

Rawlsian Assignments*

Juan S. Pereyra[†]

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Abstract

We study the assignment of indivisible objects to individuals when transfers are not allowed. Previous literature has mainly focused on efficiency (from an ex-ante and ex-post perspective), and *individually* fair assignments. Consequently, distributional concerns have been overlooked. We are inspired by the assignment of apartments in housing cooperatives where families regard the egalitarianism of the assignments as a first order requirement. They want to avoid assignments where some families get their top apartment, while others get options ranked very low in their preferences. Based on the idea of fairness by Rawls, we introduce the notion of Rawlsian assignments. We prove that there always exists a unique Rawlsian assignment, which is ordinally efficient, and satisfies the equal treatment of equals. We illustrate our analysis with preferences data from housing cooperatives. Our results show that the Rawlsian assignment significantly improves, from an egalitarian perspective, both the probabilistic serial mechanism, and the mechanism currently in use.

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1 Introduction

Economics deals with the allocation of resources in the economy. Although in many markets prices serve to coordinate this allocation, there are many allocation problems for which the use of prices is neither desirable nor feasible. This is the case of public housing where a set of apartments are to be allocated to families without the use of transfers. One challenge in such situations is how to ensure that resources are allocated efficiently. Also, the indivisibility of objects makes impossible in general to attain fairness from an ex-post perspective. As a remedy, an ex-ante perspective is generally adopted and random allocations where each agent receives a lottery over the set of objects are usually employed.

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[†]Departamento de Economía, FCS-UDELAR, Constituyente 1502, Montevideo, Uruguay. (email: juan.pereyra@cienciassociales.edu.uy)

Our primary motivation is the assignment of apartments in housing cooperatives. A housing cooperative consists of a group of families that take part in the construction of a building. Once the building is finished, the apartments have to be distributed among the families. Prices cannot be used, and the assignment is computed based on families' (ordinal) preferences. One of the main concern of the cooperatives members in Uruguay is the *egalitarianism* of the final assignment. They want to avoid unequal assignments where, for example, a family gets the most preferred apartment, while others families are assigned to apartments ranked very low. There have been many important contributions in the literature regarding the efficiency and individual fairness of the assignments. However, no much attention has been paid to this kind of *distributional* concerns. In this paper, we introduce a new concept to capture this issue and we study its relation with others properties.

We follow the idea of justice of Rawls, and evaluate an assignment by the well-being of the worst-off individuals. The analysis begins by asking, given a certain assignment, who are the worst-off agents. If we were focusing on deterministic assignments, we could consider the rank of the assigned object at each agent's preferences. The worst off agents would be those associated with the highest rank. Then, we would like to select the assignments that minimize the highest rank. In general, there are multiple such assignments, so one could apply the same criterion recursively as proposed by [Sen \(2017\)](#).

When considering random assignments, the definition of the worst-off agents is more challenging. If agents' cardinal utilities were known, one might consider the expected utility of each agent, and then consider the agent with the lowest expected utility. But in our problem, only agents' ordinal preferences are known. Thus, for each agent we consider among those objects the agent gets with positive probability, the rank of the least preferred object. Then, we focus on those agents associated with the highest rank, and among them we consider those who get the object with highest probability.

The criterion we propose first improves the assignment of the worst off-agents. In a second step, we consider those agents that receive the least and second-least preferred objects with highest probability, and choose the assignment that minimize these probabilities. We continue in this way until the first objects. The assignment that results is a **Rawlsian assignment**.

We first show that, in any problem, there always exists a **unique** Rawlsian assignment. Roughly, if there were two Rawlsian assignments, it would be possible to decrease the probability with which agents receive their least preferred objects by considering the average assignment. We also prove that the Rawlsian assignment is ordinally efficient, and satisfies a minimum requirement in terms of fairness (equal treatment of equals). The drawback is that the Rawlsian mechanism is manipulable by agents (it is not strategy-proof).

Finally, we illustrate the analysis with a set preferences from different housing cooperatives in Uruguay. We compare our solution with the outcome of the probabilistic serial (PS) by [Bogomolnaia and Moulin \(2001\)](#), and with the assignment that was implemented.¹ The results

¹The mechanism which is currently used, called the **MTAV**, was proposed by [Prino, Sánchez, and Cancela](#)

show that the Rawlsian solution improves significantly the assignment of the least favored agents.² There are winners and losers when we change from PS to the Rawlsian mechanism, but the number of agents who prefer their assignment under the Rawlsian mechanism is always (weakly) higher. Thus, our solution stands for an alternative to both the PS and the MTAV mechanisms as it significantly improves on them from an egalitarian perspective.

In the next section, we place our contributions within the related literature. Section 3 presents the model and definitions. In Section 4, we define the concept of Rawlsian assignments. In Section 5, we present our main results and discuss the relation of Rawlsian assignments with other concepts analyzed in the literature. In Section 6 we illustrate our results with data from housing cooperatives in Uruguay, while in Section 7 we conclude.

