 that each contains one agent in G and at least one agent $d_r^w \in D_r$ where $u_{d_r^w}(\pi) = 1$.*

Proof. By Lemma 34, $\pi(g_i) = \{g_i, \alpha_a, \alpha_b\}$ where $\{\alpha_a, \alpha_b\} \in E$ for any $1 \leq i \leq 12l$. It follows that there exists $12l$ triples in π of the form $\{g_i, \alpha_a, \alpha_b\}$ where $\{\alpha_a, \alpha_b\} \in E$. Let T be this set of triples. Since $\{\alpha_a, \alpha_b\} \in E$, it must be that $u_{\alpha_a}(\pi) = u_{\alpha_b}(\pi) = 1$. It follows that $\alpha_a \notin G$ and $\alpha_b \notin G$. It must be that for any $\{g_i, \alpha_a, \alpha_b\} \in T$ there exists some $1 \leq r \leq m$ where either $\alpha_a \in D_r$ or $\alpha_b \in D_r$, for otherwise the only possibility is that $\{\alpha_a, \alpha_b\} \subset W_i$ for some variable gadget W_i , which would contradict Lemma 32. We have now shown that each $t \in T$ comprises $\{g_i, d_r^w, \alpha_b\}$ where $u_{d_r^w}(\pi) = u_{\alpha_b}(\pi) = 1$ for some $1 \leq r \leq m$, $1 \leq w \leq 4$ and $\alpha_b \in N$. It remains to show that for a given r , there exist exactly three triples in T where each triple contains at least one agent in D_r .

Suppose for a contradiction that there exists some $1 \leq r \leq m$ where the number of triples in T that contain an agent in D_r is not three. Recall that $|T| = 12l$, $m = 4l$, and each $t \in T$ contains at least one d_r^w for some $1 \leq r \leq m$ and $1 \leq w \leq 4$. A counting argument shows that there must exist some $1 \leq s \leq m$ where there are at least four triples $t_1, t_2, t_3, t_4 \in T$ that each contain at least one agent in D_s . Without loss of generality, it must be that $d_s^1 \in t_1$, $d_s^2 \in t_2$, $d_s^3 \in t_3$, and $d_s^4 \in t_4$. Since we have previously established that $u_{d_s^4}(\pi) = 1$ this leads to a contradiction since $N(d_s^4) = \{d_s^1, d_s^2, d_s^3\}$ but $d_s^1 \notin t_4$, $d_s^2 \notin t_4$, and $d_s^3 \notin t_4$. It follows that for each $1 \leq r \leq m$, there exists exactly three triples in T where each triple contains at least one agent in D_r . $\square$

Lemma 36. *In the reduction, if the ASHG represented by (N, E) contains an envy-free partition into triples then C is exactly satisfiable.*

Proof. Suppose π is a wj -envy-free partition into triples in (N, E) . By Lemma 32, for any variable gadget W_i either $\sigma(W_i, \pi) = 1$ or $\sigma(W_i, \pi) = 3$. Construct a truth assignment f in C by setting $f(x_i)$ to be true if $\sigma(W_i, \pi) = 1$ and false otherwise. Each variable x_i corresponds to exactly one variable gadget W_i so it follows that f is a valid truth assignment. By the construction of (N, E) , each clause c_r corresponds to exactly one clause gadget D_r . Each clause gadget is adjacent to three variable gadgets that correspond to the three variables in that clause. To show that f is an exact model of C , it is sufficient to show that for each clause gadget D_r there exists exactly one variable gadget W_i such that D_r is adjacent to W_i and $\sigma(W_i, \pi) = 1$.

Consider an arbitrary clause gadget D_r and the corresponding clause $c_r = \{x_i, x_j, x_k\}$, labelling i, j, k such that d_r^1 is adjacent to some agent $w_i^{a_1} \in W_i$, d_r^2 is adjacent to some agent $w_j^{a_2} \in W_j$ and d_r^3 is adjacent to some agent $w_k^{a_3} \in W_k$.

By Lemma 35, there exists exactly three triples t_1, t_2, t_3 such that t_1 contains one agent $g_{h_1} \in G$ and at least one agent $d_r^{b_1} \in D_r$ where $u_{d_r^{b_1}}(\pi) = 1$; t_2 contains one agent $g_{h_2} \in G$ and at least one agent $d_r^{b_2} \in D_r$ where $u_{d_r^{b_2}}(\pi) = 1$; and t_3 contains one agent in $g_{h_3} \in G$ and at least one agent $d_r^{b_3} \in D_r$ where $u_{d_r^{b_3}}(\pi) = 1$. Consider d_r^4 . If $u_{d_r^4}(\pi) = 0$ then $\{b_1, b_2, b_3\} = \{1, 2, 3\}$ so without loss of generality we may assume that $t_1 = \{d_r^1, w_i^{a_1}, g_{h_1}\}$, $t_2 = \{d_r^2, w_j^{a_2}, g_{h_2}\}$, and $t_3 = \{d_r^3, w_k^{a_3}, g_{h_3}\}$. In this configuration d_r^4 wj-envies $w_i^{a_1}$ since $u_{d_r^4}(\{d_r^1, g_{h_1}\}) = 1$, $u_{d_r^4}(\{d_r^4, g_{h_1}\}) = u_{d_r^1}(\pi)$, and $u_{g_{h_1}}(\{d_r^4, d_r^1\}) = u_{g_{h_1}}(\pi) = 0$. It follows that $u_{d_r^4}(\pi) > 0$. From the above observation on the contents of t_1, t_2 , and t_3 it must be that $u_{d_r^4}(\pi) < 2$ and so $u_{d_r^4}(\pi) = 1$. There are three possible cases: either $d_r^1 \in \pi(d_r^4)$, $d_r^2 \in \pi(d_r^4)$, or $d_r^3 \in \pi(d_r^4)$. By the symmetry of the clause gadget, we describe only the case in which $d_r^1 \in \pi(d_r^4)$. Without loss of generality we may assume that $t_1 = \{d_r^1, d_r^4, g_{h_1}\}$. Furthermore, we may assume that $t_2 = \{d_r^2, w_j^{a_2}, g_{h_2}\}$ and $t_3 = \{d_r^3, w_k^{a_3}, g_{h_3}\}$. Consider $w_i^{a_1}$. If $u_{w_i^{a_1}}(\pi) = 0$ then $w_i^{a_1}$ wj-envies d_r^4 , since $u_{w_i^{a_1}}(\{d_r^1, g_{h_1}\}) = 1$, $u_{d_r^1}(\{w_i^{a_1}, g_{h_1}\}) = u_{d_r^1}(\pi)$, and $u_{g_{h_1}}(\{w_i^{a_1}, d_r^1\}) = u_{g_{h_1}}(\pi) = 0$. It must be that $u_{w_i^{a_1}}(\pi) > 0$. If $u_{w_i^{a_1}}(\pi) = 1$ then Lemma 32 is contradicted. It follows that $u_{w_i^{a_1}}(\pi) = 2$ and thus $\sigma(W_i, \pi) = 1$. Now consider $w_j^{a_2}$ and $w_k^{a_3}$. Since $u_{w_j^{a_2}}(\pi) = 1$ it must be that $\sigma(W_j, \pi) = 3$. Similarly, since $u_{w_k^{a_3}}(\pi) = 1$ it must be that $\sigma(W_k, \pi) = 3$. To recap, after we supposed that $d_r^1 \in \pi(d_r^4)$, we showed that $\sigma(W_i, \pi) = 1$ and $\sigma(W_j, \pi) = \sigma(W_k, \pi) = 3$, as required. The cases in which $d_r^2 \in \pi(d_r^4)$ or $d_r^3 \in \pi(d_r^4)$ are symmetric. $\square$

Theorem 4. *It is NP-complete to decide if a wj-envy-free partition into triples exists in a given ASHG, even when preferences are symmetric and binary and $\Delta_{(N,E)} = 2$*

Proof. Suppose we are given an ASHG with binary and symmetric preferences, represented by a graph (N, E) , and a partition into triples π . For any two agents $\alpha_i, \alpha_j \in N$ we can test if α_i wj-envies α_j in constant time. We can therefore decide if π is wj-envy-free in polynomial time, which shows that the above decision problem belongs to NP.

The above reduction from 3-CNF₊³-XSAT can be performed in polynomial time. By Lemmas 3 and 5, the formula C is exactly satisfiable if and only if a wj-envy-free partition into triples exists in the constructed instance represented by (N, E) . It follows that the reduction is valid and the above decision problem is NP-hard. $\square$

A.3 Justified envy-freeness

Theorem 5. *Given an ASHG with binary, but not necessarily symmetric preferences, a j-envy-free partition into triples must exist and can be found in $O(|N|^3)$ time.*

Proof. Suppose (N, V) is an ASHG with binary preferences. We shall describe an algorithm to construct a partition into triples π in (N, V) that is j-envy-free. The algorithm has worst-case time complexity $O(|N|^3)$.

From (N, V) , we shall construct another ASHG (N, V') , which has symmetric and binary preferences, by deleting asymmetric arcs. We shall use v in the context of the instance (N, V') , which has binary preferences, and v' in the context of the instance (N, V) , which has binary and symmetric preferences. Similarly, we shall use $u'_{\alpha_i}(\pi)$ to refer to $u'_{\alpha_i}(S) = \sum_{\alpha_j \in S \setminus \{\alpha_i\}} v'_{\alpha_i}(\alpha_j)$ where $S \subseteq N$. The construction of (N, V') can be performed in $O(|N|^2)$ time.

Run Algorithm **findStableUW** (see [19]) on the instance (N, V') to construct a partition into triples π . This algorithm runs in $O(|N|^3)$ time. Notice that π thus meets the following condition: if there exists a pair of agents $\alpha_i, \alpha_j \in N$ where $v'_{\alpha_i}(\alpha_j) = 1$ and $u'_{\alpha_i}(\pi) = u'_{\alpha_j}(\pi) = 0$ then no triple contains three agents each with utility 0. It remains to show that π is j-envy-free in (N, V) . In the proof, which follows, we refer to the variables and subroutines in Algorithm **findStableUW**.

For a contradiction, suppose there exists some $\alpha_{i_1} \in N$ where α_{i_1} has j-envy in π for some $\alpha_{j_1} \in N \setminus \{\alpha_{i_1}\}$. Let $\pi(\alpha_{i_1}) = \{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\}$ and $\pi(\alpha_{j_1}) = \{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}\}$. Since α_{i_1} has j-envy for α_{j_1} it must be that $u_{\alpha_{i_1}}(\{\alpha_{j_2}, \alpha_{j_3}\}) > 0$ so either $v_{\alpha_{i_1}}(\alpha_{j_2}) = 1$, $v_{\alpha_{i_1}}(\alpha_{j_3}) = 1$, or both. Suppose without loss of generality that $v_{\alpha_{i_1}}(\alpha_{j_2}) = 1$. Note that it must also be that $u_{\alpha_{j_2}}(\{\alpha_{i_1}, \alpha_{j_3}\}) > u_{\alpha_{j_2}}(\pi)$ and $u_{\alpha_{j_3}}(\{\alpha_{i_1}, \alpha_{j_2}\}) > u_{\alpha_{j_3}}(\pi)$. It follows that $v_{\alpha_{j_2}}(\alpha_{i_1}) > v_{\alpha_{j_2}}(\alpha_{j_1})$ and $v_{\alpha_{j_3}}(\alpha_{i_1}) > v_{\alpha_{j_3}}(\alpha_{j_1})$. Since preferences are binary, it must be that $v_{\alpha_{j_2}}(\alpha_{i_1}) = v_{\alpha_{j_3}}(\alpha_{i_1}) = 1$ and $v_{\alpha_{j_2}}(\alpha_{j_1}) = v_{\alpha_{j_3}}(\alpha_{j_1}) = 0$. Since $v_{\alpha_{i_1}}(\alpha_{j_2}) = 1$ and $v_{\alpha_{j_2}}(\alpha_{i_1}) = 1$ it follows that $v'_{\alpha_{i_1}}(\alpha_{j_2}) = 1$.

We now show that the triple $\{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}\}$ does not belong to M_1 . Since $v_{\alpha_{j_2}}(\alpha_{j_1}) = v_{\alpha_{j_3}}(\alpha_{j_1}) = 0$, it follows that $v'_{\alpha_{j_2}}(\alpha_{j_1}) = v'_{\alpha_{j_3}}(\alpha_{j_1}) = 0$ so $u'_{\alpha_{j_1}}(M_1) = 0$. Since M_1 is a P -matching, no triple in M_1 contains an agent with utility 0, which shows that $\{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}\} \notin M_1$. Note that it follows that $\{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}\} \subseteq U$.

Next, we show that $u'_{\alpha_{i_1}}(\pi) = 0$. Since α_{i_1} envies α_{j_1} in (N, V) it must be that $u_{\alpha_{i_1}}(\pi) < 2$ and thus that $u'_{\alpha_{i_1}}(\pi) < 2$. It remains that $u'_{\alpha_{i_1}}(\pi) \in \{0, 1\}$. We will show that $u'_{\alpha_{i_1}}(\pi) = 0$. Suppose for a contradiction that $u'_{\alpha_{i_1}}(\pi) = 1$ and without loss of generality that $v'_{\alpha_{i_1}}(\alpha_{i_2}) = 1$. It follows that $v_{\alpha_{i_1}}(\alpha_{i_2}) = 1$. It must be that $u_{\alpha_{i_1}}(\pi) = 1$. Since α_{i_1} has j-envy for α_{j_1} , meaning $u_{\alpha_{i_1}}(\{\alpha_{j_2}, \alpha_{j_3}\}) > u_{\alpha_{i_1}}(\pi) = 1$, it must be that $v_{\alpha_{i_1}}(\alpha_{j_3}) = v_{\alpha_{i_1}}(\alpha_{j_2}) = 1$. We observed earlier that $v_{\alpha_{j_3}}(\alpha_{j_1}) = 1$ so it must be that $v'_{\alpha_{i_1}}(\alpha_{j_3}) = 1$. We also observed that $\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}$ are unmatched in M_1 so $u'_{\alpha_{j_2}}(M_1) = u'_{\alpha_{j_3}}(M_1) = 0$. Now, the triple $\{\alpha_{i_1}, \alpha_{j_2}, \alpha_{j_3}\}$ blocks M_1 in (N, V') , since $u'_{\alpha_{i_1}}(M_1) = 1$, $u'_{\alpha_{j_2}}(M_1) = u'_{\alpha_{j_3}}(M_1) = 0$, and $v'_{\alpha_{i_1}}(\alpha_{j_2}) = v'_{\alpha_{i_1}}(\alpha_{j_3}) = 1$. Since M_1 is a stable partition into triples in (N, V') , by the correctness of Algorithm **findStableUW**, this is a contradiction. Note that since M_1 is a P -matching, it follows that α_{i_1} is unmatched in M_1 so $\alpha_{i_1} \in U$.

Next, we show that $v'_{\alpha_{j_2}}(\alpha_{j_3}) = 0$. Suppose for a contradiction that $v'_{\alpha_{j_2}}(\alpha_{j_3}) = 1$. We established earlier that $\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}$ are unmatched in M_1 so $u'_{\alpha_{j_2}}(M_1) = u'_{\alpha_{j_3}}(M_1) = 0$. We also established that $u'_{\alpha_{i_1}}(\pi) = 0$ hence $u'_{\alpha_{i_1}}(M_1) = 0$. We also established that $v'_{\alpha_{i_1}}(\alpha_{j_2}) = 1$, so since $v'_{\alpha_{j_2}}(\alpha_{j_3}) = 1$, by assumption, it now

follows that $\{\alpha_{i_1}, \alpha_{j_2}, \alpha_{j_3}\}$ blocks M_1 in (N, V') , which is a contradiction since M_1 is stable in (N, V') .