2 Related Literature

There have been many important contributions since the introduction of the assignment problem by Hylland and Zeckhauser (1979). In addition to the study of properties related to efficiency, fairness, and incentive compatibility, different mechanisms have been proposed. Zhou (1990) and Abdulkadiroğlu and Sönmez (1998) were the first papers to introduce the serial random dictatorship (or random priority) mechanism. This mechanism performs very well in terms of incentives, but its outcome may be stochastically dominated with respect to individual preferences (may not be ordinally efficient). Bogomolnaia and Moulin (2001) proposed another mechanism, the PS, which is ordinally efficient, but not incentive compatible. We introduce the Rawlsian mechanism which retains the efficiency of the PS and introduces an egalitarian requirement.³

The Rawlsian idea of justice (Rawls, 2020) has been applied to two-sided matching markets, see for example Masarani and Gokturk (1989) and Romero-Medina (2005). These papers study the compatibility of this idea of justice and stability, and use the Rawlsian criterion to select a stable matching that treats equally both sides of the market. The assignment problem that we study is different as it is a one-sided problem where the concept of stability does not apply. Klaus and Klijn (2010) adapt the Rawlsian criterion to the roommate problem and show that the concept is compatible with stability. Duddy (2021) introduces the notion of egalitarian assignments, also motivated by fairness concerns. We refine his definition by choosing a particular egalitarian assignment (see Appendix 8.3 for a more detailed discussion).

The construction of our concept resembles the welfarist definition of the PS mechanism by (2016). First, it selects among all the deterministic assignments, those that minimize the rank of the objects assigned to the worst-off families. Second, among the selected assignments, it considers those that maximize the sum of families' utility, assuming that the each family's utility for the apartment ranked in position k is $n - k$. Finally, if multiple assignments are selected, one of them is randomly chosen. See Appendix 8.4 for a formal description of the mechanism.

²For example, under the PS assignment and for all the cooperatives, there is at least one family who get the least preferred apartment with positive probability. In the Rawlsian assignment, this is never the case. All families get (with positive) apartments between their top 27% and top 64% of their preferences, depending on the cooperative.

³See also Basteck (2018) for a discussion of other fairness concepts.

Bogomolnaia (2015) and Bogomolnaia (2018). The author presents an alternative description of PS where at each step k it considers the agent who receives her k -top objects with lower probability, and maximizes this probability. The Rawlsian criterion performs a similar procedure but beginning with the least preferred objects, and minimizes the probability of the worst-off agent. Although this similarity, we show that both mechanisms are not equivalent.⁴

The notion of Rawlsian assignments is also related to the downward lexicographic extension of stochastic dominance (Cho and Doğan, 2016; Cho, 2018). Different from Cho (2018) who conducts the comparison from an individual perspective, that is, he compares two lotteries from an agent's perspective, we use a similar criterion but applied to the whole assignment. Also, Cho and Doğan (2016) show that upward lexicographic (ul) efficiency and ordinal efficiency are equivalent. Then, a Rawlsian assignment is ul-efficient, but not the other way around.

3 Primitives and definitions

Let I be the set of agents and O the set of objects (both of size n). Each agent $i \in I$ has preferences over the set of objects, denoted by $\succeq_i$. We assume that all objects are acceptable for the agents (every object is preferred to the non-object option). A preference profile is denoted by $\succeq = (\succeq_i)_{i \in I}$. Sometimes we will represent agent i 's preferences as a n -dimensional vector r_i , where $r_{io} = k$ means that object o is ranked k th by agent i .⁵

We study the assignment of objects to agents, where each agent is to be assigned at most one object, and each object cannot be assigned to more than one agent. An assignment problem is defined by the tuple $(I, O, \succeq)$. We fix the sets of agents and objects, and define a problem by agents' preferences $\succeq$.

A solution of an assignment problem is a (random) assignment $x = (x_i)_{i \in I}$, where each x_i is a probability distribution over O . We interpret $x_{io} \in [0, 1]$ as the probability with which agent i is allocated object o . An assignment is deterministic if all of its entries are either zero or one.⁶ We will usually describe an assignment by a matrix where the rows represent the agents and the columns the objects, and each row i has the assignment of agent i : x_i . Let X be the set of assignments (or equivalently, the set of $n \times n$ bi-stochastic matrices). A mechanism is a function ϕ that maps every problem to an assignment: for every $\succeq$, $\phi(\succeq) \in X$. We denote as $\phi_i(\succeq)$ the assignment of agent i by mechanism ϕ in problem $\succeq$.

We will use the following concept of efficiency for random assignments due to Bogomolnaia and Moulin (2001).

Definition 1 (Ordinal Efficiency). 1. An assignment x_i for agent i first order stochastic dom-

⁴The analysis of Bogomolnaia (2015) and Bogomolnaia (2018) allows for indifference classes in agents' preferences. The same extension applies in our case.

⁵These assumptions fit our main motivation. Indeed, families in the case of the cooperatives have to rank all the apartments, and ties in preferences are not allowed.

⁶We know by the Birkhoff-von Neumann theorem (Birkhoff, 1946) that any random assignment can be represented as a lottery over the set of deterministic assignments.

inates (fosd) an assignment x'_i if, for each $o \in O$:

$$\sum_{o': o' \succeq_i o} x_{io'} \geq \sum_{o': o' \succeq_i o} x'_{io'}.$$

2. An assignment x' is stochastically dominated by another assignment x if for every agent i , x_i fosd x'_i , and $x \neq x'$.
3. An assignment is ordinally efficient if it is not stochastically dominated by any assignment.

The following is a minimal requirement of fairness usually used in the literature.

Definition 2 (Equal treatment of equals). *Given a problem $(\succeq)$, an assignment x satisfies equal treatment of equals if for all $i, j \in I$ we have:*

$$\succeq_i = \succeq_j \implies x_i = x_j.$$

A stronger notion of fairness is envy-freeness, it requires that each agent should prefer her assignment over anyone's else assignment.

Definition 3 (Sd-envy-freeness). *Given a problem $(\succeq)$, an assignment x is sd-envy-free if for all $i, j \in I$, we have x_i fosd x_j .*

If instead of requiring that any individual assignment fosd the assignment of all other agents, we require that there is no other agent's assignment that fosd the individual assignment, we have a weaker notion, called *weak sd-envy-freeness*.

Finally, we define strategyproofness.

Definition 4 (Strategyproofness). *A mechanism ϕ is strategyproof if at any preference profile no agent can benefit by misreporting her preferences: for each $i \in I$, for each $\succeq$, and for each $\succeq'_i$ the following holds:*

$$\phi_i(\succeq) \text{ fosd } \phi_i(\succeq'_i, \succeq_{-i}).$$

As with sd-envy freeness, if in the last definition we require that there is no manipulation such that its outcome fosd the assignment the individual gets from truth-telling, we get a weaker notion, called *weak strategyproofness*.

4 Rawlsian assignments

In this section, we define our main concept of Rawlsian assignments. Given an assignment x , we denote by $b_i^x(k)$ the total probability with which agent i gets objects in positions n to k in her preferences. That is:

$$b_i^x(k) = \sum_{o \in O} \mathbb{1}\{r_{io} \geq k\} x_{io}.$$

Note that the previous definition is different from the standard concept of stochastic dominance where the probabilities are added from the most to the least preferred object. In particular, $b_i^x(n)$ is the probability with which agent i receives her least preferred object, and, by definition, $b_i^x(1) = 1$. We denote by b_i^x the vector with the cumulative probability from the least to the most preferred object: $b_i^x = (b_i^x(n), b_i^x(n-1), \dots, b_i^x(1) = 1)$.

Given an assignment x , and the vectors $(b_i^x)_{i \in I}$, we define the vector $B^x \in [0, 1]^{n^2}$ as follows.

1. The first elements $(B_1^x, \dots, B_n^x)$ are the elements $(b_1^x(n), \dots, b_n^x(n))$ in a non-increasing order.
2. Elements $(B_{n+1}^x, \dots, B_{2n}^x)$ are the elements $(b_1^x(n-1), \dots, b_n^x(n-1))$ in a non-increasing order.
3. In general, elements $(B_{(k-1)n+1}^x, \dots, B_{kn}^x)$ for $k = 1, \dots, n$, are the elements $(b_1^x(n-k+1), \dots, b_n^x(n-k+1))$ in a non-increasing order.

Vector B^x describes the cumulative distribution of probabilities induced by the assignment x , from the least preferred object of each agent, to her most preferred object. It orders the cumulative probabilities in a non-increasing order. Then, the first n entries of the vector are the probabilities with which each agent receives her least preferred object. And, in particular, the first entry corresponds to the agent who receives her least preferred object with (weakly) higher probability. The following example illustrates the definitions.⁷

Example 1. Consider the following problem with three agents, $I = \{1, 2, 3\}$, and three objects $O = \{a, b, c\}$. Agents' preferences are:

$\succeq_1$	$\succeq_2$	$\succeq_3$
a	a	b
b	b	c
c	c	a

Consider the assignment:

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then: $b_1^x = b_2^x = (0, \frac{1}{2}, 1)$, $b_3^x = (0, 1, 1)$, and $B^x = (0, 0, 0, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, 1)$. $\square$

⁷Note that this is different from the leximin order. B^x is not ordered in a non-increasing way but we order its elements by considering blocks of n elements, and within each block, the elements are placed in a non-increasing order.

To compare two assignments, x and y , we begin with those agents who are worst-off. We first compare the agent who receives his least preferred object with highest probability (the first entry of vectors B^x and B^y). If the probability is the same under both assignments, we compare the agent who receives her least preferred object with the second highest probability. If all probabilities associated with the least preferred object of each agent are the same, we conduct the same comparison with the least and second-least preferred object of each agent. If at some point, an entry of the vector B^x is strictly lower than the same entry of vector B^y , we say that x Rawls-dominates y . Formally, we compare the vectors B^x and B^y using a lexicographic order. This is the idea of the following definition.

Definition 5. *Given two assignments x and y , consider the vectors B^x and B^y . We say that x Rawlsian-dominates y (x R-dominates y) if there is $j \in \{1, \dots, n^2\}$ such that $B_j^x < B_j^y$, and for all $l < j$, $B_l^x = B_l^y$.*

The following is the key concept of our analysis.