We have just shown that $v'_{\alpha_{j_2}}(\alpha_{j_3}) = 0$. Note that $v'_{\alpha_{j_1}}(\alpha_{j_3}) = 0$ since $v_{\alpha_{j_3}}(\alpha_{j_1}) = 0$. Since $v'_{\alpha_{j_1}}(\alpha_{j_3}) = 0$ and $v'_{\alpha_{j_2}}(\alpha_{j_3}) = 0$ it follows that $u'_{\alpha_{j_1}}(\pi) = u'_{\alpha_{j_2}}(\pi) = u'_{\alpha_{j_3}}(\pi) = 0$. Now recall the pseudocode of Algorithm **findStableUW**. If $|Y| \geq \lfloor |U|/3 \rfloor$ then no triple in π exists in which each agent has utility 0, in (N, V') . Since $\{\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}\}$ is such a triple, it follows that $|Y| < \lfloor |U|/3 \rfloor$. From the pseudocode we can see that the algorithm entered the “else” condition. In this case, each element in Y contains two agents $\alpha_{k_1}, \alpha_{k_2}$ that are eventually added to the same triple in M_2 , so $u_{\alpha_{k_1}}(M_2) = u_{\alpha_{k_2}}(\pi) \geq 1$ and $u_{\alpha_{k_1}}(M_2) = u_{\alpha_{k_1}}(\pi) \geq 1$.

Since $\pi(\alpha_{i_1}) \neq \pi(\alpha_{j_2})$, it must be that $\{\alpha_{i_1}, \alpha_{i_2}\} \notin X$ so $\{\alpha_{i_1}, \alpha_{i_2}\} \notin Y$. By construction, Y is a maximal matching among U . We have already shown that $\{\alpha_{i_1}, \alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3}\} \subseteq U$. Since $v'_{\alpha_{i_1}}(\alpha_{j_2}) = 1$, the only explanation is that some element of Y contains either α_{i_1} or α_{j_2} . However, we established earlier that $u'_{\alpha_{i_1}}(\pi) = 0$ and $u'_{\alpha_{j_2}}(\pi) = 0$, which contradicts our conclusion in the previous paragraph that each element in Y contains two agents $\{\alpha_{k_1}, \alpha_{k_2}\}$ where $u_{\alpha_{k_1}}(M_2) = u_{\alpha_{k_2}}(M_2) = u_{\alpha_{k_1}}(\pi) = u_{\alpha_{k_2}}(\pi) \geq 1$. $\square$

A.3.1 Proof of Theorem 6

We describe the reduction, shown in Figure 2, from Directed Triangle Cover in full. Unless otherwise specified, assume that $v_{\alpha_i}(\alpha_j) = 0$ for any $\alpha_i, \alpha_j \in N$.

First construct a set of five agents $H = \{h_1, h_2, h_3, h_4, h_5\}$. The structure of the valuations between these agents has a number of symmetries. In order to simplify the description of H we introduce a function $r : \{1, \dots, 5\} \times \mathbb{N} \mapsto \{1, \dots, 5\}$ that describes a cyclic succession among agents $h_1, \dots, h_5$, where $r(i, c) = ((i - 1) \oplus_5 c) + 1$. For each i where $1 \leq i \leq 5$ let $v_{h_i}(h_{r(i,1)}) = v_{h_{r(i,1)}}(h_i) = 1$, $v_{h_i}(h_{r(i,3)}) = 1$, and $v_{h_i}(h_{r(i,2)}) = 2$.

Next, add four agents l_1, l_2, l_3, l_4 . Let $v_{l_1}(l_2) = v_{l_2}(l_1) = v_{l_3}(l_4) = v_{l_4}(l_3) = 2$ and $v_{l_1}(l_3) = v_{l_1}(l_4) = v_{l_2}(l_3) = v_{l_2}(l_4) = v_{l_3}(l_1) = v_{l_3}(l_2) = v_{l_4}(l_1) = v_{l_4}(l_2) = 1$. Let $L = \{l_1, \dots, l_4\}$.

Next, add $3q$ agents $c_1, \dots, c_{3q}$. Let $v_{c_i}(l_3) = v_{l_3}(c_i) = v_{l_4}(c_i) = 1$ and $v_{c_i}(l_4) = 2$ for each i where $1 \leq i \leq 3q$. For each i and j where $1 \leq i, j \leq 3q$ let $v_{c_i}(c_j) = 2$ if $(w_i, w_j) \in A$ and 1 otherwise. Let $C = \{c_1, \dots, c_{3q}\}$.

Finally, let $N = H \cup L \cup C$.

The structure of the valuations among agents in C reflects the directed graph G . As we shall see in Lemma 43, a partition into directed 3-cycles in G exists if and only if some j-envy-free partition into triples exists that contains q triples, each containing three agents in C corresponding to a directed 3-cycle in G .

A.3.1.1 Correctness of the reduction: first direction

Lemma 37. *In the reduction, if X is a partition into directed 3-cycles in G then a j-envy-free partition into triples π exists in (N, V) .*

Proof. Suppose X is a partition into directed 3-cycles. We shall construct a partition into triples π that is j-envy-free. First, add $\{h_1, h_2, h_3\}$, $\{h_4, l_1, l_2\}$,

and $\{h_5, l_3, l_4\}$ to π . Next, for each directed 3-cycle $X_p = \{w_i, w_j, w_k\}$ in X , add $\{c_i, c_j, c_k\}$ to π .

Suppose for a contradiction that some agent α_j exists where α_j has j-envy for some other agent α_{k_1} where $\pi(\alpha_{k_1}) = \{\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3}\}$. Recalling that $N = H \cup L \cup C$, it must be that either $\alpha_{k_1} \in H$, $\alpha_{k_1} \in L$, or $\alpha_{k_1} \in C$. We show that each case leads to a contradiction. It follows that no such α_j exists and thus that π is j-envy-free.

- Suppose $\alpha_{k_1} \in H$. By the construction of π there are two possibilities: either $\alpha_{k_1} \in \{h_1, h_2, h_3\}$ or $\alpha_{k_1} \in \{h_4, h_5\}$.
 - Suppose firstly that $\alpha_{k_1} \in \{h_4, h_5\}$ then by the construction of π either $\{\alpha_{k_2}, \alpha_{k_3}\} = \{l_1, l_2\}$ or $\{\alpha_{k_2}, \alpha_{k_3}\} = \{l_3, l_4\}$. Note that $u_{l_1}(\pi) = u_{l_2}(\pi) = u_{l_3}(\pi) = u_{l_4}(\pi) = 2$. Since $u_{l_1}(\{l_3, l_4\}) = 2$ and $u_{l_2}(\{l_3, l_4\}) = 2$, neither l_1 nor l_2 has j-envy for α_{k_1} , so $\alpha_j \notin \{l_1, l_2\}$. Similarly, since $u_{l_3}(\{l_1, l_2\}) = 2$ and $u_{l_4}(\{l_1, l_2\}) = 2$ neither l_3 nor l_4 has j-envy for α_{k_1} , so $\alpha_j \notin \{l_3, l_4\}$. Since $u_{c_i}(\pi) = 3$, $u_{c_i}(\{l_1, l_2\}) = 0$, and $u_{c_i}(\{l_3, l_4\}) = 2$ for any i where $1 \leq i \leq 3q$, it must be no agent in C has j-envy for α_{k_1} , so $\alpha_j \notin C$. It remains that $\alpha_j \in H$. Since this implies $v_{l_1}(\alpha_j) = v_{l_2}(\alpha_j) = v_{l_3}(\alpha_j) = v_{l_4}(\alpha_j) = 0$ it follows that α_j does not have j-envy for α_{k_1} and thus that $\alpha_{k_1} \notin \{h_4, h_5\}$.
 - Suppose then that $\alpha_{k_1} \in \{h_1, h_2, h_3\}$. Since α_j has j-envy for α_{k_1} it must be that $v_{\alpha_j}(\alpha_{k_1}) \geq 1$ so it follows that $\alpha_j \in \{h_4, h_5\}$. If $\alpha_{k_1} = h_1$ and $\alpha_j = h_4$ then we reach a contradiction since h_4 has j-envy for h_1 but $v_{h_3}(h_1) = 1 = v_{h_3}(h_4)$. Similarly, if $\alpha_{k_1} = h_1$ and $\alpha_j = h_5$ then we reach a contradiction since $v_{h_2}(h_1) = 1 = v_{h_2}(h_5)$. If $\alpha_{k_1} = h_2$ or $\alpha_{k_1} = h_3$ then we also reach a contradiction since $v_{h_1}(h_4) = v_{h_1}(h_5) = 1 = v_{h_1}(h_2) < 2 = v_{h_1}(h_3)$.
- Suppose $\alpha_{k_1} \in C$. By the construction of π it must be that $\alpha_{k_2} \in C$ and $\alpha_{k_3} \in C$ so we label $\alpha_{k_1} = c_{i_1}$, $\alpha_{k_2} = c_{i_2}$, and $\alpha_{k_3} = c_{i_3}$. By the construction of (N, V) in the reduction it follows that $v_{c_{i_2}}(c_{i_1}) \geq 1$ and $v_{c_{i_3}}(c_{i_1}) \geq 1$. Since α_j has j-envy for c_{i_1} it follows then that $v_{c_{i_2}}(\alpha_j) = 2$ and $v_{c_{i_3}}(\alpha_j) = 2$. By the construction of the instance there are two possibilities: either $\alpha_j = l_4$ or $\alpha_j \in C$. If $\alpha_j = l_4$ then $u_{l_4}(\{c_{i_2}, c_{i_3}\}) = 2$ which is a contradiction since by assumption l_4 has j-envy for c_{i_1} but $u_{l_4}(\pi) = 2$. If $\alpha_j \in C$ then label $\alpha_j = c_{i_4}$. Since $v_{c_{i_2}}(c_{i_4}) = 2$ and $v_{c_{i_3}}(c_{i_4}) = 2$, by the construction of C it must be that $(w_{i_2}, w_{i_4}) \in A$ and $(w_{i_3}, w_{i_4}) \in A$, where the vertices $w_{i_2}, w_{i_3}, w_{i_4}$ are the vertices in W that correspond respectively to the agents $c_{i_2}, c_{i_3}, c_{i_4}$ in C . Since G is antisymmetric, it follows that $(w_{i_4}, w_{i_2}) \notin A$ and $(w_{i_4}, w_{i_3}) \notin A$ so it must be that $v_{c_{i_4}}(c_{i_2}) = v_{c_{i_4}}(c_{i_3}) = 1$. This is also a contradiction since by assumption c_{i_4} has j-envy for c_{i_1} but $u_{c_{i_4}}(\{c_{i_2}, c_{i_3}\}) = 2$ and by the construction of π it must be that $u_{c_{i_4}}(\pi) = 3$.
- Suppose $\alpha_{k_1} \in L$. It must be that $\alpha_{k_1} = l_{i_1}$, $\alpha_{k_2} = l_{i_2}$, and $\alpha_{k_3} = l_{i_3}$, for some i_1, i_2 where $1 \leq i_1, i_2 \leq 4$ and $i_3 \in \{4, 5\}$. If $\alpha_j \in H$ then it must be that $v_{l_{i_2}}(\alpha_j) = 0$ which contradicts the supposition that α_j has j-envy for l_{i_1} . Otherwise, if $\alpha_j \notin H$ then $v_{h_{i_3}}(\alpha_j) = 0$, which also contradicts the supposition that α_j has j-envy for l_{i_1} .

□

A.3.1.2 Correctness of the reduction: second direction Suppose π is a j-envy-free partition into triples in (N, V) . We shall prove the existence of a partition into directed 3-cycles X in G .

Lemma 38. $\sigma(H, \pi) \geq 3$.

Proof. Since $|H| = 5$ it must be that $\sigma(H, \pi) \geq 2$. Suppose for a contradiction that $\sigma(H, \pi) = 2$. It must be that one triple in π contains three agents in H and one triple in π contains two agents in H . Suppose the former triple is $\{h_{i_1}, h_{i_2}, h_{i_3}\}$ and the latter triple is $\{h_{i_4}, h_{i_5}, \alpha_j\}$, where $1 \leq i_1, \dots, i_5 \leq 5$ and $\alpha_j \in N \setminus H$. There are five symmetries in H and $\binom{5}{2} = 10$ possible assignments of $\{h_{i_4}, h_{i_5}\}$ to two agents in H , so we need only consider the two assignments $i_4 = 1, i_5 = 2$ and $i_4 = 1, i_5 = 3$, which are not symmetric. If $i_4 = 1$ and $i_5 = 2$ then it remains that $\{i_1, i_2, i_3\} = \{3, 4, 5\}$. In this case, h_5 has j-envy for α_j since $u_{h_5}(\pi) = 2 < 3 \leq u_{h_5}(\{h_1, h_2\})$, $v_{h_1}(\alpha_j) = 0 < 1 = v_{h_1}(h_5)$, and $v_{h_2}(\alpha_j) = 0 < 1 = v_{h_2}(h_5)$. If $i_4 = 1$ and $i_5 = 3$ then it remains that $\{i_1, i_2, i_3\} = \{2, 4, 5\}$. In this case, h_4 has j-envy for α_j since $u_{h_4}(\pi) = 2 < 3 \leq u_{h_4}(\{h_1, h_3\})$, $v_{h_1}(\alpha_j) = 0 < 1 = v_{h_1}(h_4)$, and $v_{h_3}(\alpha_j) = 0 < 1 = v_{h_3}(h_4)$. □

Lemma 39. *At least two triples in π each contain exactly one agent in H .*

Proof. By Lemma 38, $\sigma(H, \pi) \geq 3$. If, contrary to the lemma statement, at most one triple in π contains exactly one agent in H then it must be that two triples in π each contain two agents in H and one triple in π contains exactly one agent in H . Suppose one of the two former triples is $\{h_{i_1}, h_{i_2}, \alpha_{j_1}\}$ and the latter triple is $\{h_{i_3}, \alpha_{j_2}, \alpha_{j_3}\}$, where $1 \leq i_1, i_2, i_3 \leq 5$ and $\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3} \in N \setminus H$. By the construction of the instance it must be that $v_{h_{i_1}}(\alpha_{j_1}) = v_{h_{i_2}}(\alpha_{j_1}) = 0$, $v_{h_{i_1}}(h_{i_3}) \geq 1$ and $v_{h_{i_2}}(h_{i_3}) \geq 1$. It follows that h_{i_3} has j-envy for α_{j_1} since $u_{h_{i_3}}(\pi) = 0 < 2 \leq u_{h_{i_3}}(\{h_{i_1}, h_{i_2}\})$, $v_{h_{i_1}}(\alpha_{j_1}) = 0 < 1 \leq v_{h_{i_1}}(h_{i_3})$, and $v_{h_{i_2}}(\alpha_{j_1}) = 0 < 1 \leq v_{h_{i_2}}(h_{i_3})$. This contradicts the supposition that π is j-envy-free. □

We have shown in Lemma 39 that at least two triples in π each contain exactly one agent in H . Suppose $t_\beta, t_\gamma \in \pi$ are two such triples and $t_\beta = \{h_{a_1}, \alpha_{b_1}, \alpha_{b_2}\}$ and $t_\gamma = \{h_{a_2}, \alpha_{b_3}, \alpha_{b_4}\}$.

Lemma 40. $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} = L$.

Proof. Suppose for a contradiction that $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} \neq L$.

By definition, $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} \cap H = \emptyset$ and $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} \neq L$ it must be that at least one agent in $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\}$ belongs to C . Assume without loss of generality that $\alpha_{b_1} \in C$.

Note that by construction of the instance, the valuation of any agent not in H for any other agent not in H is at least 1.

Since $\alpha_{b_2} \notin H$, it must be that $u_{\alpha_{b_2}}(\pi) = v_{\alpha_{b_2}}(\alpha_{b_1})$. By the design of the instance, since $\alpha_{b_2} \notin H$ and $\alpha_{b_1} \notin H$ it must be that $v_{\alpha_{b_2}}(\alpha_{b_1}) \in \{1, 2\}$. We consider each possibility of $u_{\alpha_{b_2}}(\pi) = v_{\alpha_{b_2}}(\alpha_{b_1})$.