Definition 6. *An assignment x is Rawlsian if it is not Rawlsian-dominated (R-dominated) by any other assignment.*

The previous definition captures the idea of justice in the spirit of Rawls in the sense that, we first improve the assignment of the agent who receives her least preferred object with the highest probability; conditionally on that, we improve the assignment of the second worst agent, and so on, and so forth.

Example 2. *In the problem of Example 1, consider the assignment:*

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

associated with the vector $B^y = (1, 0, 0, 1, 0, 0, 1, 1, 1)$. Clearly, x R-dominates y . Moreover, x is a Rawlsian assignment. Note first that if another assignment is such that $x_{3c} < 1$, then we should have $x_{1c} > 0$ or $x_{2c} > 0$. And then this new assignment will be R-dominated by x . So, if an assignment R-dominates x , it should that the $x_{1b} < \frac{1}{2}$ or $x_{2b} < \frac{1}{2}$. In the first case, then $x_{2b} > \frac{1}{2}$, and the new assignment is R-dominated by x . In the first case, then $x_{1b} > \frac{1}{2}$, and the new assignment is R-dominated by x . Finally, there is no assignment that R-dominates x .

5 Results

It is easy to see that in every problem there always exists a Rawlsian assignment.⁸ In principle, it might exist multiple Rawlsian assignments. Suppose this is indeed the case, and x and y are two

⁸Given a problem, consider an assignment x^1 , and its associated vector B^1 . If x^1 is not a Rawlsian assignment, there exists another assignment x^2 that R-dominates x^1 . Then, there is an entry j of B^2 such that $B_j^2 < B_j^1$ and

Rawlsian assignments. But then, one may consider the assignment $\frac{1}{2}x + \frac{1}{2}y$, which R-dominates x and y . Thus, as the following proposition shows, in every problem there is a unique Rawlsian assignment.⁹

Proposition 1. *Given a market, there exists a unique Rawlsian assignment.*

A minimum requirement of fairness is the equal treatment of equals: agents with the same preferences should receive the same assignment. In the next proposition we prove that the Rawlsian assignment satisfies this property.

Proposition 2. *A Rawlsian assignment satisfies equal treatment of equals.*

There is no relation between Rawlsian assignments and sd-envy-free assignments. In Example 1, the outcome of PS which is sd-envy-free is not Rawlsian. Also, as we show in the next example, a Rawlsian assignment in general is not sd-envy-free (moreover, the same example shows that it is neither weak sd-envy-free).

Example 3. *Consider the following problem with three agents, $I = \{1, 2, 3\}$, and three objects $O = \{a, b, c\}$.*

$\succeq_1$	$\succeq_2$	$\succeq_3$
a	b	b
b	a	c
c	c	a

The Rawlsian assignment is:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that agent 3's assignment is first-order-stochastic dominated by the assignment of agent 2 (considering agent 3's preferences). □

Duddy (2021) introduces a notion of egalitarian assignments. An assignment y is **inegalitarian** if there exists another assignment x and an agent j such that the assignment of all the agents at x fofd the assignment of j at y . That is, there is another assignment at which all the agents receive an assignment preferred than the assignment of j at y . An egalitarian assignment is an assignment that is not inegalitarian. In Appendix 8.3 we define this concept, and show that the Rawlsian assignment is always egalitarian, but that the converse does not hold.

$B_l^1 = B_l^2$ for all $l < j$. If x^2 is not a Rawlsian assignment, there is another assignment x^3 that R-dominates x^2 . Then, there is an entry j' of B^3 such that $B_{j'}^3 < B_{j'}^2$ and $B_l^1 = B_l^3$ for all $l < j'$. If x^3 is not a Rawlsian assignment, we continue in the same way. Note that, at some point, we should find a Rawlsian assignment because each entry of the vectors B^k is bounded from below by zero, and the number of entries is finite.

⁹Randomization is key for this result. If we restrict the analysis to deterministic assignments, there are problems with multiple Rawlsian assignments.

5.1 Efficiency

We now analyze the efficiency of the Rawlsian assignments. In contrast to many economic environments where there is a tension between fairness and efficiency, these concepts are compatible in our framework.¹⁰

Proposition 3. *A Rawlsian assignment is ordinally efficient.*

It is easy to see that the converse of the proposition is not true: there are ordinally efficient assignments that are not Rawlsian. We show this in the following example.

Example 4. *Consider the problem of Example 1. The outcome of the Probabilistic Serial¹¹ (which coincides in the case with Random Serial Dictatorship) is:*

$$y^{PS} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

The outcome of PS is ordinally efficient and for this problem is different from the Rawlsian assignment.

There is another concept of efficiency, called rank efficiency, recently proposed by Featherstone (2020). In Appendix 8.2 we give the formal definition and show that there is no relation between Rawlsian and rank efficient assignments.

We have shown that the Rawlsian mechanism, the function that maps every problem to the Rawlsian assignment, is ordinally efficient and satisfies the equal-treatment of equals. When there are least 4 agents, we know by Bogomolnaia and Moulin (2001), that no mechanism is ordinally efficient, strategyproofness, and satisfies the equal treatment of equals. Thus, we have the following corollary.

Corollary 1. *The Rawlsian mechanism is not strategyproof.*

To illustrate the possible manipulations of the Rawlsian mechanism, suppose a problem where one object o is regarded by all agents but i, j as the least preferred object. Moreover, let's say that agents i, j rank this object as the second least preferred. Then, the Rawlsian mechanism assigns object o to agents i, j . In this situation, it is easy to complete the example such that when agent i or j ranks object o higher than the second-last position, she gets a preferred assignment. This is the idea of the following example.

¹⁰This was already pointed out by Bogomolnaia and Moulin (2001) as the mechanism they propose, Probabilistic Serial, is ordinally efficient and sd-envy-free (which implies equal treatment of equals).

¹¹The Probabilistic Serial is defined as follows. Given a problem, think of each object as an infinitely divisible object with supply 1. **Step 1:** Each agent “eats away” from her favorite object at the same unit speed. Proceed to the next step when an object is completely exhausted. **Step s ,** for $s \in \{2, \dots, S\}$: Each agent eats away from her remaining favorite object at the same speed. Proceed to the next step when an object is completely exhausted. The procedure terminates when each agent has eaten exactly 1 total unit of objects (i.e., at time 1). The random allocation of an agent i is given by the amount of each object she has eaten until the algorithm terminates.

Example 5. Consider the following problem with three agents, $I = \{1, 2, 3\}$, and three objects $O = \{a, b, c\}$.¹²

$\succ_1$	$\succ_2$	$\succ_3$
a	b	b
b	c	c
c	a	a

The Rawlsian assignment is:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Suppose agent 2 reports the following preferences: $\succ'_2 = (b, a, c)$, so the Rawlsian assignment is:

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Agent 2 prefers her assignment under y than under x . $\square$

As we mentioned before, a weaker notion of strategyproofness requires that the outcome under an individual manipulation cannot dominate the assignment obtained when the agent reveals her true preferences. It is weaker because it allows for the case where the assignment the agent gets when she manipulates is not comparable with the assignment under truth-telling. The previous example shows that the Rawlsian mechanism is not weakly strategyproof.

6 Empirical Application

Our main motivation is the assignment of apartments in housing cooperatives. A housing cooperative is formed by a group of families who join to construct a building. Members of the cooperative can contribute to it with money or by spending time in the construction. Once the building is finished, the apartments have to be distributed among families. As prices cannot be used, the situation fits the assignment problem described before.

Before the mechanism which is currently in use, cooperatives assigned apartments randomly. Preferences were not considered and the assignment was taken randomly from the set of all assignments. The assignment was in general not efficient, so they decided to change it. A group of researchers from the Engineering School of the University of Uruguay, proposed a new

¹²The idea of this example is due to [Paleo \(2021\)](#).

mechanism, called the **MTAV**, which was finally adopted. More information about the current mechanism can be found in [Prino, Sánchez, and Cancela \(2016\)](#) and [Paleo \(2021\)](#).

One of the main concern of the families that participated in the cooperatives is the *distribution* of the final assignment. They do not want *inegalitarian* distributions in terms of the ranking of the assigned apartment at each family's preferences. They want to avoid a situation where, for example, a family gets the most preferred assignment, while others families get apartments very low in their preferences. The MTAV mechanism explicitly considers this demand (we define the mechanism in Appendix 8.4).

In this section, we use the data of families' preferences for four cooperatives to illustrate the differences between the Rawlsian, PS and MTAV mechanisms. We should mention that the MTAV mechanism is not strategyproof ([Paleo, 2021](#)). Nonetheless, we take the preferences submitted by the families as their true preferences. We are not aware about manipulations by the families, and in general given the information held by the families, it is very difficult to optimally manipulate the mechanism.

In the following table we compare the outcome of the three mechanisms.

	n	Worst Rawls	Worst PS	x_i^R fofd x_i^{PS}	x_i^{PS} fofd x_i^R	Envy R	Envy M
C_1	11	5th	11th	45%	27%	27%	45%
C_2	9	3rd	9th	44%	22%	11%	44%
C_3	14	9th	14th	21%	7%	50 %	71%
C_4	15	6th	14th	20%	20%	53%	47%

Table 1: Comparison between Rawlsian, PS, and MTAV mechanisms. In columns "Worst Rawls" and "Worst PS" we consider for each family, among all objects they get with positive probability, the rank of the lowest ranked object; then we take the maximum among all the families. In column x_i^R fofd x_i^{PS} (x_i^{PS} fofd x_i^R) we include the percentage of families who prefer their assignment under the Rawlsian (PS) over the assignment under the PS (Rawlsian) mechanism. Columns Envy R and Envy M, present the percentage of families who envy the assignment of other family under the Rawlsian and MTAV mechanisms, respectively.

Note first that those families who are worst-off in the Rawlsian assignment and in the PS assignment, the rank of the apartment is lower in the former, that is, they receive a preferred apartment. Although this should be always true by construction, the difference for each cooperative between both ranks is meaningful.