Firstly, suppose $u_{\alpha_{b_2}}(\pi) = 1$. As noted earlier, since $\alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4} \in N \setminus H$ it must be that $v_{\alpha_{b_2}}(\alpha_{b_3}) \geq 1$ and $v_{\alpha_{b_2}}(\alpha_{b_4}) \geq 1$. It follows that α_{b_2} has j-envy for h_{a_2} , since $u_{\alpha_{b_2}}(\pi) = 1 < 2 \leq u_{\alpha_{b_2}}(\{\alpha_{b_3}, \alpha_{b_4}\})$, $v_{\alpha_{b_3}}(h_{a_2}) = 0 < 1 \leq v_{\alpha_{b_3}}(\alpha_{b_2})$, and $v_{\alpha_{b_4}}(h_{a_2}) = 0 < 1 \leq v_{\alpha_{b_4}}(\alpha_{b_2})$. This contradicts the supposition that π is j-envy-free.

Suppose then that $u_{\alpha_{b_2}}(\pi) = 2$, so $v_{\alpha_{b_2}}(\alpha_{b_1}) = 2$. Since $\alpha_{b_1} \in C$ by assumption, by the design of the instance it must be $\alpha_{b_2} \in C$. For the remainder of this lemma only, label $\alpha_{b_1} = c_{i_1}$ and $\alpha_{b_2} = c_{i_2}$. Since $v_{c_{i_2}}(c_{i_1}) = 2$ it follows that $(w_{i_2}, w_{i_1}) \in A$. Since G is antisymmetric it must be that $(w_{i_1}, w_{i_2}) \notin A$ and thus that $v_{c_{i_1}}(c_{i_2}) = 1$. Since $\pi(c_{i_1}) = \{c_{i_1}, c_{i_2}, h_{a_1}\}$ it follows that $u_{c_{i_1}}(\pi) = v_{c_{i_1}}(c_{i_2}) = 1$. Now α_{b_1} has j-envy for h_{a_2} , since $u_{\alpha_{b_1}}(\pi) = 1 < 2 \leq u_{\alpha_{b_1}}(\{\alpha_{b_3}, \alpha_{b_4}\})$, $v_{\alpha_{b_3}}(h_{a_2}) = 0 < 1 \leq v_{\alpha_{b_3}}(\alpha_{b_1})$, and $v_{\alpha_{b_4}}(h_{a_2}) = 0 < 1 \leq v_{\alpha_{b_4}}(\alpha_{b_1})$. This contradicts the supposition that π is j-envy-free. $\square$

Lemma 41. $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_2\}, \{l_3, l_4\}\}$.

Proof. By Lemma 40, $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} = L$. There are now three possibilities: first that $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_3\}, \{l_2, l_4\}\}$, second that $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_4\}, \{l_2, l_3\}\}$, and third that $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_2\}, \{l_3, l_4\}\}$.

First suppose $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_3\}, \{l_2, l_4\}\}$. Without loss of generality assume that $\alpha_{b_1} = l_1$. Now l_1 has j-envy for h_{a_2} since $u_{l_1}(\{h_{a_1}, l_3\}) = 1 < 3 = u_{l_1}(\{l_2, l_4\})$, $v_{l_2}(h_{a_2}) = 0 < 2 = v_{l_2}(l_1)$, and $v_{l_4}(h_{a_2}) = 0 < 1 = v_{l_4}(l_1)$.

Second suppose $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_4\}, \{l_2, l_3\}\}$. Without loss of generality assume that $\alpha_{b_1} = l_1$. As before, l_1 has j-envy for h_{a_2} since $u_{l_1}(\{h_{a_1}, l_4\}) = 1 < 3 = u_{l_1}(\{l_2, l_3\})$, $v_{l_2}(h_{a_2}) = 0 < 2 = v_{l_2}(l_1)$, and $v_{l_3}(h_{a_2}) = 0 < 1 = v_{l_3}(l_1)$.

It remains that $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_2\}, \{l_3, l_4\}\}$. $\square$

By Lemma 41, either $\{\alpha_{b_1}, \alpha_{b_2}\} = \{l_1, l_2\}$ or $\{\alpha_{b_1}, \alpha_{b_2}\} = \{l_3, l_4\}$. Without loss of generality assume that $\{\alpha_{b_1}, \alpha_{b_2}\} = \{l_3, l_4\}$.

Lemma 42. $u_{c_i}(\pi) \geq 3$ for each i where $1 \leq i \leq 3q$.

Proof. Suppose, to the contrary, some $1 \leq i \leq 3q$ exists where $u_{c_i}(\pi) < 3$. Then c_i has j-envy for h_{a_1} since $u_{c_i}(\pi) \leq 2 < 3 = u_{c_i}(\{l_3, l_4\})$, $v_{l_3}(h_{a_1}) = 0 < 1 = v_{l_3}(c_i)$, and $v_{l_4}(h_{a_1}) = 0 < 1 = v_{l_4}(c_i)$. This contradicts our supposition that π is j-envy-free. $\square$

Lemma 43. A partition into directed 3-cycles X exists in G .

Proof. Lemma 42 shows that $u_{c_i}(\pi) \geq 3$ for each i where $1 \leq i \leq 3q$. By construction, it follows that $\pi(c_i)$ contains two agents c_j, c_k such that $v_{c_i}(c_j) \geq 1$ and $v_{c_i}(c_k) = 2$. Hence c_k corresponds to a vertex $w_k \in W$ where $(w_i, w_k) \in A$ and, since G is antisymmetric, $(w_k, w_i) \notin A$. Since c_i was chosen arbitrarily it follows that $\{w_i, w_j, w_k\}$ is a directed 3-cycle in G . It follows thus that there are exactly q triples in M each containing three agents $\{c_i, c_j, c_k\}$ where the three corresponding vertices $\{w_i, w_j, w_k\}$ form a directed 3-cycle in G . From these triples a partition into directed 3-cycles X can be easily constructed. $\square$

A.3.1.3 Correctness of the reduction: conclusion

Theorem 6. *It is NP-complete to decide if a j-envy-free partition into triples exists in a given ASHG, even if preferences are ternary (and not necessarily symmetric).*

Proof. Suppose we are given an instance (N, V) and a partition into triples π . For any two agents $\alpha_i, \alpha_j \in N$ we can test if α_i has j-envy for α_j in constant time. We can therefore decide if π is j-envy-free in polynomial time, which shows that the above decision problem belongs to NP.

The above reduction from DTC can be performed in polynomial time. By Lemmas 37 and 43, the simple directed antisymmetric graph G has a partition into directed 3-cycles X if and only if an j-envy-free partition into triples exists in the constructed instance (N, V) . It follows that the reduction is valid and the above decision problem is NP-hard. $\square$

A.3.2 Proof of Theorem 7

We describe the reduction from Partition Into Triangles (PIT), in full. The reduction, shown in Figure 3, constructs an ASHG (N, V) given an arbitrary instance $G = (W, E)$ of PIT. Since valuations are symmetric in (N, V) , we shall usually specify valuations in one direction only. For example, instead of writing “let $v_{\alpha_i}(\alpha_j) = v_{\alpha_j}(\alpha_i) = 1$ ” we write “let $v_{\alpha_i}(\alpha_j) = 1$ ”. Unless otherwise specified assume that $v_{\alpha_i}(\alpha_j) = 0$ for any $\alpha_i, \alpha_j \in N$.

First, construct a set of eleven agents $H = \{h_1, \dots, h_{11}\}$. The structure of the valuations between these agents has a number of symmetries. In order to simplify the description of H we introduce a function $r : \{2, \dots, 11\} \times \mathbb{N} \mapsto \{2, \dots, 11\}$ that describes a cyclic succession among agents $h_2, \dots, h_{11}$, where $r(i, c) = ((i - 2) \oplus_{10} c) + 2$. For each i where $2 \leq i \leq 11$ let $v_{h_1}(h_i) = 2$. For each i where $2 \leq i \leq 11$, let:

- $v_{h_i}(h_{r(i,1)}) = 4$ if i is even otherwise 5
- $v_{h_i}(h_{r(i,2)}) = 6$ if i is even otherwise 3
- $v_{h_i}(h_{r(i,3)}) = 1$
- $v_{h_i}(h_{r(i,4)}) = 1$
- $v_{h_i}(h_{r(i,5)}) = 3$

Next, add four agents l_1, l_2, l_3, l_4 . Let $v_{l_1}(l_2) = v_{l_3}(l_4) = 2$ and $v_{l_1}(l_3) = v_{l_1}(l_4) = v_{l_2}(l_3) = v_{l_2}(l_4) = 1$. Let $L = \{l_1, \dots, l_4\}$.

Next, add $3q$ agents $c_1, \dots, c_{3q}$. Let $v_{c_i}(l_r) = 3$ for each i and r where $1 \leq i \leq 3q$ and $1 \leq r \leq 4$. For each i and j where $1 \leq i, j \leq 3q$ let $v_{c_i}(c_j) = 3$ if $\{w_i, w_j\} \in E$ and 2 otherwise. Let $C = \{c_1, \dots, c_{3q}\}$.

Finally, let $N = H \cup L \cup C$.

The structure of the valuations among agents in C reflects the undirected graph G . As we shall see in Lemma 54, a partition into triangles in G exists if and only if some j-envy-free partition into triples exists that contains q triples each containing three agents in C corresponding to three pairwise adjacent vertices in G .

A.3.2.1 Correctness of the reduction: first direction

Lemma 44. *In the reduction, if X is a partition into triangles in G then a j -envy-free partition into triples π exists in (N, V) .*

Proof. Suppose X is a partition into triangles. We shall construct a partition into triples π that is j -envy-free. First, add $\{h_2, h_{10}, h_{11}\}$, $\{h_5, h_6, h_8\}$, $\{h_1, h_9, h_4\}$, $\{h_3, l_1, l_2\}$ and $\{h_7, l_3, l_4\}$ to π . Next, for each triangle $X_p = \{w_i, w_j, w_k\}$ in X , add $\{c_i, c_j, c_k\}$ to π .

Suppose for a contradiction that some agent α_j exists where α_j has j -envy for some other agent α_{k_1} where $\pi(\alpha_{k_1}) = \{\alpha_{k_1}, \alpha_{k_2}, \alpha_{k_3}\}$. Recalling that $N = H \cup L \cup C$, it must be that either $\alpha_{k_1} \in H$, $\alpha_{k_1} \in L$, or $\alpha_{k_1} \in C$. We show that each case leads to a contradiction. It follows that no such α_j exists and thus that π is j -envy-free.

- Suppose $\alpha_{k_1} \in H$. Either $\alpha_{k_1} \in \{h_3, h_7\}$ or $\alpha_{k_1} \in H \setminus \{h_3, h_7\}$.
 - Suppose $\alpha_{k_1} \in \{h_3, h_7\}$. Then it must be that either $\{\alpha_{k_2}, \alpha_{k_3}\} = \{l_1, l_2\}$ or $\{\alpha_{k_2}, \alpha_{k_3}\} = \{l_3, l_4\}$. Suppose firstly that $\{\alpha_{k_2}, \alpha_{k_3}\} = \{l_1, l_2\}$. We can see immediately that $\alpha_j \notin H$ since otherwise $u_{\alpha_j}(\{l_1, l_2\}) = 0$. It must also be that $\alpha_j \notin C$, since $u_{c_a}(\{l_1, l_2\}) = 6 = u_{c_a}(\pi)$ for any $c_a \in C$. Similarly, $u_{l_3}(\{l_1, l_2\}) = 2 = u_{l_3}(\pi)$ and $u_{l_4}(\{l_1, l_2\}) = 2 = u_{l_4}(\pi)$ so $\alpha_j \neq l_3$ and $\alpha_j \neq l_4$. This shows $\alpha_j \notin L$. We have shown that $\alpha_j \notin H$, $\alpha_j \notin C$, and $\alpha_j \notin L$, which is a contradiction. The proof for the case in which $\{\alpha_{k_2}, \alpha_{k_3}\} = \{l_3, l_4\}$ is symmetric and also leads to a contradiction.
 - Suppose $\alpha_{k_1} \in H \setminus \{h_3, h_7\}$. If $\alpha_{k_1} = h_1$ then $\{\alpha_{k_2}, \alpha_{k_3}\} = \{h_4, h_9\}$ so it must be that $v_{h_4}(\alpha_j) > 2 = v_{h_4}(h_1)$ and $v_{h_9}(\alpha_j) > 2 = v_{h_9}(h_1)$, which is impossible by the design of H . The proof for every other assignment of α_{k_1} is similar: if $\alpha_{k_1} = h_2$ then $v_{h_{11}}(\alpha_j) > 5$, which is impossible. If $\alpha_{k_1} = h_4$ then $v_{h_1}(\alpha_j) > 2$, which is impossible. If $\alpha_{k_1} = h_5$ then $v_{h_6}(\alpha_j) > 5$ and $v_{h_8}(\alpha_j) > 1$, which is impossible. If $\alpha_{k_1} = h_6$ then $v_{h_8}(\alpha_j) > 6$, which is impossible. If $\alpha_{k_1} = h_8$ then $v_{h_6}(\alpha_j) > 6$, which is impossible. If $\alpha_{k_1} = h_9$ then $v_{h_1}(\alpha_j) > 2$, which is impossible. If $\alpha_{k_1} = h_{10}$ then $v_{h_2}(\alpha_j) > 6$, which is impossible. If $\alpha_{k_1} = h_{11}$ then $v_{h_2}(\alpha_j) > 5$ and $v_{h_{10}}(\alpha_j) > 4$, which is impossible.
- Suppose $\alpha_{k_1} \in C$. By construction, it must be that $\alpha_{k_1} = c_{i_1}$, $\alpha_{k_2} = c_{i_2}$, and $\alpha_{k_3} = c_{i_3}$ where $\{c_{i_1}, c_{i_2}, c_{i_3}\} \subseteq C$, where the corresponding vertices $\{w_{i_1}, w_{i_2}, w_{i_3}\}$ in G are a triangle. It follows that $v_{c_{i_2}}(c_{i_1}) = 3$. By assumption, α_j has j -envy for c_{i_1} so it must be that $v_{c_{i_2}}(\alpha_j) > v_{c_{i_2}}(c_{i_1}) = 3$, which is impossible by the design of C .
- Suppose $\alpha_{k_1} \in L$. It must be that $\alpha_{k_1} = l_{i_1}$ for some i_1 where $1 \leq i_1 \leq 4$, $\alpha_{k_2} = l_{i_2}$ for some i_2 where $1 \leq i_2 \leq 4$ and $\alpha_{k_3} = h_{i_3}$ where $i_3 \in \{3, 7\}$. If $\alpha_j \in H$ then $v_{l_{i_2}}(\alpha_j) = 0$ which contradicts the supposition that α_j has j -envy for l_{i_1} . Otherwise, if $\alpha_j \notin H$ then $v_{h_{i_3}}(\alpha_j) = 0$, which also contradicts the supposition that α_j has j -envy for l_{i_1} .

□

A.3.2.2 Correctness of the reduction: second direction Suppose π is a j-envy-free partition into triples in (N, V) . We shall prove the existence of a partition into triangles X in G . In the first part of this section (up to and including Lemma 50) we focus on H . To begin, we define two possible configurations of H in π . If some triple $t \in \pi$ contains exactly one agent in H then we say that H has an *open configuration* in π . Otherwise, we say that H has a *closed configuration* in π . We shall eventually show, in Lemma 50, that the only possible configuration of H in π is an open configuration. In Lemmas 45 – 49 we prove a number of intermediary results.