Another measure of individual fairness is sd-envy-freeness. The PS assignment is sd-envy-free, so no family envies the assignment of other family. As we observed before the Rawlsian mechanism is not sd-envy-free, so there may be families with envy. In Table 1 we compute the percentage of families with envy in the Rawlsian assignment for each cooperative. In two cases, half of the families envy another assignment, while in the other two cooperatives these figures are only 27% and 11%. We compare these numbers with the implemented assignment. In three of the cooperatives the Rawlsian assignment has less families with envy than the MTAV, and in one they present a similar percentage. We should be careful with this comparison as we only

have the deterministic assignment of the MTAV, and the Rawlsian is a random assignment.¹³

Table 2 presents the expected number of agents assigned to their first, second, . . . , and last option, for each cooperative and mechanism (this is sum of the probability with which agents are assigned to their first, second, . . . , and last option). As one may presume, if we compare the Rawlsian and PS mechanism, the expected number of agents assigned to the low ranked options is lower under the former mechanism. On the other hand, PS assigns more agents to the their first options. It is interesting to note that the MTAV mechanism lays between the two other mechanisms. Indeed, the Rawlsian mechanism outperforms the MTAV under the Rawlsian criterion, but not in terms of the expected number of agents assigned to their top choice. The converse is true when we compare the MTAV and PS mechanisms.

	C_1			C_2			C_3			C_4		
	R	PS	M	R	PS	M	R	PS	M	R	PS	M
1	3.7	5.6	6	3.5	4.6	5	1.7	3.6	4	3.7	7.2	8
2	3.3	0.7	1	4.5	2.1	2	1.3	2.2	4	2.7	1.4	1
3	3	2.2	2	1	0.9	2	1.7	0.7	0	1.5	0.8	1
4	0	0.6	1		0.9		2	1.2	0	5.2	1.4	2
5	1	0.5	1		0.4		3.2	1.6	2	0	1.3	1
6		1.4					3.2	1.1	1	2	0.9	2
7							0.1	0.1	1		2	
8							0	0.5	0			
9							0	0.6	2			
10								2.3	0			

Table 2: Expected number of agents assigned to each rank, for each cooperative and mechanism. R, PS, and M stand for the Rawlsian, probabilistic serial, and MTAV mechanism, respectively.

7 Concluding Remarks

We study the assignment of indivisible objects to agents when prices cannot be used. We are inspired by the case of housing cooperatives where families are concerned about the distribution of the final assignments. In particular, assignments where some families get their first option while others get options very low in their preferences, want to be avoided. In this context, we introduce a new concept, Rawlsian assignments, under which individuals are improved beginning from those who are worst-off. We prove that there always exists a unique Rawlsian assignment which is ordinally efficient and satisfies the equal treatment of equals. Finally, we compare our proposed mechanism with the PS and the mechanism currently in use. The results show that the Rawlsian mechanisms significantly outperform the other two mechanisms from an egalitarian perspective.

¹³In terms of how much envy each family has, on average each family envies the assignment of 1.3, 1, 1.9 and 2, 3 families under the Rawlsian mechanisms in the cooperative C_1 , C_2 , C_3 , C_4 , respectively. For the MTAV mechanism these figures are always higher: 2.4, 1.5, 3.9, and 3.3.

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8 Appendix

8.1 Proofs

8.1.1 Proof of Proposition 1.

Proof. Suppose there are two Rawlsian assignments, x and y . Both are associated with the same vector B . Given an assignment z , consider a matrix P^z where agents are represented in the rows, and in column k we include the probability with which each receives the object ranked in position k .¹⁴ Starting from the last column, consider the first column where the matrices P^x and P^y differ (there is such a column as x and y are different assignments). Let's say that it is column $n - c$.

Note that the probabilities of column c in each matrix are the same, but distributed differently. Until column c , the two matrices are the same, so the same agents get the same probabilities for the corresponding objects. This implies that, all agents receive the same probability for objects ranked in positions $n - (c + 1), \dots, n$ under x and y . Therefore, they also receive the same probabilities under assignment $\frac{1}{2}(x + y)$.

Consider the highest element of column c of each of the assignments x , y , and $\frac{1}{2}(x + y)$. If the highest element of the assignments x and y corresponds to the same agent, then the highest element of the three assignments coincides. If it corresponds to a different agent, then the highest element of the assignments x and y is higher than the highest element of the assignment $\frac{1}{2}(x + y)$. As the assignments x and y differ in at least one object, the assignment $\frac{1}{2}(x + y)$ R-dominates x and y . But this contradicts the fact that x and y are Rawlsian assignments. $\square$

8.1.2 Proof of Proposition 2.

Proof. Consider a problem, and two agents, say i and j with the same preferences. Suppose, their assignments under the Rawlsian assignment differ: $x_i^R \neq x_j^R$. Consider the assignment y where all the agents receive the same lottery than in x^R , and for agents i and j , each receives $\frac{1}{2}(x_i^R + x_j^R)$. Consider the vectors B^{x^R} and $B^{\frac{1}{2}(x_i^R + x_j^R)}$, and compare them element by element. They have the same entries until the least preferred object of i and j where $x_i^R \neq x_j^R$, and this entry is lower under $B^{\frac{1}{2}(x_i^R + x_j^R)}$ than under B^{x^R} . Then, assignment y R-dominates x^R , which is a contradiction. $\square$

8.1.3 Proof of Proposition 3

Proof. We will use the equivalent definition of ordinal efficiency where we accumulate the probabilities from bottom to top: x_i fosi x'_i if the probability of being assigned to her least preferred object is larger under x' than under x , the probability of being assigned to her two least preferred objects is larger under x' than under x , and so on so forth.