Lemma 45. *No triples t_1, t_2 in π exist such that t_1 contains exactly two agents in H and t_2 contains exactly one agent in H .*

Proof. Suppose for a contradiction that some such $t_1, t_2 \in \pi$ exist. Suppose $t_1 = \{h_{i_1}, h_{i_2}, \alpha_{j_1}\}$ and $t_2 = \{h_{i_3}, \alpha_{j_2}, \alpha_{j_3}\}$ where $1 \leq i_1, i_2, i_3 \leq 11$ and $\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3} \in N \setminus H$. Now h_{i_3} has j-envy for α_{j_1} since $u_{h_{i_3}}(\pi) = 0 < 2 \leq u_{h_{i_3}}(\{h_{i_1}, h_{i_2}\})$, $v_{h_{i_1}}(\alpha_{j_1}) = 0 < 1 \leq v_{h_{i_1}}(h_{i_3})$ and $v_{h_{i_2}}(\alpha_{j_1}) = 0 < 1 \leq v_{h_{i_2}}(h_{i_3})$. This contradicts our supposition that π is j-envy-free. $\square$

Lemma 46. *If $\sigma(H, \pi) = 4$ and $u_{h_1}(\pi) < 4$ then H has an open configuration in π .*

Proof. Suppose, to the contrary, that $\sigma(H, \pi) = 4$, $u_{h_1}(\pi) < 4$, and H has a closed configuration in π . Since $\sigma(H, \pi) = 4$ it must be that three triples in π each contain exactly three agents in H and one triple in π contains exactly two agents in H . Suppose then that $t_1, t_2, t_3 \in \pi$ each contain exactly three agents in H and $t_4 \in \pi$ contains exactly two agents in H . Since $u_{h_1}(\pi) < 4$ by assumption, by the design of H it follows that $\pi(h_1)$ contains at most one agent in $\{h_2, \dots, h_{11}\}$ and therefore $h_1 \in t_4$. It follows that $t_4 = \{h_1, h_{i_1}, \alpha_j\}$ for some $2 \leq i_1 \leq 11$ and $\alpha_j \in N \setminus H$. We use a case analysis to prove a contradiction occurs for each possible assignment of i_1 .

- Suppose first $i_1 = 2$. If $u_{h_4}(\pi) \leq 7$ then h_4 has j-envy for α_j , since $u_{h_4}(\pi) \leq 7 < 8 = u_{h_4}(\{h_1, h_2\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_4)$, and $v_{h_2}(\alpha_j) = 0 < 6 = v_{h_2}(h_4)$. It follows that $u_{h_4}(\pi) \geq 8$. By the assumptions regarding the structure of π , it must be that $\pi(h_4)$ contains three agents in H . Since $t_4 = \{h_1, h_2, \alpha_j\}$ it must be that $\pi(h_4) = \{h_4, h_{i_2}, h_{i_3}\}$ where $\{i_2, i_3\} \subset \{3, 5, 6, 7, 8, 9, 10, 11\}$. Recall that $v_{h_4}(h_3) = 5$, $v_{h_4}(h_5) = 4$, $v_{h_4}(h_6) = 6$, $v_{h_4}(h_7) = 1$, $v_{h_4}(h_8) = 1$, $v_{h_4}(h_9) = 3$, $v_{h_4}(h_{10}) = 1$, and $v_{h_4}(h_{11}) = 1$. Since we established $u_{h_4}(\pi) \geq 8$ it follows that there are 5 possibilities: $\{i_2, i_3\} = \{3, 9\}$, $\{i_2, i_3\} = \{3, 5\}$, $\{i_2, i_3\} = \{3, 6\}$, $\{i_2, i_3\} = \{5, 6\}$, and $\{i_2, i_3\} = \{6, 9\}$, which we shall now consider.
 - Suppose $\{i_2, i_3\} = \{3, 9\}$. It follows that h_2 has j-envy for h_9 , since $u_{h_2}(\pi) = 2 < 10 = u_{h_2}(\{h_3, h_4\})$, $v_{h_3}(h_9) = 1 < 4 = v_{h_3}(h_2)$, and $v_{h_4}(h_9) = 3 < 6 = v_{h_4}(h_2)$. This contradicts the supposition that π is j-envy-free.

- Suppose $\{i_2, i_3\} = \{3, 5\}$. It follows that h_2 has j-envy for h_5 , since $u_{h_2}(\pi) = 2 < 10 = u_{h_2}(\{h_3, h_4\})$, $v_{h_3}(h_5) = 3 < 4 = v_{h_3}(h_2)$, and $v_{h_4}(h_5) = 4 < 6 = v_{h_4}(h_2)$.
- Suppose $\{i_2, i_3\} = \{3, 6\}$. Consider h_{11} . If $u_{h_{11}}(\pi) \leq 6$ then h_{11} has j-envy for α_j , since $u_{h_{11}}(\pi) \leq 6 < 7 = u_{h_{11}}(\{h_1, h_2\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_{11})$, and $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$. It follows that $u_{h_{11}}(\pi) \geq 7$. We have established that $h_2 \notin \pi(h_{11})$, $h_3 \notin \pi(h_{11})$, and $h_6 \notin \pi(h_{11})$ so, by the design of H , it must be that $\pi(h_{11}) = \{h_9, h_{10}, h_{11}\}$. Now h_2 has j-envy for h_9 , since $u_{h_2}(\pi) = 2 < 11 = u_{h_2}(\{h_{10}, h_{11}\})$, $v_{h_{10}}(h_9) = 5 < 6 = v_{h_{10}}(h_2)$, and $v_{h_{11}}(h_9) = 3 < 5 = v_{h_{11}}(h_2)$.
- Suppose $\{i_2, i_3\} = \{5, 6\}$. Consider h_3 . If $u_{h_3}(\pi) \leq 5$ then h_3 has j-envy for α_j , since $u_{h_3}(\pi) \leq 5 < 6 = u_{h_3}(\{h_1, h_2\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_3)$, and $v_{h_2}(\alpha_j) = 0 < 4 = v_{h_2}(h_3)$. It follows that $u_{h_3}(\pi) \geq 6$. We have established that $h_2 \notin \pi(h_3)$, $h_4 \notin \pi(h_3)$, and $h_5 \notin \pi(h_3)$ so, by the design of H , it must be that $\pi(h_3) = \{h_3, h_8, h_{11}\}$. Now h_{11} has j-envy for α_j , since $u_{h_{11}}(\pi) = 4 < 7 = u_{h_{11}}(\{h_1, h_2\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_{11})$, and $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$.
- Suppose $\{i_2, i_3\} = \{6, 9\}$. Consider h_{11} . If $u_{h_{11}}(\pi) \leq 6$ then h_{11} has j-envy for α_j , since $u_{h_{11}}(\pi) \leq 6 < 7 = u_{h_{11}}(\{h_1, h_2\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_{11})$, and $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$. It follows that $u_{h_{11}}(\pi) \geq 7$. We have established that $h_2 \notin \pi(h_{11})$, $h_6 \notin \pi(h_{11})$, and $h_9 \notin \pi(h_{11})$ so, by the design of H , it must be that $\pi(h_{11}) = \{h_3, h_{10}, h_{11}\}$. Now h_2 has j-envy for h_{10} , since $u_{h_2}(\pi) = 2 < 9 = u_{h_2}(\{h_3, h_{11}\})$, $v_{h_3}(h_{10}) = 1 < 4 = v_{h_3}(h_2)$, and $v_{h_{11}}(h_{10}) = 4 < 5 = v_{h_{11}}(h_2)$.
- Suppose next $i_1 = 3$. If $u_{h_4}(\pi) \leq 6$ then h_4 has j-envy for α_j , since $u_{h_4}(\pi) \leq 6 < 7 = u_{h_4}(\{h_1, h_3\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_4)$, and $v_{h_3}(\alpha_j) = 0 < 5 = v_{h_3}(h_4)$. It follows that $u_{h_4}(\pi) \geq 7$. Since $\pi(h_4) \neq t_4$ it must be that $\pi(h_4) = \{h_4, h_{i_2}, h_{i_3}\}$ where $\{i_2, i_3\} \subset \{2, 5, 6, 7, 8, 9, 10, 11\}$. Recall that $v_{h_4}(h_2) = 6$, $v_{h_4}(h_5) = 4$, $v_{h_4}(h_6) = 6$, $v_{h_4}(h_7) = 1$, $v_{h_4}(h_8) = 1$, $v_{h_4}(h_9) = 3$, $v_{h_4}(h_{10}) = 1$, and $v_{h_4}(h_{11}) = 1$. Since we established $u_{h_4}(\pi) \geq 7$ it follows that there are 14 possibilities: $\{i_2, i_3\} = \{2, 5\}$, $\{i_2, i_3\} = \{2, 6\}$, $\{i_2, i_3\} = \{2, 7\}$, $\{i_2, i_3\} = \{2, 8\}$, $\{i_2, i_3\} = \{2, 9\}$, $\{i_2, i_3\} = \{2, 10\}$, $\{i_2, i_3\} = \{2, 11\}$, $\{i_2, i_3\} = \{5, 6\}$, $\{i_2, i_3\} = \{5, 9\}$, $\{i_2, i_3\} = \{6, 7\}$, $\{i_2, i_3\} = \{6, 8\}$, $\{i_2, i_3\} = \{6, 9\}$, $\{i_2, i_3\} = \{6, 10\}$, and $\{i_2, i_3\} = \{6, 11\}$, which we shall now consider.
 - Suppose $\{i_2, i_3\} = \{2, 5\}$. It follows that h_3 has j-envy for h_5 , since $u_{h_3}(\pi) = 2 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(h_5) = 1 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(h_5) = 4 < 5 = v_{h_4}(h_3)$.
 - Suppose $\{i_2, i_3\} = \{2, 6\}$. If $u_{h_5}(\pi) \leq 4$ then h_5 has j-envy for α_j , since $u_{h_5}(\pi) \leq 4 < 5 = u_{h_5}(\{h_1, h_3\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_5)$, and $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_5)$. It follows that $u_{h_5}(\pi) \geq 5$. We have established that $h_3 \notin \pi(h_5)$, $h_4 \notin \pi(h_5)$, and $h_6 \notin \pi(h_5)$ so, by the design of H , it must be that $\pi(h_5) = \{h_5, h_7, h_{10}\}$. It remains that $\pi(h_{11}) = \{h_8, h_9, h_{11}\}$. Now h_{11} has j-envy for α_j , since $u_{h_{11}}(\pi) = 4 < 5 = u_{h_{11}}(\{h_1, h_3\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_{11})$, and $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_{11})$.

- Suppose $\{i_2, i_3\} = \{2, 7\}$. It follows that h_3 has j-envy for h_7 , since $u_{h_3}(\pi) = 2 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(h_7) = 3 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(h_7) = 1 < 5 = v_{h_4}(h_3)$.
- Suppose $\{i_2, i_3\} = \{2, 8\}$. It follows that h_3 has j-envy for h_8 , since $u_{h_3}(\pi) = 2 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(h_8) = 1 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(h_8) = 1 < 5 = v_{h_4}(h_3)$.
- Suppose $\{i_2, i_3\} = \{2, 9\}$. It follows that h_3 has j-envy for h_9 , since $u_{h_3}(\pi) = 2 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(h_9) = 1 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(h_9) = 3 < 5 = v_{h_4}(h_3)$.
- Suppose $\{i_2, i_3\} = \{2, 10\}$. If $u_{h_{11}}(\pi) \leq 4$ then h_{11} has j-envy for α_j , since $u_{h_{11}}(\pi) \leq 4 < 5 = u_{h_5}(\{h_1, h_3\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_{11})$, and $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_{11})$. It follows that $u_{h_{11}}(\pi) \geq 5$. We have established that $h_2 \notin \pi(h_{11})$, $h_3 \notin \pi(h_{11})$, and $h_{10} \notin \pi(h_{11})$ so, by the design of H , it must be that $\pi(h_{11}) = \{h_6, h_9, h_{11}\}$. Since $h_5 \notin t_4$, it must be that $\pi(h_5)$ contains three agents in H and thus $\pi(h_5) = \{h_5, h_7, h_8\}$. Now h_6 has j-envy for h_5 , since $u_{h_6}(\pi) = 4 < 10 = u_{h_6}(\{h_7, h_8\})$, $v_{h_7}(h_5) = 3 < 4 = v_{h_7}(h_6)$, and $v_{h_8}(h_5) = 1 < 6 = v_{h_8}(h_6)$.
- Suppose $\{i_2, i_3\} = \{2, 11\}$. If $u_{h_5}(\pi) \leq 4$ then h_5 has j-envy for α_j , since $u_{h_5}(\pi) \leq 4 < 5 = u_{h_5}(\{h_1, h_3\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_5)$, and $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_5)$. It follows that $u_{h_5}(\pi) \geq 5$. We have established that $h_3 \notin \pi(h_5)$ and $h_4 \notin \pi(h_5)$ so, by the design of H , there are three possibilities: either $\pi(h_5) = \{h_5, h_6, h_7\}$, $\pi(h_5) = \{h_5, h_6, h_{10}\}$, or $\pi(h_5) = \{h_5, h_7, h_{10}\}$.
 - * If $\pi(h_5) = \{h_5, h_6, h_7\}$ then h_4 has j-envy for h_7 , since $u_{h_4}(\pi) = 7 < 10 = u_{h_4}(\{h_5, h_6\})$, $v_{h_5}(h_7) = 3 < 4 = v_{h_5}(h_4)$, and $v_{h_6}(h_7) = 4 < 6 = v_{h_6}(h_4)$.
 - * If $\pi(h_5) = \{h_5, h_6, h_{10}\}$ then h_4 has j-envy for h_{10} , since $u_{h_4}(\pi) = 7 < 10 = u_{h_4}(\{h_5, h_6\})$, $v_{h_5}(h_{10}) = 3 < 4 = v_{h_5}(h_4)$, and $v_{h_6}(h_{10}) = 1 < 6 = v_{h_6}(h_4)$.
 - * If $\pi(h_5) = \{h_5, h_7, h_{10}\}$ then it remains that $\pi(h_6) = \{h_6, h_8, h_9\}$. Now h_6 has j-envy for h_{10} , since $u_{h_6}(\pi) = 7 < 9 = u_{h_6}(\{h_5, h_7\})$, $v_{h_5}(h_{10}) = 3 < 5 = v_{h_5}(h_6)$, and $v_{h_7}(h_{10}) = 1 < 4 = v_{h_7}(h_6)$.
- Suppose $\{i_2, i_3\} = \{5, 6\}$. If $u_{h_2}(\pi) \leq 5$ then h_2 has j-envy for α_j , since $u_{h_2}(\pi) \leq 5 < 6 = u_{h_2}(\{h_1, h_3\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_2)$, and $v_{h_3}(\alpha_j) = 0 < 4 = v_{h_3}(h_2)$. It follows that $u_{h_2}(\pi) \geq 6$. Since $h_2 \notin t_4$ it must be that $\pi(h_2)$ contains three agents in H . Since $t_4 = \{h_1, h_3, \alpha_j\}$ and $\pi(h_4) = \{h_4, h_5, h_6\}$ it follows that $\pi(h_2) = \{h_2, h_{i_4}, h_{i_5}\}$ where $\{i_4, i_5\} \subset \{7, 8, 9, 10, 11\}$. Recall that $v_{h_2}(h_7) = 3$, $v_{h_2}(h_8) = 1$, $v_{h_2}(h_9) = 1$, $v_{h_2}(h_{10}) = 6$, and $v_{h_2}(h_{11}) = 5$. Since we established $u_{h_2}(\pi) \geq 6$ it follows that there are 7 possibilities: $\{h_{i_4}, h_{i_5}\} = \{7, 10\}$, $\{h_{i_4}, h_{i_5}\} = \{7, 11\}$, $\{h_{i_4}, h_{i_5}\} = \{8, 10\}$, $\{h_{i_4}, h_{i_5}\} = \{8, 11\}$, $\{h_{i_4}, h_{i_5}\} = \{9, 10\}$, $\{h_{i_4}, h_{i_5}\} = \{9, 11\}$, and $\{h_{i_4}, h_{i_5}\} = \{10, 11\}$, which we shall now consider.
 - * If $\{h_{i_4}, h_{i_5}\} = \{7, 10\}$ then it remains that $\pi(h_{11}) = \{h_8, h_9, h_{11}\}$. Now h_{11} has j-envy for h_7 , since $u_{h_{11}}(\pi) = 4 < 9 = u_{h_{11}}(\{h_2, h_{10}\})$, $v_{h_2}(h_7) = 3 < 5 = v_{h_2}(h_{11})$, and $v_{h_{10}}(h_7) = 1 < 4 = v_{h_{10}}(h_{11})$.

- * If $\{h_{i_4}, h_{i_5}\} = \{7, 11\}$ then h_3 has j-envy for h_7 , since $u_{h_3}(\pi) = 2 < 7 = u_{h_3}(\{h_2, h_{11}\})$, $v_{h_2}(h_7) = 3 < 4 = v_{h_2}(h_3)$, and $v_{h_{11}}(h_7) = 1 < 3 = v_{h_{11}}(h_3)$.
- * If $\{h_{i_4}, h_{i_5}\} = \{8, 10\}$ then it remains that $\pi(h_7) = \{h_7, h_9, h_{11}\}$. Now h_8 has j-envy for h_{11} , since $u_{h_8}(\pi) = 7 < 9 = u_{h_8}(\{h_7, h_9\})$, $v_{h_7}(h_{11}) = 1 < 5 = v_{h_7}(h_8)$, and $v_{h_9}(h_{11}) = 3 < 4 = v_{h_9}(h_8)$.
- * If $\{h_{i_4}, h_{i_5}\} = \{8, 11\}$ then h_3 has j-envy for h_8 , since $u_{h_3}(\pi) = 2 < 7 = u_{h_3}(\{h_2, h_{11}\})$, $v_{h_2}(h_8) = 1 < 4 = v_{h_2}(h_3)$, and $v_{h_{11}}(h_8) = 1 < 3 = v_{h_{11}}(h_3)$.
- * If $\{h_{i_4}, h_{i_5}\} = \{9, 10\}$ then it remains that $\pi(h_7) = \{h_7, h_8, h_{11}\}$. Now h_9 has j-envy for h_{11} , since $u_{h_9}(\pi) = 6 < 7 = u_{h_9}(\{h_7, h_8\})$, $v_{h_7}(h_{11}) = 1 < 3 = v_{h_7}(h_9)$, and $v_{h_8}(h_{11}) = 1 < 4 = v_{h_8}(h_9)$.
- * If $\{h_{i_4}, h_{i_5}\} = \{9, 11\}$ then it remains that $\pi(h_{10}) = \{h_7, h_8, h_{10}\}$. Now h_{10} has j-envy for h_9 , since $u_{h_{10}}(\pi) = 7 < 10 = u_{h_{10}}(\{h_2, h_{11}\})$, $v_{h_2}(h_9) = 1 < 6 = v_{h_2}(h_{10})$, and $v_{h_{11}}(h_9) = 3 < 4 = v_{h_{11}}(h_{10})$.
- * If $\{h_{i_4}, h_{i_5}\} = \{10, 11\}$ then it remains that $\pi(h_7) = \{h_7, h_8, h_9\}$. Now h_{10} has j-envy for h_7 , since $u_{h_{10}}(\pi) = 10 < 11 = u_{h_{10}}(\{h_8, h_9\})$, $v_{h_8}(h_7) = 5 < 6 = v_{h_8}(h_{10})$, and $v_{h_9}(h_7) = 3 < 5 = v_{h_9}(h_{10})$.
- Suppose $\{i_2, i_3\} = \{5, 9\}$. It follows that h_3 has j-envy for h_9 , since $u_{h_3}(\pi) = 2 < 8 = u_{h_3}(\{h_4, h_5\})$, $v_{h_4}(h_9) = 3 < 5 = v_{h_4}(h_3)$, and $v_{h_5}(h_9) = 1 < 3 = v_{h_5}(h_3)$.
- Suppose $\{i_2, i_3\} = \{6, 7\}$. Consider h_5 . Since $h_3 \notin \pi(h_5)$, $h_4 \notin \pi(h_5)$, $h_6 \notin \pi(h_5)$, and $h_7 \notin \pi(h_5)$ by the design of H it must be that $u_{h_5}(\pi) \leq 4$. It follows that h_5 has j-envy for h_7 , since $u_{h_5}(\pi) \leq 4 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_7) = 1 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_7) = 4 < 5 = v_{h_6}(h_5)$.
- Suppose $\{i_2, i_3\} = \{6, 8\}$. Consider h_5 . If $u_{h_5}(\pi) \leq 4$ then h_5 has j-envy for α_j , since $u_{h_5}(\pi) \leq 4 < 5 = u_{h_2}(\{h_1, h_3\})$, $v_{h_1}(\alpha_j) = 0 < 2 = v_{h_1}(h_5)$, and $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_5)$. It follows that $u_{h_5}(\pi) \geq 5$. Since $h_3 \notin \pi(h_5)$, $h_4 \notin \pi(h_5)$, and $h_6 \notin \pi(h_5)$, the only possibility is that $\pi(h_5) = \{h_5, h_7, h_{10}\}$. It remains that $\pi(h_2) = \{h_2, h_9, h_{11}\}$. Now h_{10} has j-envy for h_9 , since $u_{h_{10}}(\pi) = 4 < 10 = u_{h_{10}}(\{h_2, h_{11}\})$, $v_{h_2}(h_9) = 1 < 6 = v_{h_2}(h_{10})$, and $v_{h_{11}}(h_9) = 3 < 4 = v_{h_{11}}(h_{10})$.
- Suppose $\{i_2, i_3\} = \{6, 9\}$. Consider h_5 . Since $h_3 \notin \pi(h_5)$, $h_4 \notin \pi(h_5)$, and $h_6 \notin \pi(h_5)$ by the design of H it must be that $u_{h_5}(\pi) \leq 6$. It follows that h_5 has j-envy for h_9 , since $u_{h_5}(\pi) \leq 6 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_9) = 3 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_9) = 1 < 5 = v_{h_6}(h_5)$.
- Suppose $\{i_2, i_3\} = \{6, 10\}$. Consider h_5 . Since $h_3 \notin \pi(h_5)$, $h_4 \notin \pi(h_5)$, $h_6 \notin \pi(h_5)$, and $h_{10} \notin \pi(h_5)$ by the design of H it must be that $u_{h_5}(\pi) \leq 4$. It follows that h_5 has j-envy for h_{10} , since $u_{h_5}(\pi) \leq 4 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_{10}) = 1 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_{10}) = 1 < 5 = v_{h_6}(h_5)$.
- Suppose $\{i_2, i_3\} = \{6, 11\}$. Consider h_5 . Since $h_3 \notin \pi(h_5)$, $h_4 \notin \pi(h_5)$, and $h_6 \notin \pi(h_5)$ by the design of H it must be that $u_{h_5}(\pi) \leq 6$. It follows that h_5 has j-envy for h_{11} , since $u_{h_5}(\pi) \leq 6 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_{11}) = 1 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_{11}) = 3 < 5 = v_{h_6}(h_5)$. $\square$

Lemma 47. *If $\sigma(H, \pi) = 4$ and $u_{h_1}(\pi) = 4$ then H has an open configuration in π .*

Proof. Suppose, to the contrary, that $\sigma(H, \pi) = 4$, $u_{h_1}(\pi) = 4$, and H has an open configuration in π . Since $\sigma(H, \pi) = 4$ it must be that three triples in π each contain exactly three agents in H and one triple in π contains exactly two agents in H . Suppose $t_1, t_2, t_3 \in \pi$ each contain exactly three agents in H and $t_4 \in \pi$ contains exactly two agents in H . Since $u_{h_1}(\pi) = 4$, by the design of H it follows that $\pi(h_1)$ contains two agents in $\{h_2, \dots, h_{11}\}$ and therefore $h_1 \notin t_4$. It follows that $t_4 = \{h_{i_1}, h_{i_2}, \alpha_j\}$ for some $2 \leq i_1, i_2 \leq 11$ and $\alpha_j \in (N \setminus H)$. We use a case analysis to prove a contradiction occurs for each possible assignment of $\{h_{i_1}, h_{i_2}\}$ where $2 \leq i_1, i_2 \leq 11$.

As before, in the proof of Lemma 46, the symmetries of H allow us to shorten the case analysis. Recall that the structure of the valuations between agents $h_2 \dots h_{11}$ has five symmetries. It follows that for any possible assignment of $\{i_1, i_2\}$ there exist in total five symmetric assignments, where the case analysis for one assignment is symmetric to the case analysis for the other four assignments. Since there are $\binom{10}{2} = 45$ possible assignments of $\{i_1, i_2\}$ where $2 \leq i_1, i_2 \leq 11$ and five symmetries we need only consider the nine assignments $\{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{3, 7\}$, of which no two are symmetric.

- Suppose $\{i_1, i_2\} = \{2, 3\}$. Since $h_2 \notin \pi(h_4)$ and $h_3 \notin \pi(h_4)$ it must be that $u_{h_4}(\pi) \leq 10$. It follows that h_4 has j-envy for α_j since $u_{h_4}(\pi) \leq 10 < 12 = u_{h_4}(\{h_2, h_3\})$, $v_{h_2}(\alpha_j) = 0 < 6 = v_{h_2}(h_4)$, and $v_{h_3}(\alpha_j) = 0 < 5 = v_{h_3}(h_4)$. This contradicts the supposition that π is j-envy-free. The following proofs concerning other possible assignments of $\{i_1, i_2\}$ are similar in technique although in some cases we must make further deductions about the utilities of agents in H .
- Suppose $\{i_1, i_2\} = \{2, 4\}$. Since $h_2 \notin \pi(h_3)$ and $h_4 \notin \pi(h_3)$ it follows that $u_{h_3}(\pi) \leq 6$. It then follows that h_3 has j-envy for α_j since $u_{h_3}(\pi) \leq 6 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(\alpha_j) = 0 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(\alpha_j) = 0 < 5 = v_{h_4}(h_3)$.
- Suppose $\{i_1, i_2\} = \{2, 5\}$. If $u_{h_4}(\pi) \leq 9$ then h_4 has j-envy for α_j , since $u_{h_4}(\pi) \leq 9 < 10 = u_{h_4}(\{h_2, h_5\})$, $v_{h_2}(\alpha_j) = 0 < 6 = v_{h_2}(h_4)$, and $v_{h_5}(\alpha_j) = 0 < 4 = v_{h_5}(h_4)$. It follows that $u_{h_4}(\pi) \geq 10$. The only possibility is that $\pi(h_4) = \{h_3, h_4, h_6\}$. Now h_3 has j-envy for α_j since $u_{h_3}(\pi) = 6 < 7 = u_{h_3}(\{h_2, h_5\})$, $v_{h_2}(\alpha_j) = 0 < 4 = v_{h_2}(h_3)$, and $v_{h_5}(\alpha_j) = 0 < 3 = v_{h_5}(h_3)$.
- Suppose $\{i_1, i_2\} = \{2, 6\}$. Since $h_2 \notin \pi(h_4)$ and $h_6 \notin \pi(h_4)$ it follows that $u_{h_4}(\pi) \leq 9$. Now h_4 has j-envy for α_j since $u_{h_4}(\pi) \leq 9 < 12 = u_{h_4}(\{h_2, h_6\})$, $v_{h_2}(\alpha_j) = 0 < 6 = v_{h_2}(h_4)$, and $v_{h_6}(\alpha_j) = 0 < 6 = v_{h_6}(h_4)$.
- Suppose $\{i_1, i_2\} = \{2, 7\}$. If $u_{h_4}(\pi) \leq 6$ then h_4 has j-envy for α_j , since $u_{h_4}(\pi) \leq 6 < 7 = u_{h_4}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 6 = v_{h_2}(h_4)$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_4)$. It follows that $u_{h_4}(\pi) \geq 7$. By the design of H , it follows that $\pi(h_4)$ contains three agents in H . Since $t_4 = \{h_2, h_7, \alpha_j\}$ it follows that $\pi(h_4) = \{h_4, h_{i_3}, h_{i_4}\}$ where $\{i_3, i_4\} \subset \{1, 3, 5, 6, 8, 9, 10, 11\}$. Recall that $v_{h_4}(h_1) = 2$, $v_{h_4}(h_3) = 5$, $v_{h_4}(h_5) = 4$, $v_{h_4}(h_6) = 6$, $v_{h_4}(h_8) = 1$, $v_{h_4}(h_9) = 3$, $v_{h_4}(h_{10}) = 1$, and $v_{h_4}(h_{11}) = 1$. Since we established $u_{h_4}(\pi) \geq 7$ it follows

that there are 11 possibilities: $\{i_3, i_4\} = \{1, 3\}$, $\{i_3, i_4\} = \{1, 6\}$, $\{i_3, i_4\} = \{3, 5\}$, $\{i_3, i_4\} = \{3, 6\}$, $\{i_3, i_4\} = \{3, 9\}$, $\{i_3, i_4\} = \{5, 6\}$, $\{i_3, i_4\} = \{5, 9\}$, $\{i_3, i_4\} = \{6, 8\}$, $\{i_3, i_4\} = \{6, 9\}$, $\{i_3, i_4\} = \{6, 10\}$, and $\{i_3, i_4\} = \{6, 11\}$, which we shall now consider.

- Suppose $\{i_3, i_4\} = \{1, 3\}$. It follows that h_2 has j-envy for h_1 since $u_{h_2}(\pi) = 3 < 10 = u_{h_2}(\{h_3, h_4\})$, $v_{h_3}(h_1) = 2 < 4 = v_{h_3}(h_2)$, and $v_{h_4}(h_1) = 2 < 6 = v_{h_4}(h_2)$.
- Suppose $\{i_3, i_4\} = \{1, 6\}$. Consider h_5 . It must be that $u_{h_5}(\pi) \leq 6$. Now h_5 has j-envy for h_1 since $u_{h_5}(\pi) \leq 6 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_1) = 2 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_1) = 2 < 5 = v_{h_6}(h_5)$.
- Suppose $\{i_3, i_4\} = \{3, 5\}$. It follows that h_2 has j-envy for h_5 since $u_{h_2}(\pi) = 3 < 10 = u_{h_2}(\{h_3, h_4\})$, $v_{h_3}(h_5) = 3 < 4 = v_{h_3}(h_2)$, and $v_{h_4}(h_5) = 4 < 6 = v_{h_4}(h_2)$.
- Suppose $\{i_3, i_4\} = \{3, 6\}$. In this case, consider $\pi(h_1)$. Since $h_1 \notin t_4$ it follows that $\pi(h_1)$ contains three agents in H . Suppose $\pi(h_1) = \{h_1, h_{i_5}, h_{i_6}\}$ where $2 \leq i_5, i_6 \leq 11$. Since we have established $\pi(h_2) = \{h_2, h_7, \alpha_j\}$ and $\pi(h_4) = \{h_3, h_4, h_6\}$ it follows that $\{h_{i_5}, h_{i_6}\} \subset \{5, 8, 9, 10, 11\}$. Thus there are $\binom{5}{2} = 10$ possible assignments of $\{h_{i_5}, h_{i_6}\}$, which we shall now consider.
 - * If $\{h_{i_5}, h_{i_6}\} = \{5, 8\}$ then h_7 has j-envy for h_1 since $u_{h_7}(\pi) = 3 < 8 = u_{h_7}(\{h_5, h_8\})$, $v_{h_5}(h_1) = 2 < 3 = v_{h_5}(h_7)$, and $v_{h_8}(h_1) = 2 < 5 = v_{h_8}(h_7)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{5, 9\}$ then h_7 has j-envy for h_1 since $u_{h_7}(\pi) = 3 < 6 = u_{h_7}(\{h_5, h_9\})$, $v_{h_5}(h_1) = 2 < 3 = v_{h_5}(h_7)$, and $v_{h_9}(h_1) = 2 < 3 = v_{h_9}(h_7)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{5, 10\}$ then $u_{h_{10}}(\pi) = 5$. It follows that h_{10} has j-envy for α_j since $u_{h_{10}}(\pi) = 5 < 7 = u_{h_{10}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 6 = v_{h_2}(h_{10})$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_{10})$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{5, 11\}$ then $u_{h_{11}}(\pi) = 3$. It follows that h_{11} has j-envy for α_j since $u_{h_{11}}(\pi) = 3 < 6 = u_{h_{11}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_{11})$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{8, 9\}$ then h_7 has j-envy for h_1 since $u_{h_7}(\pi) = 3 < 8 = u_{h_7}(\{h_8, h_9\})$, $v_{h_8}(h_1) = 2 < 5 = v_{h_8}(h_7)$, and $v_{h_9}(h_1) = 2 < 3 = v_{h_9}(h_7)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{8, 10\}$ then it remains that $\pi(h_9) = \{h_5, h_9, h_{11}\}$ and thus $u_{h_9}(\pi) = u_{\{h_5, h_{11}\}} = 4$. It follows that h_9 has j-envy for h_1 since $u_{h_9}(\pi) = 4 < 9 = u_{h_9}(\{h_8, h_{10}\})$, $v_{h_8}(h_1) = 2 < 4 = v_{h_8}(h_9)$, and $v_{h_{10}}(h_1) = 2 < 5 = v_{h_{10}}(h_9)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{8, 11\}$ then $u_{h_{11}}(\pi) = 3$. It follows that h_{11} has j-envy for α_j since $u_{h_{11}}(\pi) = 3 < 6 = u_{h_{11}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_{11})$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{9, 10\}$ then it remains that $\pi(h_9) = \{h_5, h_8, h_{11}\}$ and thus $u_{h_{11}}(\pi) = u_{\{h_5, h_8\}} = 2$. It follows that h_{11} has j-envy for α_j since $u_{h_{11}}(\pi) = 2 < 6 = u_{h_{11}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_{11})$.