¹⁴Matrix P^z is created by reordering each row based on agents' preferences. Each row of x and P^z has the same elements, but ordered differently.

Consider a Rawlsian assignment x , and suppose it is not ordinally efficient. Let y be another assignment that stochastically dominates x . Consider the first n entries of the vectors B^x and B^y : the probability with which each agent is assigned to her least preferred object, under x and y , respectively.

Let denote by $(p_{nn}, p_{(n-1)n}, \dots, p_{1n})$ and $(q_{nn}, q_{(n-1)n}, \dots, q_{1n})$, the first n elements of vectors B^x and B^y , respectively. Note that the agent corresponding to each probability p_{in} may not be same agent corresponding to q_{in} . Also, $p_{in} > p_{jn}$ and $q_{in} > q_{jn}$ for $i > j$ (the elements are ordered from the largest to the smallest).

We know that $p_{nn} \leq q_{nn}$ (because x is Rawlsian). Suppose $p_{nn} < q_{nn}$. So these elements cannot correspond to the same agent, otherwise the agent is worst off under y than under x , and y does not fofd x . So, it should be that the agent corresponding to q_{nn} has a probability $p_{jn} \leq p_{nn}$ in B^x . But then, $p_{jn} \leq p_{nn} < q_{nn}$. Then, the agent receives a larger probability of being assigned to her least preferred object under y than under x , which contradicts the fact that y fofd x . Then, $p_{nn} = q_{nn}$. Moreover, the agent associated with probability p_{nn} is the same agent who is associated with q_{nn} . If not, then, the agent associated with probability q_{nn} is associated with a probability p_{jn} , but then $p_{jn} < p_{nn} = q_{nn}$, and then the agent is assigned to her least preferred option with higher probability under y than under x , which is a contradiction.

We know that $p_{(n-1)n} \leq q_{(n-1)n}$ (because x is Rawlsian). Suppose $p_{(n-1)n} < q_{(n-1)n}$. So these elements cannot correspond to the same agent, otherwise the agent is worst off under y . By the same argument as before, it cannot correspond neither to an agent with the probability p_{jn} , $j < (n-1)$. So, it should be that it corresponds to the agent associated with p_{nn} . Consider now the agent who corresponds to q_{nn} , and let's find her probability in vector B^x . It should be an element $p_{i(n)}$ with $i > n$. But then, $q_{nn} = p_{nn} > p_{i(n)}$; then, agent i receives her least preferred object with higher probability in y than in x , a contradiction. Then, $p_{(n-1)n} = q_{(n-1)n}$. By a similar argument we prove that $p_{in} = q_{in}$ for $i = 1, \dots, n$.

We can apply the same argument as before to the vectors $(p_{n(n-1)}, p_{(n-1)(n-1)}, \dots, p_{1(n-1)})$, and $(q_{n(n-1)}, q_{(n-1)(n-1)}, \dots, q_{1(n-1)})$; $\dots$, $(p_{n1}, p_{(n-1)1}, \dots, p_{11})$, and $(q_{n1}, q_{(n-1)1}, \dots, q_{11})$ to prove that $B^x = B^y$. Moreover, by the same argument, $x = y$, which is a contradiction.

□

8.2 Relation with rank efficiency

Given an assignment x , we define $M^x(k) = \sum_{i \in I} b_i^x(k)$ for $k = 1, \dots, n$. So, for example, $M^x(n)$ is the sum of the probabilities with which each agent is assigned to her least preferred option. Equivalently, $M^x(k)$ is the expected number of agents who receive an object ranked in position k or lower in assignment x .

An assignment, x , is said to rank-dominates another assignment, y , if the rank distribution of y first-order stochastically dominates that of x , that is, if:¹⁵

¹⁵This is equivalent to the original definition of Featherstone (2020), where the sum of the probabilities are

$$M_y(k) \geq M_x(k).$$

for all ranks, k , with the inequality strict for at least one k . If an assignment is not rank-dominated by any other assignment, it is rank efficient.

The following example shows a problem where the Rawlsian assignment is not rank-efficient.

Remark 1. A rank efficient assignment may not be Rawlsian. Consider the following problem:¹⁶

$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$	$\succeq_5$
a	b	c	b	b
d	c	b	a	a
c	a	a	d	d
b	d	d	c	e
e	e	e	e	c

The following assignment is rank efficient:

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

but it is not Rawlsian. Indeed, its associated vector is

$$B^y = (0, 0, 0, 0, 0; 1, 0, 0, 0, 0; 1, 1, 0, 0, 0; 1, 1, 0, 0, 0; 1, 1, 1, 1, 1).$$

The assignment x , indicated in agents' preferences, R -dominates y .