- * If $\{h_{i_5}, h_{i_6}\} = \{9, 11\}$ then $u_{h_{11}}(\pi) = 5$. It follows that h_{11} has j-envy for α_j since $u_{h_{11}}(\pi) = 5 < 6 = u_{h_{11}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_{11})$.
- * If $\{h_{i_5}, h_{i_6}\} = \{10, 11\}$ then h_2 has j-envy for h_1 since $u_{h_2}(\pi) = 3 < 11 = u_{h_2}(\{h_{10}, h_{11}\})$, $v_{h_{10}}(h_1) = 2 < 6 = v_{h_{10}}(h_2)$, and $v_{h_{11}}(h_1) = 2 < 5 = v_{h_{11}}(h_2)$.
- Suppose $\{i_3, i_4\} = \{3, 9\}$. It follows that h_2 has j-envy for h_9 since $u_{h_2}(\pi) = 3 < 10 = u_{h_2}(\{h_3, h_4\})$, $v_{h_3}(h_9) = 1 < 4 = v_{h_3}(h_2)$, and $v_{h_4}(h_9) = 3 < 6 = v_{h_4}(h_2)$.
- Suppose $\{i_3, i_4\} = \{5, 6\}$. Consider h_3 . If $u_{h_3}(\pi) \leq 4$ then h_3 has j-envy for α_j , since $u_{h_3}(\pi) \leq 4 < 5 = u_{h_3}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 4 = v_{h_2}(h_3)$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_3)$. It follows that $u_{h_3}(\pi) \geq 5$. We have established that $h_4 \notin \pi(h_3)$, $h_5 \notin \pi(h_3)$ and $h_2 \notin \pi(h_3)$ so, by the design of H , there are three possibilities: either $\pi(h_3) = \{h_1, h_3, h_8\}$, $\pi(h_3) = \{h_1, h_3, h_{11}\}$, or $\pi(h_3) = \{h_3, h_8, h_{11}\}$.
 - * If $\pi(h_3) = \{h_1, h_3, h_8\}$ then $u_{h_8}(\pi) = 5$. It follows that h_8 has j-envy for α_j , since $u_{h_8}(\pi) = 5 < 6 = u_{h_8}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 1 = v_{h_2}(h_8)$, and $v_{h_7}(\alpha_j) = 0 < 5 = v_{h_7}(h_8)$.
 - * If $\pi(h_3) = \{h_1, h_3, h_{11}\}$ then $u_{h_{11}}(\pi) = 5$. It follows that h_{11} has j-envy for α_j , since $u_{h_{11}}(\pi) = 5 < 6 = u_{h_{11}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_{11})$.
 - * If $\pi(h_3) = \{h_3, h_8, h_{11}\}$ then $u_{h_8}(\pi) = 4$. It follows that h_8 has j-envy for α_j , since $u_{h_8}(\pi) = 4 < 6 = u_{h_8}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 1 = v_{h_2}(h_8)$, and $v_{h_7}(\alpha_j) = 0 < 5 = v_{h_7}(h_8)$.
- Suppose $\{i_3, i_4\} = \{5, 9\}$. Consider h_6 . It must be that $u_{h_6}(\pi) \leq 9$. Now h_6 has j-envy for h_9 since $u_{h_6}(\pi) \leq 9 < 11 = u_{h_6}(\{h_4, h_5\})$, $v_{h_4}(h_9) = 3 < 6 = v_{h_4}(h_6)$, and $v_{h_5}(h_9) = 1 < 5 = v_{h_5}(h_6)$.
- Suppose $\{i_3, i_4\} = \{6, 8\}$. Consider h_{10} . If $u_{h_{10}}(\pi) \leq 6$ then h_{10} has j-envy for α_j , since $u_{h_{10}}(\pi) \leq 6 < 7 = u_{h_{10}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 6 = v_{h_2}(h_{10})$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_{10})$. It follows that $u_{h_{10}}(\pi) \geq 7$. We have established that $h_2 \notin \pi(h_{10})$ and $h_8 \notin \pi(h_{10})$ so, by the design of H , there are four possibilities: either $\pi(h_{10}) = \{h_1, h_9, h_{10}\}$, $\pi(h_{10}) = \{h_5, h_9, h_{10}\}$, $\pi(h_{10}) = \{h_5, h_{10}, h_{11}\}$, or $\pi(h_{10}) = \{h_9, h_{10}, h_{11}\}$.
 - * If $\pi(h_{10}) = \{h_1, h_9, h_{10}\}$ then h_8 has j-envy for h_1 , since $u_{h_8}(\pi) = 7 < 10 = u_{h_8}(\{h_9, h_{10}\})$, $v_{h_9}(h_1) = 2 < 4 = v_{h_9}(h_8)$, and $v_{h_{10}}(h_1) = 2 < 6 = v_{h_{10}}(h_8)$.
 - * If $\pi(h_{10}) = \{h_5, h_9, h_{10}\}$ then h_8 has j-envy for h_5 , since $u_{h_8}(\pi) = 7 < 10 = u_{h_8}(\{h_9, h_{10}\})$, $v_{h_9}(h_5) = 1 < 4 = v_{h_9}(h_8)$, and $v_{h_{10}}(h_5) = 3 < 6 = v_{h_{10}}(h_8)$.
 - * If $\pi(h_{10}) = \{h_5, h_{10}, h_{11}\}$ then $u_{h_{11}}(\pi) = 5$. It follows that h_{11} has j-envy for α_j , since $u_{h_{11}}(\pi) = 5 < 6 = u_{h_{11}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_j) = 0 < 5 = v_{h_2}(h_{11})$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_{11})$.
 - * If $\pi(h_{10}) = \{h_9, h_{10}, h_{11}\}$ then h_8 has j-envy for h_{11} , since $u_{h_8}(\pi) = 7 < 10 = u_{h_8}(\{h_9, h_{10}\})$, $v_{h_9}(h_{11}) = 3 < 4 = v_{h_9}(h_8)$, and $v_{h_{10}}(h_{11}) = 4 < 6 = v_{h_{10}}(h_8)$.

- Suppose $\{i_3, i_4\} = \{6, 9\}$. It must be that $u_{h_5}(\pi) \leq 6$. Now h_5 has j-envy for h_9 since $u_{h_5}(\pi) \leq 6 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_9) = 3 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_9) = 1 < 5 = v_{h_6}(h_5)$.
- Suppose $\{i_3, i_4\} = \{6, 10\}$. It must be that $u_{h_5}(\pi) \leq 5$. Now h_5 has j-envy for h_{10} since $u_{h_5}(\pi) \leq 5 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_{10}) = 1 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_{10}) = 1 < 5 = v_{h_6}(h_5)$.
- Suppose $\{i_3, i_4\} = \{6, 11\}$. It must be that $u_{h_5}(\pi) \leq 6$. Now h_5 has j-envy for h_{11} since $u_{h_5}(\pi) \leq 6 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_{11}) = 1 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_{11}) = 3 < 5 = v_{h_6}(h_5)$.
- Suppose $\{i_1, i_2\} = \{3, 4\}$. If $u_{h_2}(\pi) \leq 9$ then h_2 has j-envy for α_j , since $u_{h_2}(\pi) \leq 9 < 10 = u_{h_2}(\{h_3, h_4\})$, $v_{h_3}(\alpha_j) = 0 < 4 = v_{h_3}(h_2)$, and $v_{h_4}(\alpha_j) = 0 < 6 = v_{h_4}(h_2)$. It follows that $u_{h_2}(\pi) \geq 10$. The only possibility is that $\pi(h_2) = \{h_2, h_{10}, h_{11}\}$. Now consider h_5 . If $u_{h_5}(\pi) \leq 6$ then h_5 has j-envy for α_j , since $u_{h_5}(\pi) \leq 6 < 7 = u_{h_5}(\{h_3, h_4\})$, $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_5)$, and $v_{h_4}(\alpha_j) = 0 < 4 = v_{h_4}(h_5)$. It follows that $u_{h_5}(\pi) \geq 7$. Since we have established $\pi(h_2) = \{h_2, h_{10}, h_{11}\}$ and $t_4 = \{h_3, h_4, \alpha_j\}$, there are just two possibilities: either $\pi(h_5) = \{h_1, h_5, h_6\}$ or $\pi(h_5) = \{h_5, h_6, h_7\}$.
 - If $\pi(h_5) = \{h_1, h_5, h_6\}$ then h_4 has j-envy for h_1 since $u_{h_4}(\pi) = 5 < 10 = u_{h_4}(\{h_5, h_6\})$, $v_{h_5}(h_1) = 2 < 4 = v_{h_5}(h_4)$, and $v_{h_6}(h_1) = 2 < 6 = v_{h_6}(h_4)$.
 - If $\pi(h_5) = \{h_5, h_6, h_7\}$ then h_4 has j-envy for h_7 since $u_{h_4}(\pi) = 5 < 10 = u_{h_4}(\{h_5, h_6\})$, $v_{h_5}(h_7) = 3 < 4 = v_{h_5}(h_4)$, and $v_{h_6}(h_7) = 4 < 6 = v_{h_6}(h_4)$.
- Suppose $\{i_1, i_2\} = \{3, 5\}$. If $u_{h_4}(\pi) \leq 8$ then h_4 has j-envy for α_j , since $u_{h_4}(\pi) \leq 8 < 9 = u_{h_4}(\{h_3, h_5\})$, $v_{h_3}(\alpha_j) = 0 < 5 = v_{h_3}(h_4)$, and $v_{h_5}(\alpha_j) = 0 < 4 = v_{h_5}(h_4)$. It follows that $u_{h_4}(\pi) \geq 9$. There are three possibilities: either $\pi(h_4) = \{h_2, h_4, h_6\}$, $\pi(h_4) = \{h_2, h_4, h_9\}$, or $\pi(h_4) = \{h_4, h_6, h_9\}$.
 - Suppose $\pi(h_4) = \{h_2, h_4, h_6\}$. In this case, consider $\pi(h_1)$. Since $h_1 \notin t_4$ it follows that $\pi(h_1)$ contains three agents in H . Suppose $\pi(h_1) = \{h_1, h_{i_5}, h_{i_6}\}$ where $1 \leq i_5, i_6 \leq 11$. Since we have established $\pi(h_3) = \{h_3, h_5, \alpha_j\}$ and $\pi(h_2) = \{h_2, h_4, h_6\}$ it follows that $\{h_{i_5}, h_{i_6}\} \subset \{7, 8, 9, 10, 11\}$. Thus there are $\binom{5}{2} = 10$ possible assignments of $\{h_{i_5}, h_{i_6}\}$, which we shall now consider.
 - * If $\{h_{i_5}, h_{i_6}\} = \{7, 8\}$ then h_6 has j-envy for h_1 , since $u_{h_6}(\pi) = 7 < 10 = u_{h_6}(\{h_7, h_8\})$, $v_{h_7}(h_1) = 2 < 4 = v_{h_7}(h_6)$, and $v_{h_8}(h_1) = 2 < 6 = v_{h_8}(h_6)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{7, 9\}$ then it remains that $\pi(h_8) = \{h_8, h_{10}, h_{11}\}$. It follows that h_8 has j-envy for h_1 , since $u_{h_8}(\pi) = 7 < 9 = u_{h_8}(\{h_7, h_9\})$, $v_{h_7}(h_1) = 2 < 5 = v_{h_7}(h_8)$, and $v_{h_9}(h_1) = 2 < 4 = v_{h_9}(h_8)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{7, 10\}$ then h_7 has j-envy for α_j , since $u_{h_7}(\pi) = 3 < 4 = u_{h_7}(\{h_3, h_5\})$, $v_{h_3}(\alpha_j) = 0 < 1 = v_{h_3}(h_7)$, and $v_{h_5}(\alpha_j) = 0 < 3 = v_{h_5}(h_7)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{7, 11\}$ then h_7 has j-envy for α_j , since $u_{h_7}(\pi) = 3 < 4 = u_{h_7}(\{h_3, h_5\})$, $v_{h_3}(\alpha_j) = 0 < 1 = v_{h_3}(h_7)$, and $v_{h_5}(\alpha_j) = 0 < 3 = v_{h_5}(h_7)$.

- * If $\{h_{i_5}, h_{i_6}\} = \{8, 9\}$ then it remains that $\pi(h_7) = \{h_7, h_{10}, h_{11}\}$. It follows that h_7 has j-envy for h_1 , since $u_{h_7}(\pi) = 2 < 8 = u_{h_7}(\{h_8, h_9\})$, $v_{h_8}(h_1) = 2 < 5 = v_{h_8}(h_7)$, and $v_{h_9}(h_1) = 2 < 3 = v_{h_9}(h_7)$.
- * If $\{h_{i_5}, h_{i_6}\} = \{8, 10\}$ then it remains that $\pi(h_9) = \{h_7, h_9, h_{11}\}$. It follows that h_9 has j-envy for h_1 , since $u_{h_9}(\pi) = 6 < 9 = u_{h_9}(\{h_8, h_{10}\})$, $v_{h_8}(h_1) = 2 < 4 = v_{h_8}(h_9)$, and $v_{h_{10}}(h_1) = 2 < 5 = v_{h_{10}}(h_9)$.
- * If $\{h_{i_5}, h_{i_6}\} = \{8, 11\}$ then h_{11} has j-envy for α_j , since $u_{h_{11}}(\pi) = 3 < 4 = u_{h_{11}}(\{h_3, h_5\})$, $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_{11})$, and $v_{h_5}(\alpha_j) = 0 < 1 = v_{h_5}(h_{11})$.
- * If $\{h_{i_5}, h_{i_6}\} = \{9, 10\}$ then it remains that $\pi(h_{11}) = \{h_7, h_8, h_{11}\}$. It follows that h_{11} has j-envy for h_1 , since $u_{h_{11}}(\pi) = 2 < 7 = u_{h_{11}}(\{h_9, h_{10}\})$, $v_{h_9}(h_1) = 2 < 3 = v_{h_9}(h_{11})$, and $v_{h_{10}}(h_1) = 2 < 4 = v_{h_{10}}(h_{11})$.
- * If $\{h_{i_5}, h_{i_6}\} = \{9, 11\}$ then it remains that $\pi(h_{10}) = \{h_7, h_8, h_{10}\}$. It follows that h_{10} has j-envy for h_1 , since $u_{h_{10}}(\pi) = 7 < 9 = u_{h_{10}}(\{h_9, h_{11}\})$, $v_{h_9}(h_1) = 2 < 5 = v_{h_9}(h_{10})$, and $v_{h_{11}}(h_1) = 2 < 4 = v_{h_{11}}(h_{10})$.
- * If $\{h_{i_5}, h_{i_6}\} = \{10, 11\}$ then h_2 has j-envy for h_1 , since $u_{h_2}(\pi) = 7 < 11 = u_{h_2}(\{h_{10}, h_{11}\})$, $v_{h_{10}}(h_1) = 2 < 6 = v_{h_{10}}(h_2)$, and $v_{h_{11}}(h_1) = 2 < 5 = v_{h_{11}}(h_2)$.
- Suppose $\pi(h_4) = \{h_2, h_4, h_9\}$. It follows that h_3 has j-envy for h_9 , since $u_{h_3}(\pi) = 3 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(h_9) = 1 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(h_9) = 3 < 5 = v_{h_4}(h_3)$.
- Suppose $\pi(h_4) = \{h_4, h_6, h_9\}$. It follows that h_5 has j-envy for h_9 , since $u_{h_5}(\pi) = 3 < 9 = u_{h_5}(\{h_4, h_6\})$, $v_{h_4}(h_9) = 3 < 4 = v_{h_4}(h_5)$, and $v_{h_6}(h_9) = 1 < 5 = v_{h_6}(h_5)$.
- Suppose $\{i_1, i_2\} = \{3, 6\}$. It must be that $u_{h_4}(\pi) \leq 10$. It follows that h_4 has j-envy for α_j since $u_{h_4}(\pi) \leq 10 < 11 = u_{h_4}(\{h_3, h_6\})$, $v_{h_3}(\alpha_j) = 0 < 5 = v_{h_3}(h_4)$, and $v_{h_6}(\alpha_j) = 0 < 6 = v_{h_6}(h_4)$.
- Suppose $\{i_1, i_2\} = \{3, 7\}$. If $u_{h_8}(\pi) \leq 7$ then h_8 has j-envy for α_j , since $u_{h_8}(\pi) \leq 7 < 8 = u_{h_8}(\{h_3, h_7\})$, $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_8)$, and $v_{h_7}(\alpha_j) = 0 < 5 = v_{h_7}(h_8)$. It follows that $u_{h_8}(\pi) \geq 8$. By the design of H , it follows that $\pi(h_8)$ contains three agents in H . Since $t_4 = \{h_3, h_7, \alpha_j\}$ it must be that $\pi(h_8) = \{h_8, h_{i_3}, h_{i_4}\}$ where $\{i_3, i_4\} \subset \{1, 2, 4, 5, 6, 9, 10, 11\}$. Recall that $v_{h_8}(h_1) = 2$, $v_{h_8}(h_2) = 1$, $v_{h_8}(h_4) = 1$, $v_{h_8}(h_5) = 1$, $v_{h_8}(h_6) = 6$, $v_{h_8}(h_9) = 4$, $v_{h_8}(h_{10}) = 6$, and $v_{h_8}(h_{11}) = 1$. Since we established $u_{h_8}(\pi) \geq 8$ it follows that there are five possibilities: $\{i_3, i_4\} = \{1, 6\}$, $\{i_3, i_4\} = \{1, 10\}$, $\{i_3, i_4\} = \{6, 9\}$, $\{i_3, i_4\} = \{6, 10\}$, and $\{i_3, i_4\} = \{9, 10\}$, which we shall now consider.
 - Suppose $\{i_3, i_4\} = \{1, 6\}$. It follows that h_7 has j-envy for h_1 , since $u_{h_7}(\pi) = 1 < 9 = u_{h_7}(\{h_6, h_8\})$, $v_{h_6}(h_1) = 2 < 4 = v_{h_6}(h_7)$, and $v_{h_8}(h_1) = 2 < 5 = v_{h_8}(h_7)$.
 - Suppose $\{i_3, i_4\} = \{1, 10\}$. Consider h_9 . It follows that $u_{h_9}(\pi) \leq 6$. Now h_9 has j-envy for h_1 , since $u_{h_9}(\pi) \leq 6 < 9 = u_{h_9}(\{h_8, h_{10}\})$, $v_{h_8}(h_1) = 2 < 4 = v_{h_8}(h_9)$, and $v_{h_{10}}(h_1) = 2 < 5 = v_{h_{10}}(h_9)$.

- Suppose $\{i_3, i_4\} = \{6, 9\}$. It follows that h_7 has j-envy for h_9 , since $u_{h_7}(\pi) = 1 < 9 = u_{h_7}(\{h_6, h_8\})$, $v_{h_6}(h_9) = 1 < 4 = v_{h_6}(h_7)$, and $v_{h_8}(h_9) = 4 < 5 = v_{h_8}(h_7)$.
- Suppose $\{i_3, i_4\} = \{6, 10\}$. In this case, consider $\pi(h_1)$. Since $h_1 \notin t_4$ it follows that $\pi(h_1)$ contains three agents in H . Suppose $\pi(h_1) = \{h_1, h_{i_5}, h_{i_6}\}$ where $2 \leq i_5, i_6 \leq 11$. Since we have established $\pi(h_3) = \{h_3, h_7, \alpha_j\}$ and $\pi(h_6) = \{h_6, h_8, h_{10}\}$ it follows that $\{h_{i_5}, h_{i_6}\} \subset \{2, 4, 5, 9, 11\}$. Thus there are $\binom{5}{2} = 10$ possible assignments of $\{h_{i_5}, h_{i_6}\}$, which we shall now consider.
 - * If $\{h_{i_5}, h_{i_6}\} = \{2, 4\}$ then h_3 has j-envy for h_1 , since $u_{h_3}(\pi) = 1 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(h_1) = 2 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(h_1) = 2 < 5 = v_{h_4}(h_3)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{2, 5\}$ then h_3 has j-envy for h_1 , since $u_{h_3}(\pi) = 1 < 7 = u_{h_3}(\{h_2, h_5\})$, $v_{h_2}(h_1) = 2 < 4 = v_{h_2}(h_3)$, and $v_{h_5}(h_1) = 2 < 3 = v_{h_5}(h_3)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{2, 9\}$ then it remains that $\pi(h_{11}) = \{h_4, h_5, h_{11}\}$. Now h_3 has j-envy for h_{11} , since $u_{h_3}(\pi) = 1 < 8 = u_{h_3}(\{h_4, h_5\})$, $v_{h_4}(h_{11}) = 1 < 5 = v_{h_4}(h_3)$, and $v_{h_5}(h_{11}) = 1 < 3 = v_{h_5}(h_3)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{2, 11\}$ then h_3 has j-envy for h_1 , since $u_{h_3}(\pi) = 1 < 7 = u_{h_3}(\{h_2, h_{11}\})$, $v_{h_2}(h_1) = 2 < 4 = v_{h_2}(h_3)$, and $v_{h_{11}}(h_1) = 2 < 3 = v_{h_{11}}(h_3)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{4, 5\}$ then h_3 has j-envy for h_1 , since $u_{h_3}(\pi) = 1 < 8 = u_{h_3}(\{h_5, h_5\})$, $v_{h_4}(h_1) = 2 < 5 = v_{h_4}(h_3)$, and $v_{h_5}(h_1) = 2 < 3 = v_{h_5}(h_3)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{4, 9\}$ then h_4 has j-envy for α_j , since $u_{h_4}(\pi) = 5 < 6 = u_{h_4}(\{h_3, h_7\})$, $v_{h_3}(\alpha_j) = 0 < 5 = v_{h_3}(h_4)$, and $v_{h_7}(\alpha_j) = 0 < 1 = v_{h_7}(h_4)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{4, 11\}$ then h_3 has j-envy for h_1 , since $u_{h_3}(\pi) = 1 < 8 = u_{h_3}(\{h_4, h_{11}\})$, $v_{h_4}(h_1) = 2 < 5 = v_{h_4}(h_3)$, and $v_{h_{11}}(h_1) = 2 < 3 = v_{h_{11}}(h_3)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{5, 9\}$ then h_5 has j-envy for α_j , since $u_{h_5}(\pi) = 3 < 6 = u_{h_5}(\{h_3, h_7\})$, $v_{h_3}(\alpha_j) = 0 < 3 = v_{h_3}(h_5)$, and $v_{h_7}(\alpha_j) = 0 < 3 = v_{h_7}(h_5)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{5, 11\}$ then h_3 has j-envy for h_1 , since $u_{h_3}(\pi) = 1 < 6 = u_{h_3}(\{h_5, h_{11}\})$, $v_{h_5}(h_1) = 2 < 3 = v_{h_5}(h_3)$, and $v_{h_{11}}(h_1) = 2 < 3 = v_{h_{11}}(h_3)$.
 - * If $\{h_{i_5}, h_{i_6}\} = \{9, 11\}$ then h_{10} has j-envy for h_1 , since $u_{h_{10}}(\pi) = 7 < 9 = u_{h_{10}}(\{h_9, h_{11}\})$, $v_{h_9}(h_1) = 2 < 5 = v_{h_9}(h_{10})$, and $v_{h_{11}}(h_1) = 2 < 4 = v_{h_{11}}(h_{10})$.
- Suppose $\{i_3, i_4\} = \{9, 10\}$. Consider h_2 . If $u_{h_2}(\pi) \leq 6$ then h_2 has j-envy for α_j , since $u_{h_2}(\pi) \leq 6 < 7 = u_{h_2}(\{h_3, h_7\})$, $v_{h_3}(\alpha_j) = 0 < 4 = v_{h_3}(h_2)$, and $v_{h_7}(\alpha_j) = 0 < 3 = v_{h_7}(h_2)$. It follows that $u_{h_2}(\pi) \geq 7$. We have established that $h_3 \notin \pi(h_2)$, $h_7 \notin \pi(h_2)$, and $h_{10} \notin \pi(h_2)$ so, by the design of H , there are three possibilities: either $\pi(h_2) = \{h_1, h_2, h_4\}$, $\pi(h_2) = \{h_1, h_2, h_{11}\}$, or $\pi(h_2) = \{h_2, h_4, h_{11}\}$.

- * If $\pi(h_2) = \{h_1, h_2, h_4\}$ then h_3 has j-envy for h_1 , since $u_{h_3}(\pi) = 1 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(h_1) = 2 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(h_1) = 2 < 5 = v_{h_4}(h_3)$.
- * If $\pi(h_2) = \{h_1, h_2, h_{11}\}$ then h_3 has j-envy for h_1 , since $u_{h_3}(\pi) = 1 < 7 = u_{h_3}(\{h_2, h_{11}\})$, $v_{h_2}(h_1) = 2 < 4 = v_{h_2}(h_3)$, and $v_{h_{11}}(h_1) = 2 < 3 = v_{h_{11}}(h_3)$.
- * If $\pi(h_2) = \{h_2, h_4, h_{11}\}$ then it remains that $\pi(h_1) = \{h_1, h_5, h_6\}$. Now h_7 has j-envy for h_1 , since $u_{h_7}(\pi) = 1 < 7 = u_{h_7}(\{h_5, h_6\})$, $v_{h_5}(h_1) = 2 < 3 = v_{h_5}(h_7)$, and $v_{h_6}(h_1) = 2 < 4 = v_{h_6}(h_7)$. $\square$

Lemma 48. *If $\sigma(H, \pi) = 4$ then H has an open configuration in π .*

Proof. Suppose $\sigma(H, \pi) = 4$. Consider $u_{h_1}(\pi)$. By the design of H , it must be that $2 \leq u_{h_1}(\pi) \leq 4$. If $u_{h_1}(\pi) < 4$ then Lemma 46 shows that H has an open configuration in π . If $u_{h_1}(\pi) = 4$ then Lemma 47 shows that H has an open configuration in π . $\square$

Lemma 49. *If $\sigma(H, \pi) = 5$ then H has an open configuration in π .*

Proof. Suppose, to the contrary, that $\sigma(H, \pi) = 5$ and H has a closed configuration in π . Then it must be that four triples in π each contain exactly two agents in H and one triple in π contains exactly three agents in H . Suppose $t_1, t_2, t_3, t_4 \in \pi$ each contain exactly two agents in H and $t_5 \in \pi$ contains exactly three agents in H , where $t_1 = \{h_{i_1}, h_{i_2}, \alpha_{j_1}\}$ for some $1 \leq i_1, i_2 \leq 11$ and $\alpha_{j_1} \in N \setminus H$. We use a case analysis on $v_{h_{i_1}}(h_{i_2})$ to prove a contradiction. Note that by the design of H it must be that $1 \leq v_{h_{i_1}}(h_{i_2}) \leq 6$.

- Suppose $v_{h_{i_1}}(h_{i_2}) = 6$. By the symmetry of H , assume without loss of generality that $\{i_1, i_2\} = \{2, 4\}$. It follows that $u_{h_3}(\pi) \leq 6$. Now h_3 has j-envy for α_{j_1} since $u_{h_3}(\pi) \leq 6 < 9 = u_{h_3}(\{h_2, h_4\})$, $v_{h_2}(\alpha_{j_1}) = 0 < 4 = v_{h_2}(h_3)$, and $v_{h_4}(\alpha_{j_1}) = 0 < 5 = v_{h_4}(h_3)$. This contradicts the supposition that π is j-envy-free. We shall use a similar technique to prove a contradiction when considering the other cases of $v_{h_{i_1}}(h_{i_2})$.
- Suppose $v_{h_{i_1}}(h_{i_2}) = 5$. Assume without loss of generality that $\{i_1, i_2\} = \{3, 4\}$. If $u_{h_2}(\pi) \leq 9$ then h_2 has j-envy for α_{j_1} since $u_{h_2}(\pi) \leq 9 < 10 = u_{h_2}(\{h_3, h_4\})$, $v_{h_3}(\alpha_{j_1}) = 0 < 4 = v_{h_3}(h_2)$, and $v_{h_4}(\alpha_{j_1}) = 0 < 6 = v_{h_4}(h_2)$. It follows that $u_{h_2}(\pi) \geq 10$. The only possibility is that $\pi(h_2) = \{h_2, h_{10}, h_{11}\}$. It must be that $\pi(h_2) = t_5$. We now consider h_6 . Since $h_6 \notin t_5$ it must be that either $h_6 \in t_2$, $h_6 \in t_3$, or $h_6 \in t_4$. Assume without loss of generality that $h_6 \in t_2$ and that $t_2 = \{h_6, h_{i_3}, \alpha_{j_2}\}$ for some $1 \leq i_3 \leq 11$ and $\alpha_{j_2} \in N \setminus H$. If $u_{h_6}(\pi) \leq 6$ then h_6 has j-envy for α_{j_1} since $u_{h_6}(\pi) \leq 6 < 7 = u_{h_6}(\{h_3, h_4\})$, $v_{h_3}(\alpha_{j_1}) = 0 < 1 = v_{h_3}(h_6)$, and $v_{h_4}(\alpha_{j_1}) = 0 < 6 = v_{h_4}(h_6)$. It follows that $u_{h_6}(\pi) \geq 7$. Since $v_{h_6}(\alpha_{j_2}) = 0$ it follows that $v_{h_6}(h_{i_3}) = u_{h_6}(\pi) \geq 7$, which is a contradiction.
- Suppose $v_{h_{i_1}}(h_{i_2}) = 4$. Assume without loss of generality that $\{i_1, i_2\} = \{2, 3\}$. If $u_{h_4}(\pi) \leq 10$ then h_4 has j-envy for α_{j_1} since $u_{h_4}(\pi) \leq 10 < 11 = u_{h_4}(\{h_2, h_3\})$, $v_{h_2}(\alpha_{j_1}) = 0 < 6 = v_{h_2}(h_4)$, and $v_{h_3}(\alpha_{j_1}) = 0 < 5 = v_{h_3}(h_4)$. It follows that $u_{h_4}(\pi) \geq 11$ which, since $h_2 \notin \pi(h_4)$, is impossible. This contradicts our supposition that $v_{h_{i_1}}(h_{i_2}) = 4$.