Remark 2. A Rawlsian assignment may not be rank-efficient.

Proof. Consider the following problem:

$\succeq_1$	$\succeq_2$	$\succeq_3$
a	a	b
b	b	a
c	c	c

The following is the Rawlsian assignment of the problem:

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

computed from the most preferred object to k .

¹⁶This example is due to [Paleo \(2021\)](#).

Consider the following assignment:

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Compute $M_x(k)$ and $M_y(k)$:

	$M_j(3)$	$M_j(2)$	$M_j(1)$
x	1	$\frac{4}{3}$	3
y	1	1	3

Assignment y Rank-dominates assignment x .

□

8.3 Relation with egalitarian assignments (Duddy, 2021)

In this section, we define the concept of egalitarian assignments, and show that the Rawlsian assignment is always egalitarian, but that the converse does not hold. Let $p_i^x(\text{Top } k)$ be the probability that i receives an object from their top k objects, under assignment x .

Definition 7. An assignment x is *inegalitarian* if there exists another assignment x' and an agent j such that:

$$p_i^x(\text{Top } k) \geq p_j^{x'}(\text{Top } k) \iff 1 - b_i^{x'}(n - k) \geq 1 - b_j^x(n - k) \iff b_j^x(n - k) \geq b_i^{x'}(n - k)$$

for all i and $k = 1, \dots, n$ (with the inequality being strict at least for one k , for every i).

Definition 8. An assignment is *egalitarian* if it is not inegalitarian.

Remark 3. An egalitarian assignment may not be Rawlsian. Consider the following problem.

$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$
a	b	a	b
b	c	b	a
c	a	c	d
d	d	d	c

The following assignment is egalitarian:

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If not, there should be another assignment y' such that agent 4 is assigned to object d with probability less than 1 (all agents in y are assigned to an object in 3rd position or higher). But in that case, one of the other agents, let's say $i \neq 4$, should be assigned to object d with positive probability. Then,

$$p_i^{y'}(\text{Top } 3) < 1 = p_i^y(\text{Top } 3) \quad \forall i,$$

which is a contradiction.

The Rawlsian assignment of this problem is:

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \square$$

Proposition 4. A Rawlsian assignment is egalitarian.

Proof. Consider a Rawlsian assignment x , and suppose it is not egalitarian. Then, there exists another assignment x' and an agent j such that: $p_i^{x'}(\text{Top } k) \geq p_j^x(\text{Top } k)$ for all i and k (with the inequality being strict at least for one k , for every i).

Let's consider the agent who receives the larger positive probability of her least preferred object in assignment x . This is the first positive element in vector B^x . Suppose this is agent l , and her least preferred object that she receives with positive probability is in position $\tilde{k}$.

Note that l cannot be agent j because in that case the first non-zero element in $B^{x'}$ is strictly lower than the element in the same position in B^x , but this contradicts that x is Rawlsian. In particular, we should have that $p_j^x(\text{Top } \tilde{k}) > p_l^{x'}(\text{Top } \tilde{k})$ (if not, the same argument applies: every agent in x' should be better than agent j in x , and then all that agents who receive the largest probability for the object in position $\tilde{k}$ will be better. This implies that all the first non-zero elements in B^x will shift part of the probabilities to more preferred objects, which contradicts that it is a Rawlsian assignment.)

Then, for every agent i we have that: $p_i^{x'}(\text{Top } \tilde{k}) \geq p_j^x(\text{Top } \tilde{k}) > p_l^x(\text{Top } \tilde{k})$. This implies that the element in the same position where B^x has its first positive entry, is strictly lower in x' , which is a contradiction. $\square$

8.4 The MTAV mechanism.

In this section we define the MTAV mechanism following [Paleo \(2021\)](#). Let $\mathcal{M}$ be the set of deterministic assignment. Each assignment in $\mathcal{M}$ is represented by a $n \times n$ matrix M where $m_{ij} = 1$ if, and only if, agent i receives object j . Given two matrices M and M' in $\mathcal{M}$, we denote by $M \odot M'$ the matrix where each element is the product of the corresponding elements of M and M' : $(M \odot M')_{ij} = (m_{ij})(m'_{ij})$. Also, denote as $\max(M)$ and $\text{sum}(M)$ the maximum and the sum of the elements of M , respectively.

Given a problem $\succeq$, define the matrix P with agents' preferences, where P_{ij} is the rank of object j in i 's preferences (equivalently, $P_{ij} = r_{ij}$). The MTAV is defined as follows.

1. For each $M \in \mathcal{M}$ computes $P \odot M$.
2. Compute $\max(P \odot M)$.
3. Select the assignments that minimize $\max(P \odot M)$.
4. Among the assignments selected in the previous step, select those that minimizes $\text{sum}(P \odot M)$.
5. If more than one assignment are selected in the last step, take one assignment at random.