- Suppose $v_{h_{i_1}}(h_{i_2}) = 3$. We assume without loss of generality that either $\{i_1, i_2\} = \{3, 11\}$ or $\{i_1, i_2\} = \{2, 7\}$.
 - Suppose $\{i_1, i_2\} = \{3, 11\}$. If $u_{h_2}(\pi) \leq 8$ then h_2 has j-envy for α_{j_1} since $u_{h_2}(\pi) \leq 8 < 9 = u_{h_2}(\{h_3, h_{11}\})$, $v_{h_3}(\alpha_{j_1}) = 0 < 4 = v_{h_3}(h_2)$, and $v_{h_{11}}(\alpha_{j_1}) = 0 < 5 = v_{h_{11}}(h_2)$. It follows that $u_{h_2}(\pi) \geq 9$. By the design of H , it follows that $\pi(h_2) = t_5$. Now consider h_1 . Since $h_1 \notin t_5$ it must be that either $h_1 \in t_2$, $h_1 \in t_3$, or $h_1 \in t_4$. Assume without loss of generality that $h_1 \in t_2$ and that $t_2 = \{h_1, h_{i_3}, \alpha_{j_2}\}$ for some $1 \leq i_3 \leq 11$ and $\alpha_{j_2} \in N \setminus H$. Since $v_{h_1}(\alpha_{j_2}) = 0$ it follows that $u_{h_1}(\pi) = v_{h_1}(h_{i_3})$. By the design of H , it must be that $v_{h_1}(h_{i_3}) = 2$ so $u_{h_1}(\pi) = 2$. Now h_1 has j-envy for α_{j_1} , since $u_{h_1}(\pi) = 2 < 4 = u_{h_1}(\{h_3, h_{11}\})$, $v_{h_3}(\alpha_{j_1}) = 0 < 2 = v_{h_3}(h_1)$, and $v_{h_{11}}(\alpha_{j_1}) = 0 < 2 = v_{h_{11}}(h_1)$. This is a contradiction.
 - Suppose $\{i_1, i_2\} = \{2, 7\}$. Consider h_1 . As before, if $u_{h_1}(\pi) \leq 3$ then h_1 has j-envy for α_{j_1} , since $u_{h_1}(\pi) \leq 3 < 4 = u_{h_1}(\{h_2, h_7\})$, $v_{h_2}(\alpha_{j_1}) = 0 < 2 = v_{h_2}(h_1)$, and $v_{h_7}(\alpha_{j_1}) = 0 < 2 = v_{h_7}(h_1)$. It follows that $u_{h_1}(\pi) \geq 4$. By the design of H , it must be that $\pi(h_1) = \{h_1, h_{i_3}, h_{i_4}\}$ for some $2 \leq i_3, i_4 \leq 11$. It follows that $\pi(h_1) = t_5$. Now consider h_4 . If $u_{h_4}(\pi) \leq 6$ then h_4 has j-envy for α_{j_1} , since $u_{h_4}(\pi) \leq 6 < 7 = u_{h_4}(\{h_2, h_7\})$, $v_{h_2}(\alpha_{j_1}) = 0 < 6 = v_{h_2}(h_4)$, and $v_{h_7}(\alpha_{j_1}) = 0 < 1 = v_{h_7}(h_4)$. It follows that $u_{h_4}(\pi) \geq 7$ so, similarly, $\pi(h_4)$ must contain three agents in H and thus $h_4 \in t_5$. Assume without loss of generality that $h_4 = h_{i_3}$ so $t_5 = \{h_1, h_4, h_{i_4}\}$. Since $u_{h_4}(\pi) \geq 7$ and $v_{h_4}(h_1) = 2$ it must be that $v_{h_4}(h_{i_4}) \geq 5$ and thus that either $i_4 = 3$ or $i_4 = 6$. Consider h_{10} . If $u_{h_{10}}(\pi) \leq 6$ then h_{10} has j-envy for α_{j_1} , since $u_{h_{10}}(\pi) \leq 6 < 7 = u_{h_{10}}(\{h_2, h_7\})$, $v_{h_2}(\alpha_{j_1}) = 0 < 6 = v_{h_2}(h_{10})$, and $v_{h_7}(\alpha_{j_1}) = 0 < 1 = v_{h_7}(h_{10})$. It follows that $u_{h_{10}}(\pi) \geq 7$. Since $h_{10} \notin t_5$ it must be that either $h_{10} \in t_2$, $h_{10} \in t_3$, or $h_{10} \in t_4$. Assume without loss of generality that $h_{10} \in t_2$ and that $t_2 = \{h_{10}, h_{i_5}, \alpha_{j_2}\}$ for some $1 \leq i_5 \leq 11$ and $\alpha_{j_2} \in N \setminus H$. Since $v_{h_{10}}(\alpha_{j_2}) = 0$ it follows that $v_{h_{10}}(h_{i_5}) = u_{h_{10}}(\pi) \geq 7$, which is a contradiction.
- Suppose $v_{h_{i_1}}(h_{i_2}) = 2$. It follows that either $i_1 = 1$ or $i_2 = 1$. Assume without loss of generality that $i_1 = 1$. Note that $2 \leq i_2 \leq 11$. Consider h_{i_2} . Note that since $v_{h_{i_2}}(\alpha_{j_1}) = 0$ it must be that $u_{h_{i_2}}(\pi) = v_{h_{i_2}}(h_1) = 2$. By the design of H , for each possible assignment of i_2 , namely $2 \leq i_2 \leq 11$, there exist five agents $h_{i_3}, h_{i_4}, h_{i_5}, h_{i_6}, h_{i_7}$ such that $v_{h_{i_2}}(h_{i_k}) > 2$ for $3 \leq k \leq 7$. A counting argument shows that at least one of these five agents does not belong to t_5 and hence must belong to either t_2 , t_3 , or t_4 . Assume without loss of generality that $h_{i_3} \in t_2$ and $t_2 = \{h_{i_3}, h_{i_8}, \alpha_{j_2}\}$ for some $2 \leq i_8 \leq 11$ and $\alpha_{j_2} \in N \setminus H$. Recall that $v_{h_{i_2}}(h_{i_3}) > 2$. By the design of H it follows that $u_{h_{i_2}}(\{h_{i_3}, h_{i_8}\}) > 3$. Now h_{i_2} has j-envy for α_{j_2} since $u_{h_{i_2}}(\pi) = 2 < 3 < u_{h_{i_2}}(\{h_{i_3}, h_{i_8}\})$, $v_{h_{i_3}}(\alpha_{j_2}) = 0 < 1 \leq v_{h_{i_3}}(h_{i_2})$, and $v_{h_{i_8}}(\alpha_{j_2}) = 0 < 1 \leq v_{h_{i_8}}(h_{i_2})$.
- Suppose $v_{h_{i_1}}(h_{i_2}) = 1$. Without loss of generality we assume that either $\{i_1, i_2\} = \{2, 5\}$ or $\{i_1, i_2\} = \{2, 6\}$.
 - Suppose $\{i_1, i_2\} = \{2, 5\}$. If $u_{h_4}(\pi) \leq 9$ then h_4 has j-envy for α_{j_1} , since $u_{h_4}(\pi) \leq 9 < 10 = u_{h_4}(\{h_2, h_5\})$, $v_{h_2}(\alpha_{j_1}) = 0 < 6 = v_{h_2}(h_4)$, and

- $v_{h_5}(\alpha_{j_1}) = 0 < 4 = v_{h_5}(h_4)$. It follows that $u_{h_4}(\pi) \geq 10$. The only possibility is that $\pi(h_4) = \{h_3, h_4, h_6\}$ and hence $\pi(h_4) = t_5$. Consider h_1 . If $u_{h_1}(\pi) \leq 3$ then h_1 has j-envy for α_{j_1} , since $u_{h_1}(\pi) \leq 3 < 4 = u_{h_1}(\{h_2, h_5\})$, $v_{h_2}(\alpha_{j_1}) = 0 < 2 = v_{h_2}(h_1)$, and $v_{h_5}(\alpha_{j_1}) = 0 < 2 = v_{h_5}(h_1)$. It follows that $u_{h_1}(\pi) \geq 4$. By the design of H , it follows that $\pi(h_1)$ must contain three agents in H , which is a contradiction since $h_1 \notin t_5$.
- Suppose $\{i_1, i_2\} = \{2, 6\}$. It follows that $u_{h_4}(\pi) \leq 9$ so h_4 has j-envy for α_{j_1} since $u_{h_4}(\pi) \leq 9 < 12 = u_{h_4}(\{h_2, h_6\})$, $v_{h_2}(\alpha_{j_1}) = 0 < 6 = v_{h_2}(h_4)$, and $v_{h_6}(\alpha_{j_1}) = 0 < 6 = v_{h_6}(h_4)$.

□

Lemma 50. H has an open configuration in π .

Proof. By definition, $4 \leq \sigma(H, \pi) \leq 11$. If $\sigma(H, \pi) \leq 5$ then H has an open configuration in π , by Lemmas 48 and 49. If $6 \leq \sigma(H, \pi) \leq 11$ then, by a counting argument, at least one triple in π must contain exactly one agent in H . In other words, H has an open configuration in π . □

We have shown, in Lemma 50, that H has an open configuration in π . By definition, some triple t_β in π contains exactly one agent in H . Since $|H| = 11$, if t_β is the only triple in π to contain exactly one agent in H then there must exist some triple in π that contains exactly two agents in H . By Lemma 45, this is a contradiction. It follows that at least two triples in π exist that each contain exactly one agent in H . Suppose $t_\beta, t_\gamma \in \pi$ are two such triples and $t_\beta = \{h_{a_1}, \alpha_{b_1}, \alpha_{b_2}\}$ and $t_\gamma = \{h_{a_2}, \alpha_{b_3}, \alpha_{b_4}\}$.

Lemma 51. $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} = L$.

Proof. Suppose for a contradiction that $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} \neq L$.

By definition, $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} \cap H = \emptyset$ and $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} \neq L$ it must be that at least one agent in $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\}$ belongs to C . Assume without loss of generality that $\alpha_{b_1} \in C$.

We have already shown that t_β contains exactly one agent in H . Since $v_{\alpha_{b_1}}(h_{a_1}) = 0$, by the design of the instance it must be that $u_{\alpha_{b_1}}(\pi) = v_{\alpha_{b_1}}(\alpha_{b_2}) \leq 3$. By the design of the instance $v_{\alpha_{b_1}}(\alpha_{b_3}) \geq 2$ and $v_{\alpha_{b_1}}(\alpha_{b_4}) \geq 2$ so $u_{\alpha_{b_1}}(\{\alpha_{b_3}, \alpha_{b_4}\}) \geq 4$. Now α_{b_1} has j-envy for h_{a_2} since $u_{\alpha_{b_1}}(\pi) \leq 3 < 4 \leq u_{\alpha_{b_1}}(\{\alpha_{b_3}, \alpha_{b_4}\})$, $v_{\alpha_{b_3}}(h_{a_2}) = 0 < 2 \leq v_{\alpha_{b_3}}(\alpha_{b_1})$, and $v_{\alpha_{b_4}}(h_{a_2}) = 0 < 2 \leq v_{\alpha_{b_4}}(\alpha_{b_1})$. □

Lemma 52. $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_2\}, \{l_3, l_4\}\}$.

Proof. By Lemma 51, $\{\alpha_{b_1}, \alpha_{b_2}, \alpha_{b_3}, \alpha_{b_4}\} = L$. There are now three possibilities: first that $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_3\}, \{l_2, l_4\}\}$, second that $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_4\}, \{l_2, l_3\}\}$, and third that $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_2\}, \{l_3, l_4\}\}$.

First suppose $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_3\}, \{l_2, l_4\}\}$. Now l_1 has j-envy for h_{a_2} since $u_{l_1}(\{h_{a_1}, l_3\}) = 1 < 3 \leq u_{l_1}(\{l_2, l_4\})$, $v_{l_2}(h_{a_2}) = 0 < 2 = v_{l_2}(l_1)$, and $v_{l_4}(h_{a_2}) = 0 < 1 \leq v_{l_4}(l_1)$.

Second suppose $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_4\}, \{l_2, l_3\}\}$. As before, l_1 has j-envy for h_{a_2} since $u_{l_1}(\{h_{a_1}, l_4\}) = 1 < 3 \leq u_{l_1}(\{l_2, l_3\})$, $v_{l_2}(h_{a_2}) = 0 < 2 = v_{l_2}(l_1)$, and $v_{l_3}(h_{a_2}) = 0 < 1 \leq v_{l_3}(l_1)$.

It remains that $\{\{\alpha_{b_1}, \alpha_{b_2}\}, \{\alpha_{b_3}, \alpha_{b_4}\}\} = \{\{l_1, l_2\}, \{l_3, l_4\}\}$. $\square$

By Lemma 52, either $\{\alpha_{b_1}, \alpha_{b_2}\} = \{l_1, l_2\}$ or $\{\alpha_{b_1}, \alpha_{b_2}\} = \{l_3, l_4\}$. Without loss of generality assume that $\{\alpha_{b_1}, \alpha_{b_2}\} = \{l_1, l_2\}$.

Lemma 53. $u_{c_i}(\pi) = 6$ for each i where $1 \leq i \leq 3q$.

Proof. Suppose, to the contrary, some $1 \leq i \leq 3q$ exists where $u_{c_i}(\pi) < 6$. Then c_i has j-envy for h_{a_1} since $u_{c_i}(\pi) \leq 5 < 6 \leq u_{c_i}(\{l_1, l_2\})$, $v_{l_1}(h_{a_1}) = 0 < 3 = v_{l_1}(c_i)$, and $v_{l_2}(h_{a_1}) = 0 < 3 = v_{l_2}(c_i)$. This contradicts our supposition that π is j-envy-free. $\square$

Lemma 54. A partition into triangles X exists in G .

Proof. Lemma 53 shows that $u_{c_i}(\pi) = 6$ for each i where $1 \leq i \leq 3q$. By construction, it follows that $\pi(c_i)$ contains two agents c_j, c_k such that $v_{c_i}(c_j) = v_{c_i}(c_k) = 3$. By construction, c_j and c_k therefore correspond to vertices $w_j, w_k \in W$ where $\{w_i, w_j\} \in E$ and $\{w_i, w_k\} \in E$. It follows thus that there are exactly q triples in M each containing three agents $\{c_i, c_j, c_k\}$, where the three corresponding vertices w_i, w_j, w_k are pairwise adjacent in G . From these triples a partition into triangles X can be easily constructed. $\square$

A.3.2.3 Correctness of the reduction: conclusion

Theorem 7. It is NP-complete to decide if a j-envy-free partition into triples exists in a given ASHG, even when preferences are symmetric and the maximum possible valuation is 6.

Proof. Suppose we are given an instance (N, V) and a partition into triples π . For any two agents $\alpha_i, \alpha_j \in N$ we can test if α_i has j-envy for α_j in constant time. We can therefore decide if π is j-envy-free in polynomial time, which shows that the above decision problem belongs to NP.

The above reduction from PIT can be performed in polynomial time. By Lemmas 44 and 54, the undirected graph G has a partition into triangles X if and only if an j-envy-free partition into triples exists in the constructed instance (N, V) . It follows that the reduction is valid and the above decision problem is NP-hard. $\square$